Homework 6

1. Prove that if f is entire and one-to-one then f must be linear.

2. If $\Omega \subset \mathbb{C}$ is a domain, let $\mathrm{Aut}(\Omega)$ be the collection of all conformal mappings of Ω to Ω. $\mathrm{Aut}(\Omega)$ is the automorphism group of Ω. Prove that $\mathrm{Aut}(\Omega)$ is a group with group operation being composition of functions. Let Ω_1 and Ω_2 be domains in $\mathbb{C}$. Suppose that $\Phi : \Omega_1 \to \Omega_2$ is a conformal map. Give a relationship between $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_2)$.

3. Let $\{f_\alpha\}$ be a normal family of holomorphic functions on a domain U. Prove that $\{f'_\alpha\}$ is a normal family.

4. Let $\Omega \subset \mathbb{C}$ be a bounded domain and let $\{f_j\}$ be a sequence of holomorphic functions on Ω. Assume that:

$$\int_\Omega |f_j(z)|^2 \, dxdy < C < \infty.$$

where C is an absolute constant independent of j. Prove that $\{f_j\}$ is a normal family.

5. In this problem you will describe the automorphism group of an annulus and determine when two annuli are confromally equivalent. Let $\mathbb{A}_{a,b}(z_0) = \{z : a < |z - z_0| < b\}$ denote the annulus with center z_0 and inner radius a and outer radius b. Define the modulus of the annulus to be $\frac{1}{2\pi} \log \frac{b}{a}$.

(a) Show that any conformal map from one annulus centered at the origin to another such annulus extends to a conformal map of $\mathbb{C} \setminus \{0\}$.

(b) Show that there is a conformal map of one annulus onto another if and only if the annuli have the same moduli.

(c) Show that any automorphism of the annulus $\mathbb{A}_{a,b}(0)$ is either a rotation $z \mapsto e^{i\phi} z$ or a rotation followed by the inversion $z \mapsto \frac{ab}{z}$.

6. Prove Vitali's Theorem. Suppose that G is an open connected set. Assume that there is a locally bounded collection $\{f_n\}$ of holomorphic functios on G and a function f that is holomorphic on G such that the set:

$$A = \{z \in G : \lim_{n \to \infty} f_n(z) = f(z)\}$$

has a limit point in G. Show that $f_n \to f$.

7. A holomorphic function $f : U \to \mathbb{C}$ is called a "branch of $\log z$" on U if $e^{f(z)} = z$ for all $z \in U$. Prove:

(a) There is a branch of $\log z$ defined on any open disc not containing the origin;

(b) There is a branch of $\log z$ defined on $\mathbb{C} \setminus (\{0\} \cup L)$ where L is an open half line emanating from the origin;

(c) There is no branch of $\log z$ defined on any open set U containing $\{z : |z| = 1\}$;

(d) If there is a continuous function $g : U \to \mathbb{C}$ such that $e^{g(z)} = z$ for $u \in U$, then g is necessarily holomorphic and hence a branch of $\log z$ on U.