Homework 3

1. Show that the conclusion to Morera's Theorem still holds if it is assumed only that the integral of f around the boundary of every rectangle in U or every triangle in U is 0. Hint: It suffices to treat the case of U a disc centered at the origin. Then it suffices to show that the integral from $(0,0)$ to $(x,0)$ followed by the integral from $(x,0)$ to (x,y) equals the integral from $(0,0)$ to $(0,y)$ followed by the integral from $(0,y)$ to (x,y). This follows from using two triangles.

2. Show that in exercise 1, you can replace triangle by disc. Hint: Green's Theorem.

3. Prove a version of l'Hopital's rule for holomorphic functions: If

$$\lim_{z\to p}\frac{f(z)}{g(z)}$$

is an indeterminate expression (you must decide what this means and state your hypotheses) for f and g holomorphic, then the limit may be evaluated by considering

$$\lim_{z\to p}\frac{\frac{\partial f}{\partial z}(z)}{\frac{\partial g}{\partial z}(z)}$$

4. Suppose that f and g are entire functions and that g never vanishes. If $|f(z)| \leq |g(z)|$ for all z, then prove that there is a constant C such that $f(z) = Cg(z)$. What if g does have zeros?

5. Let $U \subset \mathbb{C}$ be an open set. Let $f : U \to \mathbb{C}$ be holomorphic and bounded. Let $p \in U$. Prove that:

$$\left|\frac{\partial^k f}{\partial z^k}(p)\right| \leq \frac{k!}{r^k}\sup_U |f|$$

where r is the distance of p to the complement of U in $\mathbb{C}$.

6. If $f_j : U \to \mathbb{C}$ are holomorphic and $|f_j| \leq 2^{-j}$, then prove that $\sum_{j=0}^{\infty} f_j$ converges to a holomorphic function on U.

7. Let $X = \{f \in C(\overline{U}) : f \in \mathrm{Hol}(U)\}$. Define a norm on X by $\|f\|_X = \sup_{\overline{U}} |f|$. Prove that X with this norm is a Banach space and that for any $p \in U$ and $k \in \mathbb{N}$ the map $f \mapsto \frac{\partial^k f}{\partial z^k}(p)$ is a bounded linear functional.

8. Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence. Hint: Use the Binomial Theorem with the fact that $z = z_0 + (z - z_0)$.

9. Prove the following:

- The power series $\sum_n nz^n$ does not converge on any point of the unit circle.
- The power series $\sum_n \frac{z^n}{n^2}$ converges at every point of the unit circle.

- The power series $\sum_n \frac{z^n}{n}$ converges at every point of the unit circle except $z = 1$. Hint: Use summation by parts.

10. Let $\{p_j\}$ be holomorphic polynomials and assume that for all j the degree of p_j does not exceed some fixed N. If $\{p_j\}$ converges uniformly on compact sets, prove that the limit function is a holomorphic polynomial of degree not exceeding N.