

PROBLEM SET 7

(Hand in all.)

- (1) [Jacobson p. 91 #2] Show that a domain contains no idempotents ($e^2 = e$) except $e = 0$ and $e = 1$. An element z is called *nilpotent* if $z^n = 0$ for some $n \in \mathbb{Z}_{>0}$. Show that 0 is the only nilpotent in a domain.
- (2) [Jacobson p. 91 #6] Let u be an element of a ring that has a right inverse. Prove that the following conditions on u are equivalent: (1) u has more than one right inverse; (2) u is not a unit; (3) u is a left zero-divisor.
- (3) [Jacobson p. 91 #7] Prove that if an element of a ring has more than one right inverse then it has infinitely many. Construct a counterexample to show that this does not hold for monoids.
- (4) [Jacobson p. 100 #7] Let m and n be non-zero integers and let R be the subset of $M_2(\mathbb{C})$ consisting of the matrices of the form

$$\begin{pmatrix} a + b\sqrt{m} & c + d\sqrt{m} \\ n(c - d\sqrt{m}) & a - b\sqrt{m} \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Q}$. Show that R is a subring of $M_2(\mathbb{C})$ and that R is a division ring if and only if the rational numbers x, y, z, t satisfying the equation $x^2 - my^2 - nz^2 + mnt^2 = 0$ are $x = y = z = t = 0$. Give a choice of m, n for which R is a division ring and a choice of m, n for which R is not a division ring. [N.B. These rings are called “rational quaternion algebras”.]

- (5) Find the group of units in all number rings $\mathbb{Z}[\sqrt{d}]$ for integers $d < 0$ and $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ for $d < 0$ and $d \equiv 1 \pmod{4}$.
- (6) Show that $R = \mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ is Euclidean.
- (7) [Jacobson p. 97 #1] Show that the matrix

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \\ -3 & -6 & -8 \end{pmatrix}$$

is invertible in $M_3(\mathbb{Z})$ and find its inverse.

- (8) [Jacobson p. 97 #2] Prove that if R is a commutative ring then $AB = \mathbf{1}$ in $M_n(R)$ implies $BA = \mathbf{1}$. (This is not always true for noncommutative R .)