Homework 4

1. Let f be holomorphic on $D_r(p) \setminus \{p\}$ and suppose that f has a pole of order k at p. Show that the Laurent series coefficients a_j of f expanded about the point p for $j = -k, -k+1, \ldots$ are given by the formula

$$a_j = \frac{1}{(k+j)!} \partial^{k+j} \left((z-p)^k f \right) \Big|_{z=p}.$$

2. Let $R(z) = \frac{p(z)}{q(z)}$ where p and q are holomorphic polynomials. Let f be holomorphic on $\mathbb{C} \setminus \{p_1, \ldots, p_k\}$ and suppose that f has a pole at each of the points $\{p_1, \ldots, p_k\}$. Finally, assume that $|f(z)| \leq |R(z)|$ for all z at which $f(z)$ and $R(z)$ are defined. Prove that f is a constant multiple of R, in particular that f is rational. Hint: Think about $\frac{f}{R}$.

3. Prove that if $f : D_r(p) \setminus \{p\} \to \mathbb{C}$ has an essential singularity at p, then for each positive integer N, there is a sequence $\{z_n\} \subset D_r(p) \setminus \{p\}$ with $\lim z_n = p$ and:

$$\left| (z_n - p)^N f(z_n) \right| \geq N.$$

4. Calculate the annulus of convergence of the following Laurent series. Also determine convergence at any boundary points.

   1. $\sum_{j=-\infty}^{\infty} 2^{-j} z^j$
   2. $\sum_{j=-\infty, j\neq 0}^{\infty} \frac{z^j}{j^j}$

5. Let $p = 0$, classify each of the following as having a removable singularity, a pole, or an essential singularity.

   1. $\frac{1}{z}$
   2. $\sin \frac{1}{z}$
   3. $\frac{\sin z}{z}$
   4. $\frac{\cos z}{z}$

6. Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Define $g(z) = f(1/z)$. Prove that f is a polynomial if and only if g has a pole at 0.

7. Let $\{a_j : j \in \mathbb{Z}\}$ be given. Fix $k > 0$. Prove that if $\sum_{j=0}^{\infty} a_j z^j$ converges on $D_r(0)$ for some $r > 0$, then $\sum_{j=-k+1}^{\infty} a_j a^{j+k}$ converges on $D_r(0)$.

8. Use the calculus of residues to compute the following integrals:

   1. $\frac{1}{2\pi i} \oint_{\partial B_5(0)} \frac{z}{(z+1)(z+2i)} dz.$
   2. $\frac{1}{2\pi i} \oint_{\gamma} \frac{e^z}{z(z+1)(z+2)} dz$ where γ is the negatively oriented triangle with vertices $1 \pm i$ and -3.
   3. $\frac{1}{2\pi i} \oint_{\gamma} \frac{e^z}{(z+3i)^2(z+3)^2(z+4)} dz$ where γ is the positively oriented rectangle with vertices $2 \pm i$ and $-8 \pm i$.

9. Use the calculus of residues to evaluate the following integrals:

   1. $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$.

   2. $\int_0^{\infty} \frac{1}{p(x)} dx$ where $p(x)$ is any polynomial with no zeros on the nonnegative real axis.

10. Let $f(z) = e^{z+\frac{1}{z}}$. Prove that $\text{Res}_f(0) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}$.