Homework 5

1. Let U and V be open subsets of $\mathbb{C}$. Suppose that $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ are continuous functions such that $f(U) \subset V$ and $g(f(z)) = z$ for all $z \in U$. If g is differentiable and $g'(z) \neq 0$, then f is differentiable and $f'(z) = \frac{1}{g'(f(z))}$.

2. Let U and V be open subsets of $\mathbb{C}$. Suppose that $f : U \to V$ is holomorphic, one-to-one, and onto. Prove that f^{-1} is a holomorphic function on V.

3. Estimate the number of zeros of the given function in the given region:

- $f(z) = z^8 + 5z^7 - 20$ on $U = D_6(0)$;
- $f(z) = z^2 e^z - z$ on $U = D_2(0)$.

4. Give another proof of the Fundamental Theorem of Algebra as follows. Let $p(z)$ be a nonconstant polynomial. Fix $q \in \mathbb{C}$ and consider:

$$\oint_{\partial D_R(q)} \frac{p'(z)}{p(z)} dz.$$

Argue that as $R \to \infty$ this expression tends to a non-zero constant.

5. Suppose that f is analytic on $D_1(0)$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions (counting multiplicity) of the equation $f(z) = z^n$ where n is an integer that is larger than or equal to 1.

6. Suppose that $|f(z)| \leq 1$ for $|z| < 1$ and f is a non-constant analytic function. Prove that:

$$\frac{|f(0)| - |z|}{1 + |f(0)|\,|z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)|\,|z|}.$$

Hint: Consider $g(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)}$.

7. Let f be analytic in $D_1(0)$ and suppose that $|f(z)| \leq M$ for all $z \in D_1(0)$. If $f(z_k) = 0$ for $1 \leq k \leq n$, show that:

$$|f(z)| \leq M \prod_{j=1}^{n} \frac{|z - z_j|}{|1 - \overline{z_j} z|}, \quad |z| < 1.$$

8. Prove the maximum principle for harmonic functions. Namely,

(a) If u is a non-constant real-valued harmonic function in a region Ω then u cannot attain a maximum (or minimum) in Ω.

(b) Suppose that Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$ then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} \setminus \Omega} |u(z)|.$$

Hint: Assume u us achieves a local maximum at a point z_0. Let f be holomorphic near z_0 with $u = \operatorname{Re} f$. Show that f is not open.