

What can Hilbert spaces tell us about bounded functions in the bidisk?

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Dedicated to the memory of Paul R. Halmos

Abstract. We discuss various theorems about bounded analytic functions on the bidisk that were proved using operator theory.

1. Introduction

Much of modern single operator theory (as opposed to the study of operator algebras) rests on a foundation of complex analysis. Every cyclic operator can be represented as multiplication by the independent variable on the completion of the polynomials with respect to some norm. The nicest way to have a norm is in $L^2(\mu)$, and then one is led to the study of subnormal operators, introduced by P. Halmos in [34]. The study of cyclic subnormal operators becomes the study of the spaces $P^2(\mu)$, the closure of the polynomials in $L^2(\mu)$, and the theory of these spaces relies on a blend of complex analysis and functional analysis; see J. Conway's book [27] for an exposition. Alternatively, one can start with a Hilbert space that is amenable to complex analysis, such as the Hardy space H^2 , and study classes of operators on that space that have a good function theoretic representation, such as Toeplitz, Hankel or composition operators. All of these classes of operators have

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a rich theory, which depends heavily on function theory — for expositions, see *e.g.* [41, 43] and [29].

The traffic, of course, goes both ways. There are many questions in function theory that have either been answered or illuminated by an operator theory approach. The purpose of this article is to describe how operator theory has fared when studying $H^\infty(\mathbb{D}^2)$, the algebra of bounded analytic functions on the bidisk \mathbb{D}^2 . We focus on function theory results that were proved, originally at least, using operator theory.

For the topics in Sections 2 to 7, we shall first describe the situation on the disk \mathbb{D} , and then move on to the bidisk. The topics in Sections 8 to 9 do not really have analogues in one dimension. For simplicity, we shall stick to scalar-valued function theory, though many of the results have natural matrix-valued analogues.

We shall use the notation that points in the bidisk are called λ or ζ , and their coordinates will be given by superscripts: $\lambda = (\lambda^1, \lambda^2)$. We shall use z and w to denote the coordinate functions on \mathbb{D}^2 . The closed unit ball of $H^\infty(\mathbb{D})$ will be written $H_1^\infty(\mathbb{D})$ and the closed unit ball of $H^\infty(\mathbb{D}^2)$ as $H_1^\infty(\mathbb{D}^2)$.

2. Realization Formula

The realization formula is a way of associating isometries (or contractions) with functions in the ball of $H^\infty(\mathbb{D})$. In one dimension, it looks like the following; see *e.g.* [19] or [9] for a proof.

Theorem 2.1. *The function ϕ is in the closed unit ball of $H^\infty(\mathbb{D})$ if and only if there is a Hilbert space \mathcal{H} and an isometry $V : \mathbb{C} \oplus \mathcal{H} \rightarrow \mathbb{C} \oplus \mathcal{H}$, such that, writing V as*

$$V = \begin{array}{c} \mathbb{C} \quad \mathcal{H} \\ \mathbb{C} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ \mathcal{H} \end{array}, \quad (2.2)$$

one has

$$\phi(\lambda) = A + \lambda B(I - \lambda D)^{-1}C. \quad (2.3)$$

This formula was generalized to the bidisk in [3]. It becomes

Theorem 2.4. *The function ϕ is in the closed unit ball of $H^\infty(\mathbb{D}^2)$ if and only if there are auxiliary Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 and an isometry*

$$V : \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$$

such that, if $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$, V is written as

$$V = \begin{array}{c} \mathbb{C} \quad \mathcal{H} \\ \mathbb{C} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \end{array} \quad (2.5)$$

and $\mathcal{E}_\lambda = \lambda^1 I_{\mathcal{H}_1} \oplus \lambda^2 I_{\mathcal{H}_2}$, then

$$\phi(\lambda) = A + B\mathcal{E}_\lambda(I_{\mathcal{H}} - D\mathcal{E}_\lambda)^{-1}C. \quad (2.6)$$

There is a natural generalization of (2.6) to functions of d variables. One chooses d Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_d$, lets $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_d$, lets $\mathcal{E}_\lambda = \lambda^1 I_{\mathcal{H}_1} \oplus \dots \oplus \lambda^d I_{\mathcal{H}_d}$, and then, for any isometry V as in (2.5), let

$$\psi(\lambda) = A + B\mathcal{E}_\lambda(I_{\mathcal{H}} - D\mathcal{E}_\lambda)^{-1}C. \quad (2.7)$$

The set of all functions ψ that are realizable in this way is exactly the Schur-Agler class, a class of analytic functions of d variables that can also be defined as

$$\{\psi : \|\psi(T_1, \dots, T_d)\| \leq 1 \quad \forall \text{ commuting contractive matrices } (T_1, \dots, T_d)\}. \quad (2.8)$$

Von Neumann's inequality [58] is the assertion that for $d = 1$ the Schur-Agler class equals $H_1^\infty(\mathbb{D})$; Andô's inequality [16] is the analogous equality for $d = 2$. Once $d > 2$, the Schur-Agler class is a proper subset of the closed unit ball of $H^\infty(\mathbb{D}^d)$ [57, 30]. Many of the results in Sections 3 to 7 are true, with similar proofs, for the Schur-Agler class in higher dimensions (or rather the norm for which this is the unit ball)¹, but it is not known how to generalize them to $H^\infty(\mathbb{D}^d)$.

The usefulness of the realization formula stems primarily not from its ability to represent functions, but to produce functions with desired properties with the

¹Specifically, Theorems 3.8, 4.3, 6.4 and (i) \Leftrightarrow (iii) of Theorem 7.7.

aid of a suitably chosen isometry V (dubbed a *lurking isometry* by Joe Ball). An example of a lurking isometry argument is the proof of Pick's theorem in Section 3.

It is well-known that equality occurs in the Schwarz lemma on the disk only for Möbius transformations. The Schwarz lemma on \mathbb{D}^d reads as follows (see [47] for a proof).

Proposition 2.9. *If f is in $H_1^\infty(\mathbb{D}^d)$, then*

$$\sum_{r=1}^d (1 - |\lambda^r|^2) \left| \frac{\partial f}{\partial \lambda^r}(\lambda) \right| \leq 1 - |f(\lambda)|^2. \quad (2.10)$$

G. Knese [38] proved that equality in (2.10) resulted in a curious form of the realization formula. (Notice that for $d \geq 3$, the hypothesis is that f lie in $H_1^\infty(\mathbb{D}^d)$, but the conclusion means f must be in the Schur-Agler class.)

Theorem 2.11. *Suppose $f \in H_1^\infty(\mathbb{D}^d)$ is a function of all d variables, and equality holds in (2.10) everywhere on \mathbb{D}^d . This occurs if and only if f has a representation as in (2.7) where each space \mathcal{H}_r is one-dimensional, and the unitary V is symmetric (equal to its own transpose).*

Analyzing the realization formula, J.M. Anderson, M. Dritschel and J. Rovnyak were able to obtain the following higher derivative version of the Schwarz lemma [15]:

Theorem 2.12. *Let f be in $H_1^\infty(\mathbb{D}^2)$, and n_1, n_2 non-negative integers with $n = n_1 + n_2$. Let $\lambda = (z, w)$ be in \mathbb{D}^2 , with $|\lambda| = \max(|z|, |w|)$. Then*

$$\left| \frac{\partial^n f}{\partial^{n_1} z \partial^{n_2} w} \right| \leq (n-2)! \frac{1 - |f(\lambda)|^2}{(1 - |\lambda|)^{n-1}} \left[\frac{n_1^2 - n_1}{1 - |z|^2} + \frac{2n_1 n_2}{\sqrt{1 - |z|^2} \sqrt{1 - |w|^2}} + \frac{n_2^2 - n_2}{1 - |w|^2} \right].$$

3. Pick Problem

The Pick problem on the disk is to determine, given N points $\lambda_1, \dots, \lambda_N$ in \mathbb{D} and N complex numbers w_1, \dots, w_N , whether there exists $\phi \in H_1^\infty(\mathbb{D})$ such that

$$\phi(\lambda_i) = w_i, \quad i = 1, \dots, N.$$

G. Pick proved [44] that the answer is yes if and only if the N -by- N matrix

$$\left(\frac{1 - w_i \bar{w}_j}{1 - \lambda_i \bar{\lambda}_j} \right) \quad (3.1)$$

is positive semi-definite.

D. Sarason realized in [49] that Pick's theorem can be proved by showing that operators that commute with the backward shift on an invariant subspace can be lifted with preservation of norm to operators that commute with it on all of H^2 ; this result was then generalized by B. Sz-Nagy and C. Foiaş to the commutant lifting theorem [52]. Here is a proof of Pick's theorem using a lurking isometry. We shall let

$$k_\lambda^S(\zeta) = \frac{1}{1 - \bar{\lambda}\zeta}$$

denote the Szegő kernel.

PROOF: (Necessity) If such a ϕ exists, then $I - M_\phi M_\phi^*$ is a positive operator on all of H^2 . In particular, for any scalars c_1, \dots, c_N , we have

$$0 \leq \langle (I - M_\phi M_\phi^*) \sum c_j k_{\lambda_j}^S, \sum c_i k_{\lambda_i}^S \rangle = \sum c_j \bar{c}_i \frac{1 - w_i \bar{w}_j}{1 - \lambda_i \bar{\lambda}_j}. \quad (3.2)$$

Therefore (3.1) is positive semi-definite.

(Sufficiency) If (3.1) is positive semi-definite of rank M , then one can find vectors $\{g_i\}_{i=1}^N$ in \mathbb{C}^M such that

$$\frac{1 - w_i \bar{w}_j}{1 - \lambda_i \bar{\lambda}_j} = \langle g_i, g_j \rangle_{\mathbb{C}^M}. \quad (3.3)$$

We can rewrite (3.3) as

$$1 + \langle \lambda_i g_i, \lambda_j g_j \rangle_{\mathbb{C}^M} = w_i \bar{w}_j + \langle g_i, g_j \rangle_{\mathbb{C}^M}. \quad (3.4)$$

The lurking isometry $V : \mathbb{C} \oplus \mathbb{C}^M \rightarrow \mathbb{C} \oplus \mathbb{C}^M$ is defined by

$$V : \begin{pmatrix} 1 \\ \lambda_i g_i \end{pmatrix} \mapsto \begin{pmatrix} w_i \\ g_i \end{pmatrix}. \quad (3.5)$$

We extend linearly to the span of

$$\left\{ \begin{pmatrix} 1 \\ \lambda_i g_i \end{pmatrix} : i = 1, \dots, N \right\}, \quad (3.6)$$

and if this is not the whole space $\mathbb{C} \oplus \mathbb{C}^M$, we extend V arbitrarily so that it remains isometric. Write V as

$$V = \begin{array}{c} \mathbb{C} \quad \mathbb{C}^M \\ \mathbb{C} \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \\ \mathbb{C}^M \end{array},$$

and define ϕ by

$$\phi(\lambda) = A + \lambda B(I - \lambda D)^{-1}C. \quad (3.7)$$

By the realization formula Thm. 2.1, ϕ is in $H_1^\infty(\mathbb{D})$. Moreover, as (3.5) implies that

$$\begin{aligned} A + B\lambda_i g_i &= w_i \\ C + D\lambda_i g_i &= g_i, \end{aligned}$$

we get that

$$(I - \lambda_i D)^{-1}C = g_i,$$

and hence

$$\phi(\lambda_i) = A + \lambda_i B g_i = w_i,$$

so ϕ interpolates. □

(It is not hard to show that ϕ is actually a Blaschke product of degree M).

A similar argument using Thm. 2.4 solves the Pick problem on the bidisk. The theorem was first proved, by a different method, in [2].

Theorem 3.8. *Given points $\lambda_1, \dots, \lambda_N$ in \mathbb{D}^2 and complex numbers w_1, \dots, w_N , there is a function $\phi \in H_1^\infty(\mathbb{D}^2)$ that maps each λ_i to the corresponding w_i if and only if there are positive semi-definite matrices Γ^1 and Γ^2 such that*

$$1 - w_i \bar{w}_j = (1 - \lambda_i^1 \bar{\lambda}_j^1) \Gamma_{ij}^1 + (1 - \lambda_i^2 \bar{\lambda}_j^2) \Gamma_{ij}^2. \quad (3.9)$$

On the polydisk, a necessary condition to solve the Pick problem analagous to (3.9) has recently been found by A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov and H. Woerdeman [33]. As of this writing, it is unknown if the condition is also sufficient, but we would conjecture that it is not.

Theorem 3.10. *Given points $\lambda_1, \dots, \lambda_N$ in \mathbb{D}^d and complex numbers w_1, \dots, w_N , a necessary condition for there to be a function $\phi \in H_1^\infty(\mathbb{D}^d)$ that maps each λ_i to the corresponding w_i is: For every $1 \leq p < q \leq d$, there are positive semi-definite matrices Γ^p and Γ^q such that*

$$1 - w_i \bar{w}_j = \prod_{r \neq q} (1 - \lambda_i^r \bar{\lambda}_j^r) \Gamma_{ij}^q + \prod_{r \neq p} (1 - \lambda_i^r \bar{\lambda}_j^r) \Gamma_{ij}^p. \quad (3.11)$$

4. Nevanlinna Problem

If the Pick matrix (3.1) is singular (i.e. if $M < N$) then the solution is unique; otherwise it is not. R. Nevanlinna found a parametrization of all solutions in this latter case [40] (see also [20] for a more modern approach).

Theorem 4.1. *If (3.1) is invertible, there is a 2-by-2 contractive matrix-valued function*

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

such that the set of all solutions of the Pick problem is given by

$$\{\phi = G_{11} + G_{12} \frac{\psi G_{21}}{1 - G_{22} \psi} : \psi \in H_1^\infty(\mathbb{D})\}.$$

On the bidisk, we shall discuss uniqueness in Section 8 below. Consider now the non-unique case. Let ϕ be in $H_1^\infty(\mathbb{D}^2)$, and so by Theorem 2.4 it has a representation as in (2.6). Define vector-valued functions F_1, F_2 by

$$\begin{matrix} \mathbb{C} \\ \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \begin{pmatrix} F_1(\lambda) \\ F_2(\lambda) \end{pmatrix} := (I_{\mathcal{H}} - D\mathcal{E}_\lambda)^{-1} C. \quad (4.2)$$

For a given solvable Pick problem with a representation as (3.9), say that ϕ is affiliated with (Γ^1, Γ^2) if, for some representation of ϕ and F_1, F_2 as in (4.2),

$$\begin{aligned} F_1(\lambda_i)^* F_1(\lambda_j) &= \Gamma_{ij}^1 \\ F_2(\lambda_i)^* F_2(\lambda_j) &= \Gamma_{ij}^2 \end{aligned}$$

for $i, j = 1, \dots, N$. The situation is complicated by the fact that for a given ϕ , the pairs (Γ^1, Γ^2) with which it is affiliated *may or may not* be unique. J. Ball and T. Trent [21] proved:

Theorem 4.3. *Given a solvable Pick problem, with a representation as in (3.9), there is a matrix-valued function G*

$$G = \begin{array}{c} \mathbb{C} \quad \mathbb{C}^M \\ \mathbb{C} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \\ \mathbb{C}^M \end{array}$$

in the closed unit ball of $H^\infty(\mathbb{D}^2, B(\mathbb{C} \oplus \mathbb{C}^M, \mathbb{C} \oplus \mathbb{C}^M))$, such that the function ϕ solves the Pick problem and is affiliated with (Γ^1, Γ^2) if and only if it can be written as

$$\phi = G_{11} + G_{12}\Psi(I - G_{22}\Psi)^{-1}G_{21} \quad (4.4)$$

for some Ψ in $H_1^\infty(\mathbb{D}^2, B(\mathbb{C}^M, \mathbb{C}^M))$.

5. Takagi Problem

The case where the Pick matrix (3.1) has some negative eigenvalues was first studied by T. Takagi [54], and later by many other authors [1, 42, 20]. See the book [19] for an account. The principal difference is that if one wishes to interpolate with a unimodular function (*i.e.* a function that has modulus one on the unit circle \mathbb{T}), then one has to allow poles inside \mathbb{D} . A typical result is

Theorem 5.1. *Suppose the Pick matrix is invertible, and has π positive eigenvalues and ν negative eigenvalues. Then there exists a meromorphic interpolating function ϕ that is unimodular, and is the quotient of a Blaschke product of degree π by a Blaschke product of degree ν .*

If Γ is not invertible, the problem is degenerate. It turns out that there is a big difference between solving the problem of finding Blaschke products f, g such that

$$f(\lambda_i) = w_i g(\lambda_i)$$

and the problem of solving

$$f(\lambda_i)/g(\lambda_i) = w_i.$$

(The difference occurs if f and g both vanish at some node λ_i ; in the first problem the interpolation condition becomes vacuous, but in the second one needs a relation on the derivatives). The first problem is more easily handled as the limit of non-degenerate problems; see the paper [23] for recent developments on this approach. The second version of the problem was been solved by H. Woracek using Pontryagin spaces [59].

Question 1. *What is the right version of Theorem 5.1 on the bidisk?*

6. Interpolating Sequences

Given a sequence $\{\lambda_i\}_{i=1}^\infty$ in the polydisk \mathbb{D}^d , we say it is interpolating for $H^\infty(\mathbb{D}^d)$ if, for any bounded sequence $\{w_i\}_{i=1}^\infty$, there is a function ϕ in $H^\infty(\mathbb{D}^d)$ satisfying $\phi(\lambda_i) = w_i$. L. Carleson characterized interpolating sequences on \mathbb{D} in [24].

Before stating his theorem, let us introduce some definitions. A *kernel* on \mathbb{D}^d is a positive semi-definite function $k : \mathbb{D}^d \times \mathbb{D}^d \rightarrow \mathbb{C}$, *i.e.* a function such that for any choice of $\lambda_1, \dots, \lambda_N$ in \mathbb{D}^d and any complex numbers a_1, \dots, a_N , we have

$$\sum a_i \bar{a}_j k(\lambda_i, \lambda_j) \geq 0.$$

Given any kernel k on \mathbb{D}^d , a sequence $\{\lambda_i\}_{i=1}^\infty$ has an associated Gramian G^k , where

$$[G^k]_{ij} = \frac{k(\lambda_i, \lambda_j)}{\sqrt{k(\lambda_i, \lambda_i) k(\lambda_j, \lambda_j)}}.$$

We think of G^k as an infinite matrix, representing an operator on ℓ^2 (that is not necessarily bounded). When k is the Szegő kernel on \mathbb{D}^d ,

$$k^S(\zeta, \lambda) = \frac{1}{(1 - \zeta^1 \bar{\lambda}^1)(1 - \zeta^2 \bar{\lambda}^2) \dots (1 - \zeta^d \bar{\lambda}^d)}, \quad (6.1)$$

we call the associated Gramian the *Szegő Gramian*. The Szegő kernel is the reproducing kernel for the Hardy space $H^2(\mathbb{D}^d) = P^2(m)$, where m is d -dimensional Lebesgue measure on the distinguished boundary \mathbb{T}^d of \mathbb{D}^d .

An analogue of the pseudo-hyperbolic metric on the polydisk is the *Gleason distance*, defined by

$$\rho(\zeta, \lambda) := \sup\{|\phi(\zeta)| : \|\phi\|_{H^\infty(\mathbb{D}^d)} \leq 1, \phi(\lambda) = 0\}.$$

We shall call a sequence $\{\lambda_i\}_{i=1}^\infty$ *weakly separated* if there exists $\varepsilon > 0$ such that, for all $i \neq j$, the Gleason distance $\rho(\lambda_i, \lambda_j) \geq \varepsilon$. We call the sequence *strongly separated* if there exists $\varepsilon > 0$ such that, for all i , there is a function ϕ_i in $H_1^\infty(\mathbb{D})$ such that

$$\phi_i(\lambda_j) = \begin{cases} \varepsilon, & j = i \\ 0, & j \neq i \end{cases}$$

In \mathbb{D} , a straightforward argument using Blaschke products shows that a sequence is strongly separated if and only if

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \quad \forall i.$$

We can now state Carleson's theorem. Let us note that he proved it using function theoretic methods, but later H. Shapiro and A. Shields [51] found a Hilbert space approach, which has proved to be more easily generalized, *e.g.* to characterizing interpolating sequences in the multiplier algebra of the Dirichlet space [39].

Theorem 6.2. *On the unit disk, the following are equivalent:*

(1) *There exists $\varepsilon > 0$ such that*

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \quad \forall i.$$

(2) *The sequence $\{\lambda_i\}_{i=1}^\infty$ is an interpolating sequence for $H^\infty(\mathbb{D})$.*

(3) *The sequence $\{\lambda_i\}_{i=1}^\infty$ is weakly separated and the associated Szegő Gramian is a bounded operator on ℓ^2 .*

In 1987 B. Berndtsson, S.-Y. Chang and K.-C. Lin proved the following theorem [22]:

Theorem 6.3. *Let $d \geq 2$. Consider the three statements*

(1) *There exists $\varepsilon > 0$ such that*

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \quad \forall i.$$

(2) *The sequence $\{\lambda_i\}_{i=1}^\infty$ is an interpolating sequence for $H^\infty(\mathbb{D}^d)$.*

(3) *The sequence $\{\lambda_i\}_{i=1}^\infty$ is weakly separated and the associated Szegő Gram-mian is a bounded operator on ℓ^2 .*

Then (1) implies (2) and (2) implies (3). Moreover the converses of these implications are false.

We call the kernel k on \mathbb{D}^d *admissible* if, for each $1 \leq r \leq d$, the function

$$(1 - \zeta^r \bar{\lambda}^r)k(\zeta, \lambda)$$

is positive semidefinite. (This is the same as saying that multiplication by each coordinate function on the Hilbert function space with reproducing kernel k is a contraction.)

On the unit disk, all admissible kernels are in some sense compressions of the Szegő kernel, and so to prove theorems about $H^\infty(\mathbb{D})$ one can often just use the fact that it is the multiplier algebra of H^2 . On the bidisk, there is no single dominant kernel, and one must look at a huge family of them. That is the key idea needed in Theorems 2.4 and 3.8, and it allows a different generalization of Theorem 6.2, which was proved in [8]. (If this paragraph seems cryptic, there is a more detailed exposition of this point of view in [9]).

For the following theorem, let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for ℓ^2 .

Theorem 6.4. *Let $\{\lambda_i\}_{i=1}^\infty$ be a sequence in \mathbb{D}^2 . The following are equivalent:*

- (i) *$\{\lambda_i\}_{i=1}^\infty$ is an interpolating sequence for $H^\infty(\mathbb{D}^2)$.*
 - (ii) *The following two conditions hold.*
- (a) *For all admissible kernels k , their normalized Grammians are uniformly bounded:*

$$G^k \leq MI$$

for some positive constant M .

(b) For all admissible kernels k , their normalized Grammians are uniformly bounded below:

$$NG^k \geq I$$

for some positive constant N .

(iii) The sequence $\{\lambda_i\}_{i=1}^\infty$ is strongly separated and condition (a) alone holds.

(iv) Condition (b) alone holds.

Moreover, Condition (a) is equivalent to both (a') and (a''):

(a'): There exists a constant M and positive semi-definite infinite matrices Γ^1 and Γ^2 such that

$$M\delta_{ij} - 1 = \Gamma_{ij}^1(1 - \bar{\lambda}_i^1\lambda_j^1) + \Gamma_{ij}^2(1 - \bar{\lambda}_i^2\lambda_j^2).$$

(a''): There exists a function Φ in $H^\infty(\mathbb{D}^2, B(\ell^2, \mathbb{C}))$ of norm at most \sqrt{M} such that $\Phi(\lambda_i)e_i = 1$.

Condition (b) is equivalent to both (b') and (b''):

(b'): There exists a constant N and positive semi-definite infinite matrices Δ^1 and Δ^2 such that

$$N - \delta_{ij} = \Delta_{ij}^1(1 - \bar{\lambda}_i^1\lambda_j^1) + \Delta_{ij}^2(1 - \bar{\lambda}_i^2\lambda_j^2).$$

(b''): There exists a function Ψ in $H^\infty(\mathbb{D}^2, B(\mathbb{C}, \ell^2))$ of norm at most \sqrt{N} such that $\Psi(\lambda_i) = e_i$.

Neither Theorem 6.3 nor 6.4 are fully satisfactory. For example, the following is still an unsolved problem:

Question 2. If a sequence on \mathbb{D}^2 is strongly separated, is it an interpolating sequence?

7. Corona Problem

The corona problem on a domain Ω asks whether, whenever one is given ϕ_1, \dots, ϕ_N in $H^\infty(\Omega)$ satisfying

$$\sum_{i=1}^N |\phi_i(\lambda)|^2 \geq \varepsilon > 0, \quad (7.1)$$

there always exist ψ_1, \dots, ψ_N in $H^\infty(\Omega)$ satisfying

$$\sum_{i=1}^N \phi_i \psi_i = 1. \quad (7.2)$$

If the answer is affirmative, the domain is said to have no corona.

Carleson proved that the disk has no corona in [25]. The most striking example of our ignorance about the bidisk is that the answer there is still unknown.

Question 3. *Is the corona theorem true for \mathbb{D}^2 ?*

The best result known is due to T. Trent [55], who proved that a solution can be found with the ψ_i 's in a specific Orlicz space $\exp(L^{1/3})$, which is contained in $\cap_{p<\infty} H^p(m)$.

There is a version of the corona theorem, the Toeplitz-corona theorem, proved at various levels of generality by several authors [17], [53], [50], [46]. We use k^S as in (6.1) (with $d = 1$).

Theorem 7.3. *Let ϕ_1, \dots, ϕ_N be in $H^\infty(\mathbb{D})$ and $\delta > 0$. Then the following are equivalent:*

(i) *The function*

$$\left[\sum_{i=1}^N \phi_i(\zeta) \overline{\phi_i(\lambda)} - \delta \right] k^S(\zeta, \lambda) \quad (7.4)$$

is positive semi-definite on $\mathbb{D} \times \mathbb{D}$.

(ii) *The multipliers M_{ϕ_i} on H^2 satisfy the inequality*

$$\sum_{i=1}^N M_{\phi_i} M_{\phi_i}^* \geq \delta I. \quad (7.5)$$

(iii) *There exist functions ψ_1, \dots, ψ_N in $H^\infty(\mathbb{D})$ such that*

$$\sum_{i=1}^N \psi_i \phi_i = 1$$

and

$$\sup_{\lambda \in \mathbb{D}} \left[\sum_{i=1}^N |\psi_i(\lambda)|^2 \right] \leq \frac{1}{\delta}. \quad (7.6)$$

The Toeplitz-corona theorem is often considered a weak version of the corona theorem, because the proof is easier and the hypothesis (7.5) is more stringent than (7.1). It does, however, have a stronger conclusion: condition (iii) gives the exact best bound for the norm of the ψ_i 's, whereas the corona theorem asserts that if (7.1) holds, then (iii) holds for *some* $\delta > 0$. (Moreover, in practice, checking the hypothesis (7.5) is an eigenvalue problem, and so quite feasible with polynomial data. Checking (7.1) is a minimization problem over a function on the disk that one would expect to have many local minima, even if the ϕ_i 's are polynomials of fairly low degree).

The Toeplitz-corona theorem does generalize to the bidisk, but again it is not enough to check (7.4) for a single kernel (or (7.5) on a single Hilbert function space), but rather one must find a uniform lower bound that works for all admissible kernels. For details see [21, 4].

Theorem 7.7. *Let ϕ_1, \dots, ϕ_N be in $H^\infty(\mathbb{D}^2)$ and $\delta > 0$. Then the following are equivalent:*

(i) *The function*

$$\left[\sum_{i=1}^N \phi_i(\zeta) \overline{\phi_i(\lambda)} - \delta \right] k(\zeta, \lambda) \quad (7.8)$$

is positive semi-definite for all admissible kernels k .

(ii) *For every measure μ on \mathbb{T}^2 , the multipliers M_{ϕ_i} on $P^2(\mu)$ satisfy the inequality*

$$\sum_{i=1}^N M_{\phi_i} M_{\phi_i}^* \geq \delta I. \quad (7.9)$$

(iii) *There exist functions ψ_1, \dots, ψ_N in $H^\infty(\mathbb{D}^2)$ such that*

$$\sum_{i=1}^N \psi_i \phi_i = 1$$

and

$$\sup_{\lambda \in \mathbb{D}^2} \left[\sum_{i=1}^N |\psi_i(\lambda)|^2 \right] \leq \frac{1}{\delta}. \quad (7.10)$$

Although Theorem 7.7 seems to depend on the specific properties of the bidisk, (indeed, using Theorem 2.4 one can prove the equivalence of (i) and (iii) in the Schur-Agler norm on the polydisk), there is a remarkable generalization by E. Amar that applies not only to the polydisk, but to any smooth convex domain [14].

Theorem 7.11. *Let Ω be a bounded convex domain in \mathbb{C}^d containing the origin, and assume that either Ω is \mathbb{D}^d or its boundary is smooth. Let X be \mathbb{T}^d in the former case, the boundary of Ω in the latter. Let ϕ_1, \dots, ϕ_N be in $H^\infty(\Omega)$ and $\delta > 0$. Then the following are equivalent:*

(i) *There exist functions ψ_1, \dots, ψ_N in $H^\infty(\mathbb{D})$ such that*

$$\sum_{i=1}^N \psi_i \phi_i = 1$$

and

$$\sup_{\lambda \in \Omega} \left[\sum_{i=1}^N |\psi_i(\lambda)|^2 \right] \leq \frac{1}{\delta}.$$

(ii) *For every measure μ on X , the multipliers M_{ϕ_i} on $P^2(\mu)$ satisfy the inequality*

$$\sum_{i=1}^N M_{\phi_i} M_{\phi_i}^* \geq \delta I. \tag{7.12}$$

(iii) *For every measure μ on X , and every f in $P^2(\mu)$, there exist functions ψ_1, \dots, ψ_N in $P^2(\mu)$ such that*

$$\sum_{i=1}^N \psi_i \phi_i = f$$

and

$$\sum_{i=1}^N \|\psi_i\|^2 \leq \frac{1}{\delta} \|f\|^2, \tag{7.13}$$

where the norms on both sides of (7.13) are in $P^2(\mu)$.

In [56], T. Trent and B. Wick have shown that in Amar's theorem it is sufficient to consider measures μ that are absolutely continuous and whose derivatives are bounded away from zero.

8. Distinguished and Toral Varieties

A Pick problem is called *extremal* if it is solvable with a function of norm 1, but not with anything smaller. In one dimension, this forces the solution to be unique. (In the notation of Section 3, this corresponds to the Pick matrix (3.1) being singular, the vectors in (3.6) spanning \mathbb{C}^{1+M} , and the unique solution being (3.7).) On the bidisk, problems can be extremal in either one or two dimensions. For example, consider the problems: $w_1 = 0, w_2 = 1/2, \lambda_1 = (0, 0)$ and λ_2 either $(1/2, 0)$ or $(1/2, 1/2)$. The first problem has the unique solution z ; the latter problem has a unique solution on the one-dimensional set $\{z = w\}$, but is not unique off this set. Indeed, Theorem 4.3 says in this case that the general solution is given by

$$\phi(z, w) = tz + (1-t)w + t(1-t)(z-w)^2 \frac{\Psi}{1 - [(1-t)z + tw]\Psi},$$

where Ψ is any function in $H_1^\infty(\mathbb{D}^2)$ and t is any number in $[0, 1]$.

If an extremal Pick problem on \mathbb{D}^2 does not have a solution that is unique on all \mathbb{D}^2 , then the set on which it is unique must be a variety² (the zero set of a polynomial). But this is not an arbitrary variety — it has special properties.

Let \mathbb{E} be the exterior of the closed disk, $\mathbb{C} \setminus \overline{\mathbb{D}}$. Say a variety V in \mathbb{C}^2 is *toral* if every irreducible component intersects \mathbb{T}^2 in an infinite set, and say it is *distinguished* if

$$V \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$

Distinguished varieties first appeared implicitly in the paper [48] by W. Rudin, and later in the operator theoretic context of sharpening Andô's inequality for matrices [11]; they turn out to be intimately connected to function theory on \mathbb{D}^2 (see Theorem 9.2, for example). Toral varieties are related to inner functions [12] and to symmetry of a variety with respect to the torus [13]. The uniqueness variety was partially described in [11, 12]:

²We use the word variety where algebraic geometers would say algebraic set — *i.e.* we do not require that a variety be irreducible.

Theorem 8.1. *The uniqueness set for an extremal Pick problem on \mathbb{D}^2 is either all of \mathbb{D}^2 or a toral variety. In the latter case, it contains a distinguished variety.*

It is perhaps the case that the uniqueness set is all of \mathbb{D}^2 whenever the data is in some sense “generic” (see *e.g.* [7]), but how is that made precise?

Question 4. *When is the uniqueness set all of \mathbb{D}^2 ?*

Distinguished varieties have a determinantal representation. The following theorem was proved in [11], and, more constructively, in [36].

Theorem 8.2. *A variety V is a distinguished variety if and only if there is a pure matrix-valued rational inner function Ψ on the disk such that*

$$V \cap \mathbb{D}^2 = \{(z, w) \in \mathbb{D}^2 : \det(\Psi(z) - wI) = 0\}.$$

Another way to picture distinguished varieties is by taking the Cayley transform of both variables; then they become varieties in \mathbb{C}^2 with the property that when one coordinate is real, so is the other.

9. Extension property

If G is a subset of \mathbb{D}^2 , we shall say that a function f defined on G is holomorphic if, for every point P in G , there is an open ball $B(P, \varepsilon)$ in \mathbb{D}^2 and a holomorphic function on the ball whose restriction to G is f . Given such a holomorphic function f , one can ask whether there is a single function F on \mathbb{D}^2 that extends it, and, if so, whether F can be chosen with additional properties.

H. Cartan proved that if G is a subvariety, then a global extension F always exists [26] (indeed he proved this on any pseudo-convex domain, the bidisk being just a special case). If f is bounded, one can ask whether one can find an extension F with the same sup-norm. If G is an analytic retract of \mathbb{D}^2 , *i.e.* there is an analytic map $r : \mathbb{D}^2 \rightarrow G$ that is the identity on G , then $F = f \circ r$ will work. (All retracts of \mathbb{D}^2 are either singletons, embedded disks, or the whole bidisk [47].) It turns out that extending without increasing the norm is only possible for retracts [10].

Theorem 9.1. *Let $G \subseteq \mathbb{D}^2$ and assume that G is relatively polynomially convex (i.e. $G^\wedge \cap \mathbb{D}^2 = G$ where G^\wedge denotes the polynomially convex hull of G). If every polynomial f on G has an extension to a function F in $H^\infty(\mathbb{D}^2)$ of the same norm, then G is a retract.*

Let us remark that although the theorem can be proved without using operator theory, it was discovered by studying pairs of commuting operators having G as a spectral set.

One can also ask if a bounded function f has a bounded extension F , but with a perhaps greater norm. G. Henkin and P. Polyakov proved that this can always be done if G is a subvariety of the polydisk that exits transversely [35]. In the case of a distinguished variety, Knese showed how to bound the size of the extension even when there are singularities on \mathbb{T}^2 [36, 37] (he also gives a construction of the function C below):

Theorem 9.2. *Let V be a distinguished variety. Then there is a rational function $C(z)$ with no poles in \mathbb{D} such that, for every polynomial $f(z, w)$ there is a rational function F which agrees with f on $V \cap \mathbb{D}^2$ and satisfies the estimate*

$$|F(z, w)| \leq |C(z)| \sup_{(z, w) \in V} |f(z)|.$$

If V has no singularities on \mathbb{T}^2 , then C can be taken to be a constant.

10. Conclusion

Paul Halmos contributed in many ways to the development of operator theory. The purpose of this article is to show that recasting many known results about $H^\infty(\mathbb{D})$ in terms of operator theory has been extremely fruitful in understanding $H^\infty(\mathbb{D}^2)$. So far, however, it has not helped very much in understanding $H^\infty(\mathbb{B}_2)$, where \mathbb{B}_2 is the ball in \mathbb{C}^2 . There is another kernel on the ball,

$$k(\zeta, \lambda) = \frac{1}{1 - \zeta^1 \bar{\lambda}^1 - \zeta^2 \bar{\lambda}^2},$$

introduced by S. Drury [31], and operator theory has been very effective in studying this kernel [5, 6, 18, 28, 32, 45].

Question 5. *What is the correct Pick theorem on $H^\infty(\mathbb{B}_2)$?*

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