Problem Set 2

Hand in all.

(1) Show that $\exp(\mathfrak{S}_n) = \text{lcm}[1, ..., n]$. Find a counterexample to the " $\Longleftarrow$ " part of Corollary (II.D.15) if G is not assumed abelian.

(2) Find a counterexample to Prop. (II.D.13) for G nonabelian.

(3) Using definition (II.E.10), prove that $H \times K$ is a direct product of H and K. Then prove that, up to isomorphism, it is the unique direct product of H and K.

(4) Show, carefully, that the order of the element $\bar{a} = a + n\mathbb{Z}$ of $\mathbb{Z}_n$ is $n/(n,a)$.

(5) Which of the following groups are isomorphic: $\mathbb{Z}_4 \times \mathbb{Z}_6$, D_{12}, $\mathbb{Z}_{12} \times \mathbb{Z}_2$, $\mathfrak{A}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_{35}^*$, $D_6 \times \mathbb{Z}_2$?

(6) Check that conjugation induces automorphisms, and that for any permutation α, $\alpha(i_1 i_2 \cdots i_r)\alpha^{-1} = (\alpha(i_1)\alpha(i_2)\cdots\alpha(i_r))$.

(7) Compute the group of automorphisms (self-isomorphisms) of $\mathbb{Z}$, $\mathbb{Z}_m$, and $\mathfrak{S}_3$. [Hint: any homomorphism is determined by where it sends a generating set (why?), and any isomorphism sends elements to elements of the same order.] By "compute", I mean construct an isomorphism from some group we have written down already to $\text{Aut}(G)$ in each case.

(8) [Jacobson p. 53 #3] Let H_1 and H_2 be subgroups of G. Show that any right coset relative to $H_1 \cap H_2$ is the intersection of a right coset of H_1 with a right coset of H_2. Use this to prove *Poincaré's Theorem* that if H_1 and H_2 have finite index in G then so has $H_1 \cap H_2$.

(9) [Herstein] Let G be a finite group, $\alpha \in \text{Aut}(G)$ be an automorphism of G, and suppose that the subset $I := \{x \in G \mid \alpha(x) = x^{-1}\}$ has cardinality $|I| > \frac{3}{4}|G|$. Show that G is abelian.