

PROBLEM SET 9

(Hand in all.)

- (1) [Jacobson p. 111 #17] Determine the ideals and maximal ideals and prime ideals of $\mathbb{Z}_{60} \cong \mathbb{Z}/(60)$.
- (2) [Jacobson p. 118 #5] Let R be a commutative ring, and S a submonoid of the multiplicative monoid of R . In $R \times S$, define $(a, s) \sim (b, t)$ if there exists a $u \in S$ such that $u(at - bs) = 0$. Show that this is an equivalence relation in $R \times S$. Denote the equivalence class of (a, s) as a/s and the quotient set consisting of these classes as RS^{-1} . Show that RS^{-1} becomes a ring relative to $a/s + b/t = (at + bs)/st$, $(a/s)(b/t) = ab/st$, $0 = 0/1$, and $1 = 1/1$. Show that $a \mapsto a/1$ is a homomorphism of R into RS^{-1} , and that this is a monomorphism if and only if no element of S is a zero-divisor in R . Show that the elements $s/1, s \in S$, are units in RS^{-1} .
- (3) [Jacobson p. 126 #2] Show that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$ and that the real numbers $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are linearly independent over \mathbb{Q} . Show that $u = \sqrt{2} + \sqrt{3}$ is algebraic and determine an ideal I such that $\mathbb{Q}[x]/I \cong \mathbb{Q}[u]$.
- (4) [Jacobson p. 133 #9] Show that the ideal $(3, x^3 - x^2 + 2x - 1)$ in $\mathbb{Z}[x]$ is not principal.
- (5) [Jacobson p. 133 #13] Prove Wilson's theorem: if p is a prime in \mathbb{Z} , then we have $(p-1)! \equiv_{(p)} -1$.
- (6) A commutative ring is *local* if it has a unique maximal ideal.
 - (a) Using problem (2) above with $R = \mathbb{Z}, S = \{b \in \mathbb{Z} \mid p \nmid b\}$, prove that RS^{-1} is local.
 - (b) If \mathcal{R} is a local ring with unique maximal ideal \mathfrak{m} , prove that $a \in \mathcal{R}$ is a unit if and only if $a \notin \mathfrak{m}$. [Hint: you will need to use (III.D.27).]
- (7) Find an inverse (fractional ideal) for

$$I = (15 + 25\sqrt{-10}, 25 - 40\sqrt{-10}) \subset \mathbb{Z}[\sqrt{-10}] = R.$$
- (8) [Jacobson p. 140 #3] (Newton's identities. See Jacobson. Statement of problem is long but solution need not be.)