

PROBLEM SET 8

(Hand in all.)

- (1) [Jacobson p. 97 #7] Determine $C(S)$ (the set of all elements commuting with S) where S is the single matrix $N = \mathbf{e}_{12} + \mathbf{e}_{23} + \cdots + \mathbf{e}_{n-1,n}$.
- (2) Show that the set of ideals in a ring R forms a monoid under multiplication. Do they satisfy the distributive law $I(J + K) = IJ + IK$?
- (3) State and prove a uniqueness result for Hermite normal form. (We know it isn't unique as a matrix, unlike rref for fields. So, what attributes are unique (i.e. constitute invariants of the original matrix)?
- (4) Prove that ± 3 are not of the form $m^2 - 10n^2$ ($m, n \in \mathbb{Z}$).
- (5) [Jacobson p. 103 #6] Let R be a commutative ring and N denote the set of nilpotent elements of R . Show that N is an ideal and that R/N contains no non-zero nilpotent elements.
- (6) [Jacobson p. 109 #2] (a) Given $u \in R^*$ and $\eta: R \rightarrow R'$ a homomorphism, show that $\eta(u)$ is a unit in R' . (b) Suppose η is surjective. Does this imply that (under η) the group of units R^* surjects onto $(R')^*$?
- (7) [Jacobson p. 110 #4] Show that if R is a commutative ring of prime characteristic p , then $\phi: R \rightarrow R$ defined by $a \mapsto a^p$ is a homomorphism (also called an "endomorphism of R "). Is this an automorphism?
- (8) [Jacobson p. 110 #5] Let F be a finite field of characteristic p (a prime). Show that $p - 1 \mid |F| - 1$. Hence conclude that if $|F|$ is even then the characteristic is 2. (We shall see later that $|F|$ is a power of p .)
- (9) [Jacobson p. 110 #7] If S is a subset of a ring (field) R then the *subring (subfield) generated by S* is defined to be the intersection of all the subrings (subfields) containing S . If this is R itself then S is called a *generating set* for R . Show that if η_1 and η_2 are homomorphisms from R to R' which agree on the elements of a generating set $S \subset R$, then $\eta_1 = \eta_2$.
- (10) Let $R = \mathbb{Z}[\alpha]$, where $\alpha := \frac{1 + \sqrt{-19}}{2}$. Show that R is *not* Euclidean, in contrast to problem (6) from last week. [Hint: let s be a nonzero nonunit of minimal $\delta(s)$, and consider the image of α in $R/(s)$. Use the fact that $\alpha^2 - \alpha + 5 = 0$, and the result of last week's problem (5).]