On the Product of Functions in $\boldsymbol{BMO}$ and $\boldsymbol{H}^1$

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Abstract

The point-wise product $\mathfrak{b} \cdot \mathfrak{h}$ of functions $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ and $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ need not be locally integrable. However, in view of the duality between $\boldsymbol{BMO}$ and $\boldsymbol{H}^1$, we are able to give a meaning to $\mathfrak{b} \cdot \mathfrak{h}$ as a Schwartz distribution, denoted by $\mathfrak{b} \times \mathfrak{h} \in \mathscr{D}'(\mathbb{R}^n)$. The central question is concerned with the regularity of $\mathfrak{b} \times \mathfrak{h} \in \mathscr{D}'(\mathbb{R}^n)$. We prove a decomposition:

$$\mathfrak{b} \times \mathfrak{h} \;=\; \alpha + \beta\,,$$

where α is a function in $\boldsymbol{L}^1(\mathbb{R}^n)$ while β is a distribution in a Hardy-Orlicz space. Precisely this means that its maximal function $\mathscr{M}\beta$ satisfies

$$\int_{\mathbb{R}^n} \frac{\mathscr{M}\beta}{\log(e + \mathscr{M}\beta)}\, d\mu < \infty\,, \qquad \text{where} \quad d\mu = \frac{dx}{\log(e + |x|)}$$

The *Jacobian determinants* and more general *div-curl* products come to a play as atoms.