Problem Set 3

Hand in all.

(1) Prove Burnside's lemma:[1] *given a finite group G acting on a finite set X, write X^g for the fixed-point set of $g \in G$ and X/G for the set of orbits. Then we have*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Prove this by decomposing in two different ways [Hint: stabilizers and fixed-point sets] the subset $S \subset G \times X$ consisting of all ordered pairs (g, x) with $g.x = x$, and using the Orbit-Stabilizer Theorem.

(2) Let $X = \{1, 2, 3, 4\}$ and G be the subgroup of the symmetric group $\mathfrak{S}_4$ generated by (1234) and (24). Work out the orbits and stabilizers for the diagonal action of G on $X \times X$ (i.e. $g.(x, y) := (g.x, g.y)$); also verify Burnside in this case.

(3) Let n be even, and α and β be an $(n-1)$-cycle resp. an $(n-3)$-cycle in the alternating group $\mathfrak{A}_n$. Compute the orders of $\mathrm{ccl}_{\mathfrak{A}_n}(\alpha)$ and $\mathrm{ccl}_{\mathfrak{A}_n}(\beta)$.

(4) Show that every group of order $4n + 2$ contains a subgroup of order $2n + 1$. [Hint: use Cayley's theorem and Cauchy's theorem and think odd and even.]

(5) Think of $\mathfrak{A}_4$ as the group of rotational symmetries of a regular tetrahedron T, and let $E_1, \ldots, E_6$ be the edges of T. Each element of $\mathfrak{A}_4$ permutes $E_1, \ldots, E_6$ and therefore gives us an element of $\mathfrak{S}_6$. Work out the 12 elements of $\mathfrak{S}_6$ which occur in this way, starting with the cycle structure. Explain why they must form a subgroup. (Use a result rather than checking it by hand.)

[1] or, as has been suggested, "not-Burnside's lemma" since Burnside attributed it to Frobenius...