Homework 8

1. If $b_n > 1$ for all n, the prove that $\prod_n b_n$ converges if and only if $\sum_n \log b_n < \infty$.

2. Let $\{a_j\} \subset \mathbb{C} \setminus \{-1\}$. Prove that if

$$\lim_{n\to\infty} \prod_{j=1}^{n} (1 + a_j)$$

exists and is nonzero, then $a_j \to 0$.

3. If $|z| < R$, then prove that

$$\prod_{n=0}^{\infty} \left(\frac{R^{2^n} + z^{2^n}}{R^{2^n}} \right) = \frac{R}{R - z}.$$

4. Calculate explicitly

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2} \right).$$

5. Suppose that $\sum |\alpha_n - \beta_n| < \infty$. Determine the largest set of z such that:

$$\prod_{n=1}^{\infty} \frac{z - \alpha_n}{z - \beta_n}$$

converges normally.

6. Let $z_0 \in D_r(0)$ be fixed. Let f be holomorphic on a neighborhood of $\overline{D_r(0)}$. Let $a_1, \dots, a_n$ be the zeros of f in $D_r(0)$ and assume that no zero lies on $\partial D_r(0)$. Let $\phi(z) = \frac{r^2 z + r z_0}{r + z \overline{z_0}}$. Apply Jensen's formula to $f \circ \phi$ and do a change of variable in the integral to obtain the formula:

$$\log |f(z_0)| + \sum_{k=1}^{n} \log \left| \frac{r^2 - \overline{a_k} z_0}{r(z_0 - a_k)} \right| = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(\frac{re^{i\theta} + z_0}{re^{i\theta} - z_0} \right) \log \left| f(re^{i\theta}) \right| d\theta.$$

7. Prove that if f is holomorphic in the unit disc, bounded and not identically zero, and if $z_1, z_2, \dots, z_n, \dots$ are its zeros $|z_k| < 1$, then:

$$\sum_n (1 - |z_n|) < \infty.$$