PROBLEM SET 12

(Hand in #s 1, 3, 4, 7, 8, 9, 10, 11.)

(1) [Jacobson p. 163 #4,5] (i) Show that $\mathrm{End}(\mathbb{Q}) \cong \mathbb{Q}$, where on the left hand side, $\mathbb{Q}$ is considered *as an abelian group* under addition, and on the right, as a ring. [Hint: there is a natural map from right to left. Apply the Fundamental Theorem IV.B.9.]
(ii) Replacing $\mathbb{Q}$ by an arbitrary field R, does this remain true (i.e. $\mathrm{End}(R) \cong R$)?

(2) [Jacobson p. 165 #2] Let M be a left R-module and let $B = \{b \in R \mid bx = 0\,(\forall x \in M)\}$. (i) Show that B is an ideal in R. (ii) Show that if C is any ideal contained in B then M becomes a left R/C-module by defining $(a + C)x := ax$.

(3) [Jacobson p. 166 #5] Let $V = \mathbb{R}^n$, and define a linear transformation $T\colon V \to V$ by $T(x_1, \ldots, x_n) := (x_n, x_1, \ldots, x_{n-1})$. Consider V as a left $\mathbb{R}[\lambda]$-module as in Example IV.A.2(h), and define $B \subset R$ as in (2). Describe B explicitly.

(4) [Jacobson p. 166 #8] Let M be a (nonzero) finite abelian group. Can M be made into a $\mathbb{Q}$-module?

(5) [Jacobson p. 169 #2] Determine $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$ for $m, n \in \mathbb{Z}_{>0}$.

(6) [Jacobson p. 169 #5,6] (i) Show that, for a left module M over a ring R, $\mathrm{End}_R(M)$ is the centralizer in $\mathrm{End}(M)$ of the set of group endomorphisms ℓ_r, $r \in R$. [Remark: $\mathrm{End}(M)$ with no subscript means abelian group homomorphisms; with the subscript R, it means R-module homomorphisms.]
(ii) Do we have $\ell_r \in \mathrm{End}_R(M)$?

(7) Regarding $\mathbb{Q}^2$ as a module over $\mathbb{Q}[i]$ by $P(i).\vec{v} := P\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\vec{v}$, compute $End_{\mathbb{Q}[i]}(\mathbb{Q}^2)$ explicitly as a subring of $End(\mathbb{Q}^2) = M_2(\mathbb{Q})$. [Here $i = \sqrt{-1}$.]

(8) [Jacobson p. 170 #8] A left ideal I of R is called maximal if $R \neq I$ and there exist no left ideals I' such that $I \subsetneq I' \subsetneq R$. Show that a module M is irreducible if and only if $M \cong R/I$ where I is a maximal left ideal of R.

(9) [Jacobson p. 175 #3] Let R_n denote a free *right* R-module with base $\{e_1, \ldots, e_n\}$. Let $\eta \in \mathrm{End}_R(R_n)$ and write $\eta(e_i) = \sum_{j=1}^n e_j a_{ji}$. Show that $\eta \mapsto A = [a_{ij}]$ yields an isomorphism of $\mathrm{End}_R(R_n)$ with $M_n(R)$.

(10) [Jacobson p. 175 #4] Let R be commutative. Show that if η is a surjective endomorphism of R^n (as R-module), then η is bijective. Does the same conclusion hold if η is injective?

(11) [Jacobson p. 179 #2] Let M be a (left) module (over some ring R), and $M_1, \ldots, M_n$ be submodules such that $M = \sum_i M_i$ and the "triangular" set of conditions

$$M_1 \cap M_2 = 0$$
$$(M_1 + M_2) \cap M_3 = 0$$
$$\vdots$$
$$(M_1 + \cdots + M_{n-1}) \cap M_n = 0$$

hold. Show that $M = \oplus_i M_i$.

(12) [Jacobson p. 179 #3] Show that $\mathbb{Z}_{p^e}$, p a prime, $e > 0$, regarded as a $\mathbb{Z}$-module, is not a direct sum of any two nonzero submodules. Does this hold for $\mathbb{Z}$? Does it hold for $\mathbb{Z}_n$ for other positive integers n?