Clark measures with specified spectral types

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. For $\alpha \in \partial\mathbb{D}$ the Clark measure μ_α is defined to be the unique regular Borel measure on $\partial\mathbb{D}$ such that

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \int \frac{1 - |z|^2}{|\zeta - z|^2} d\mu_\alpha(\zeta) \qquad \forall z \in \mathbb{D}$$

If φ is inner then all the μ_α's will be singular. Via a well-known construction due to Douglas N. Clark these measures can be used to study the rank-one unitary perturbations of unitary operators or, using the Cayley transform, the rank-one self-adjoint perturbations of self-adjoint operators. Because of this, Clark measures have important applications in Mathematical Physics.

We construct an inner function φ whose Clark measures satisfy:

μ_α is purely atomic for $\alpha \in C$

μ_α is continuous singular for $\alpha \notin C$

where C can be any specified closed subset of $\partial\mathbb{D}$. The spectrum of φ will be all of $\partial\mathbb{D}$ if the interior of C is empty. This answers a question of Barry Simon.