Homework 9

1. Let U be a domain and $p \in U$. Let γ be a closed curve in U that begins and ends at p. Show that $\gamma \cdot \gamma^{-1}$ and $\gamma^{-1} \cdot \gamma$ are homotopic to the constant curve at p. Hint: For the first one, define

$$H(s,t) = \begin{cases} \gamma(2ts) & : \ 0 \leq t \leq \frac{1}{2} \\ \gamma(2s(1-t)) & : \ \frac{1}{2} < t \leq 1. \end{cases}$$

2. Let U be a domain and $p \in U$. Let $\gamma_1, \gamma_2, \gamma_3$ be closed curves based at p. Show that it γ_1 is homotopic to γ_2, and γ_2 is homotopic to γ_3 then γ_1 is homotopic to γ_3. Hint: If H_1 is the homotopy between γ_1 and γ_2 and H_2 is the homotopy between γ_2 and γ_3 then define:

$$H(s,t) = \begin{cases} H_1(2s,t) & : \ 0 \leq s \leq \frac{1}{2} \\ H_2(2s-1,t) & : \ \frac{1}{2} < s \leq 1. \end{cases}$$

3. Consider the holomorphic function $f(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^2}$. Determine the largest open set to which f can be analytically continued. Can you calculate a closed formula for f?

4. Let ϕ be a continuous function with compact support on $(0, \infty) \subset \mathbb{R}$ (i.e. ϕ is continuous on all of $\mathbb{R}$, but ϕ vanishes off some compact set). Define:

$$F(z) = \int_0^{\infty} \phi(t) e^{tz} dt.$$

For which values of z is the function well defined and holomoprhic? Can one use integration by parts to extend the domain of the function? What is the maximal domain of definition?