PROBLEM SET 11

(Hand in all.)

(1) Show (i) $x^5 - 4x + 2$ and (ii) $x^4 + 4x^3 + 10x^2 + 12x + 7$ are irreducible in $\mathbb{Q}[x]$. [Hint: for one of them, apply a linear change of variable.]

(2) [Jacobson p. 154 #3] Show that if p is a prime (in $\mathbb{Z}$) then the polynomial obtained by replacing x by $x+1$ in $x^{p-1} + x^{p-2} + \cdots + 1 = (x^p - 1)/(x-1)$ is irreducible in $\mathbb{Q}[x]$. Hence prove that the *cyclotomic polynomial* $x^{p-1} + x^{p-2} + \cdots + 1$ is irreducible in $\mathbb{Q}[x]$.

(3) [Jacobson p. 154 #4] Obtain factorizations into irreducible factors in $\mathbb{Z}[x]$ of the following polynomials: $x^3 - 1$, $x^4 - 1$, $x^5 - 1$, $x^6 - 1$, $x^7 - 1$, $x^8 - 1$, $x^9 - 1$, $x^10 - 1$.

(4) Let $K = \mathbb{Q}[\sqrt{-29}]$. From III.L.26 we know that $[\mathfrak{P}_5] \in Cl(K)$ has order 3, and we also note that $(2) = (2, 1 + \sqrt{-29})^2 =: \mathfrak{P}_2^2$. Show that $\mathcal{O}_K$ has ideals of norm 3 and 11 of order 6 in $Cl(K)$. [Hint: start by looking for principal ideals of norm 30 and 33.]

(5) We explained how odd primes p decompose in number rings. What about the even prime? Let $K = \mathbb{Q}[\sqrt{d}]$, and show that (in $\mathcal{O}_K$)

$$\begin{aligned}
d \underset{(4)}{\equiv} 2 &\implies (2) = (2, \sqrt{d})^2 \\
d \underset{(4)}{\equiv} 3 &\implies (2) = (2, 1 + \sqrt{d})^2 \\
d \underset{(8)}{\equiv} 1 &\implies (2) = (2, \tfrac{1+\sqrt{d}}{2})(2, \tfrac{1-\sqrt{d}}{2}) \\
d \underset{(8)}{\equiv} 5 &\implies (2) \text{ prime}
\end{aligned}$$

(6) Let $K = \mathbb{Q}[\sqrt{-26}]$. Find all non-principal ideals of norm 30 in $\mathcal{O}_K$. [Hint: here are some of your tools: Prop. III.L.25, Pell's equation (i.e. using solutions of $x^2 + 26y^2 = m$ to test whether there exists a principal ideal of norm m), uniqueness of ideal factorization, and Caesar.]

(7) Show that $X^2 = Y^3 - 14$ has no solution with $X, Y \in \mathbb{Z}$. You may assume that $h_{\mathbb{Q}(\sqrt{-14})} = 4$. [Hint: if (X, Y) is a solution, put $\alpha := X + \sqrt{-14}$ (not $X + Y\sqrt{-14}$!!). Turn the equation into an equation of ideals, decompose both sides into prime factors, and use uniqueness of ideal factorization to deduce that α is a cube in $\mathbb{Z}[\sqrt{-14}]$.]

(8) Let $K = \mathbb{Q}[\sqrt{d}]$, $d \underset{(4)}{\equiv} 2$ or 3, and $I \subset \mathcal{O}_K$ an ideal. Show that $\mathfrak{N}(I) = |\mathcal{O}_K / I|$, where $\mathfrak{N}(I)$ is defined via Hurwitz. [Hint: first compute $|\mathcal{O}_K / I|$ as a determinant.]