Part I: Circle your answer (2 points each).

1. If the function $f(x)$ is negative for $1 \leq x \leq 5$ then $\int_{2}^{4} f(x) \, dx$ must be negative.
   
   \[ \text{True} \] \hspace{1cm} \text{False} \\

2. $\int \frac{d}{dx} x^2 (\cos x)^3 \, dx = x^2 (\cos x)^3 + C$
   
   \[ \text{True} \] \hspace{1cm} \text{False} \\

3. The improper integral $\int_{2}^{\infty} \frac{1}{x} \, dx$ converges.
   
   \[ \int \frac{1}{x} \, dx = \ln x \]
   
   \[ \ln x \bigg|_{2}^{M} = \ln M - \ln 2 \quad \rightarrow \infty \quad \text{as} \quad M \rightarrow \infty \]
   
   \[ \text{True} \] \hspace{1cm} \text{False} \\

4. The midpoint rule for estimating integrals generally gives more accuracy than Simpson's rule used with the same value of $n$.
   
   \[ \text{True} \] \hspace{1cm} \text{False} \\

5. $\lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^2 \cos \left( \frac{i}{n} \right) \frac{1}{n} = \int_{0}^{1} x^2 \cos x \, dx$
   
   \[ \text{True} \] \hspace{1cm} \text{False}
Part II: Evaluate the following integrals, show enough work to make it clear how you got your answer (10 points each).

1. \[ \int_{-1}^{1} 2x^5 - x^3 \, dx = \frac{2x^6}{6} - \frac{x^4}{4} \bigg|_{-1}^{1} = \frac{2}{6} - \frac{1}{4} - \left( \frac{2}{6} - \frac{1}{4} \right) \]

2. \[ \int_{0}^{1} x\sqrt{1+x^2} \, dx = \frac{1}{2} \int \sqrt{u} \, 2x \, dx \\
\quad = \frac{1}{2} \int \sqrt{u} \, du \\
\quad = \frac{1}{2} \left[ \frac{1}{2} \sqrt{u} \right]_{0}^{1} \\
\quad = \frac{1}{2} \left[ \frac{1}{2} \sqrt{1+x^2} \right]_{0}^{1} \\
\quad = \frac{1}{4} \left( \sqrt{2} - 1 \right) \\
\quad = \frac{1}{4} \left( \frac{1}{\sqrt{2}} - 1 \right) \]
3. \[ \int x \cos 2x \, dx \]

\[ u = x \quad v = \frac{1}{2} \sin 2x \]
\[ du = dx \quad dv = \cos 2x \, dx \]

\[ \int = x \frac{1}{2} \sin 2x - \int \frac{1}{2} \sin 2x \, dx \]

\[ = x \frac{1}{2} \sin 2x - \left( -\frac{1}{2} \cos 2x \right) \]

\[ = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \]

Check:
\[ \frac{d}{dx} \left( \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right) \]

\[ = \frac{1}{2} \left( x \cos 2x \cdot 2 + \sin 2x \right) + \frac{1}{4} \left( -\sin 2x \cdot 2 \right) \]

\[ = x \cos 2x + \frac{1}{2} \sin 2x \]

\[ = x \cos 2x \]

4. \[ \int_0^M e^{-6x} \, dx \]

\[ = \lim_{M \to \infty} \left[ -\frac{1}{6} e^{-6x} \right]_0^M \]

\[ = \lim_{M \to \infty} \left( -\frac{1}{6} e^{-6M} - (-\frac{1}{6}) \right) \]

\[ = \frac{1}{6} - \lim_{M \to \infty} \frac{1}{6} e^{6M} \]

\[ = \frac{1}{6} - 0 = \frac{1}{6} \]
Part III: The answers to the following questions can be found using integrals. Give the answers as expressions involving integrals, including limits of integration. You do not need to evaluate the integrals (10 points each).

1. What is the area between the curves $y = 3 - x$ and $y = \frac{2}{x}$.

\[ \text{Cross at } \frac{2}{x} \]
\[ \text{so} \]
\[ 3x - x^2 = \frac{2}{x} \]
\[ 3x - x^2 = 2 \]
\[ x^2 - 3x + 2 = 0 \]
\[ (x - 1)(x - 2) = 0 \]
\[ x = 1, 2 \]

Cross at 1, 2, in between of, say, $\frac{3}{2}$.

LHS is $\frac{3}{2}$, RHS is $\frac{2}{x}$ smaller so

\[ \int_{\frac{1}{2}}^{\frac{3}{2}} 3 - x - \frac{2}{x} \, dx \]

2. A tank of water contains 300 gal. of water at time $t = 0$. Water flows out of the tank at the rate of $f(t) = 10/(1 + t^3)$ gal. per min. What is the amount of water in the tank after 8 min?

Water lost is $\int_0^8 \frac{10}{1 + t^3} \, dt$

Water remaining is $300 - \int_0^8 \frac{10}{1 + t^3} \, dt$

Or

\[ \text{Water end} - \text{Water start} = \int_0^8 \frac{-10}{1 + t^3} \, dt \]

Total change = \int_0^8 \frac{-10}{1 + t^3} \, dt (minus because decreasing)

So

\[ \text{Ans} - 300 = \int_0^8 \frac{-10}{1 + t^3} \, dt \]

3. What is the volume of the solid you obtain if you rotate the region above the horizontal axis and below the curve $y = x - x^2$ about the horizontal axis?

Crosses axis where $x - x^2 = 0$ so $x(1 - x) = 0$

$x = 0, 1$

\[ V = \int_0^1 \pi (x - x^2)^2 \, dx \]
Part IV: (10 points each)

1. The graph of \( y = f(x) \) is given, on the second set of axes sketch the graph of \( y = \int_0^x f(t) \, dt \). Try to get the shape, scale, intercepts, and turning points right. Don't be overly concerned about the numerical values but do indicate the scale on your axes (i.e. put in labels "1", "2" etc.).

The crossing points of  
are the turning points of  

The graph of \( y = f(x) \)

Increasing  dec  inc

Sketch of the graph of \( y = \int_0^x f(t) \, dt \)
For the function $f(x)$ whose graph is shown:

a. Write down an approximating sum for evaluating $\int_{a}^{b} f(x) \, dx$ using $n = 4$. There are several ways to do this, select any you care to but be explicit about what you are doing. Evaluate your sum.

$$\sum_{n=3}^{6} f(x_i) \Delta x = (8 + 5 + 3 + 2) \frac{1}{2} = 18$$

b. The number you computed in the previous part has a geometric interpretation. With a few words, and marking up the graph if you care to, tell what that interpretation is.

18 is the shaded area. It is an approximation to the area under the curve.