Homework 7

1. Prove that if u is harmonic on $\mathbb{C}$ and bounded on all of $\mathbb{C}$, then u is constant. Hint: Use Liouville's Theorem for holomorphic functions.

2. Let $\mathcal{L}$ be a partial differential operator of the form:

$$\mathcal{L} = a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + c\frac{\partial^2}{\partial x \partial y}$$

with a, b, c constants. Assume that $\mathcal{L}$ commutes with rotations in the sense that $\mathcal{F}f \circ \rho_\theta = \mathcal{L}(f \circ \rho_\theta)$ where $\rho_\theta(z) = e^{i\theta}z$. Prove that $\mathcal{L}$ is a constant multiple of the Laplacian.

3. Compute a formula for the Poisson integral formula for the region $U = \{z : \operatorname{Im} z > 0\}$. Hint: Use the formula for the disc and do a conformal map.

4. Use Montel's theorem to prove that if $U \subset \mathbb{C}$ is a domain and if $\mathcal{F}$ is a family of harmonic functions such that $|f_\alpha(z)| \leq M < \infty$ for all $z \in U$ then there is a subsequence f_{α_j} that converges uniformly on compact subsets of U. Hint: Choose g_α so that $f_\alpha + ig_\alpha$ is holomorphic. Consider $e^{f_\alpha + ig_\alpha}$.

5. Let $u : U \to \mathbb{C}$ be harmonic and $\overline{D_r(p)} \subset U$. Prove the following two corollaries of the mean value property as corollaries of the mean value property proved in class.

- $u(p) = \frac{1}{2\pi r}\int_{\partial D_r(p)} u(\xi)ds(\xi)$ where ds is arc length measure on $\partial D_r(p)$.
- $u(p) = \frac{1}{\pi r^2}\int_{D(p,r)} u(x,y)dxdy$.

6. Prove that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane.

7. Suppose that Ω is a simply connected domain that is bounded by a piecewise-smooth closed curve γ. Then any conformal map of $\mathbb{D}$ to Ω extends to a continuous bijection of $\overline{\mathbb{D}}$ to $\overline{\Omega}$. Hint: Consult Theorem 4.2 and Exercise 18 from Stein and Shakarchi.