The use of kernel functions in solving the Pick interpolation problem

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Abstract

The original Pick interpolation problem asks when an analytic function from the disk to the half-plane can interpolate certain prescribed values. This was solved by G. Pick in 1916. This article discusses this theorem, and generalizations of it to other domains.

1 Introduction

In 1916, G. Pick [49] considered the following question.

Question 1.1. *Given points $\lambda_1, \dots, \lambda_N$ in the unit disk $\mathbb{D}$, and numbers $w_1, \dots, w_N$ in the right-half plane Ω, does there exist a holomorphic function ϕ on $\mathbb{D}$ that has positive real part and satisfies the interpolation conditions*

$$\phi(\lambda_i) \;=\; w_i, \quad 1 \leq i \leq N?$$

He answered the question with the following theorem:

Theorem 1.2. [Pick] *Question 1.1 has an affirmative answer if and only if the matrix*

$$\mathfrak{P} \;=\; \left[\frac{w_i + \bar{w}_j}{1 - \lambda_i \bar{\lambda}_j}\right]_{i,j=1}^{N}$$

is positive semi-definite. Moreover the solution is unique if and only if the rank of $\mathfrak{P}$ is some number $k < n$. In this event, the unique solution is a k-to-1 rational mapping of $\mathbb{D}$ onto Ω.

R. Nevanlinna considered the same problem, and got a partial solution in [45]. He returned to the problem in [46], where he gave a parametrization of all solutions in the non-unique case. Since then, problems of this type have been called Pick interpolation or Nevanlinna-Pick interpolation problems.

For the purpose of this article, it is convenient to change the problem slightly, and consider functions that map $\mathbb{D}$ to $\mathbb{D}$, rather than $\mathbb{D}$ to Ω. Of course, this is just a Cayley transform of the image, so in principle nothing has changed. In practice, something important has been given up. The extreme points of the holomorphic functions from $\mathbb{D}$ to Ω, normalized to map 0 to 1, are all of the form

$$z \mapsto \frac{e^{i\theta} + z}{e^{i\theta} - z}, \tag{1.3}$$

and Herglotz's theorem says that every function of positive real part that maps 0 to 1 is an integral of functions of the form (1.3). The utility of extreme points when studying the Pick problem on finitely connected domains instead of $\mathbb{D}$ is shown in the paper [33] by S. Fisher and D. Khavinson. The extreme points of the set of holomorphic functions mapping $\mathbb{D}$ to $\mathbb{D}$ are the functions ϕ for which $\log(1 - |\phi|^2)$ is not integrable on the unit circle [37, p. 138]; this is a much larger set, and does not seem to help in studying the Pick problem.

Changing the codomain does have some benefits, however, as shown below. In the new context, Pick's theorem becomes

Theorem 1.4. *Given points $\lambda_1, \ldots, \lambda_N$ in the unit disk $\mathbb{D}$, and numbers $w_1, \ldots, w_N$ in $\mathbb{D}$, there exists a holomorphic function $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ that satisfies the interpolation conditions*

$$\phi(\lambda_i) \;=\; w_i, \quad 1 \le i \le N, \tag{1.5}$$

if and only if the Pick matrix

$$\mathfrak{P} \;=\; \left[\frac{1 - w_i \bar{w}_j}{1 - \lambda_i \bar{\lambda}_j}\right]_{i,j=1}^{N} \tag{1.6}$$

is positive semi-definite. Moreover the solution is unique if and only if the rank of $\mathfrak{P}$ is some number $k < n$. In this event, the unique solution is a Blaschke product of degree k.

One way to prove Theorem 1.4 is by Schur reduction. The idea is that if $\lambda_1 = 0$ and $w_1 = 0$, then (1.5) is satisfied if and only if $\phi(z) = z\psi(z)$, where $\psi : \mathbb{D} \to \overline{\mathbb{D}}$ is holomorphic and satisfies

$$\psi(\lambda_i) \ = \ \frac{w_i}{\lambda_i}, \quad 2 \leq i \leq N.$$

This reduces the N point problem to an $N-1$ point problem. In general, of course, one will not have $\lambda_1 = 0$ and $w_1 = 0$; but one can achieve this by pre- and post-composing with Möbius transformations of the disk. The details of the proof by Schur reduction can be found in [35, Thm. I.2.2].

2 Sarason's Approach

In [52], D. Sarason gave a different proof of Pick's theorem, using properties of the Hardy space H^2 (the Hilbert space of holomorphic functions on $\mathbb{D}$ whose Taylor coefficients at 0 are square-summable). Here are some facts about the Hardy space (these facts and their proofs can be found in many places, such as [9, 27, 31, 40]). Let $k_\lambda(z)$ be the Szegő kernel function

$$k_\lambda(z) \ = \ \frac{1}{1 - \bar{\lambda} z}.$$

The multiplier algebra of H^2 is H^∞, the bounded analytic functions on $\mathbb{D}$, and the multiplier norm is the same as the supremum of the modulus. Moreover, the reproducing property $\langle f, k_\lambda \rangle = f(\lambda)$ means that every Szegő kernel function is an eigen-vector for the adjoint of every multiplier. Indeed, writing M_ϕ for the operator of multiplication by ϕ, one has

$$\begin{aligned} \langle f, M_\phi^* k_\lambda \rangle &= \langle M_\phi f, k_\lambda \rangle \\ &= \phi(\lambda) f(\lambda) \\ &= \langle f, \overline{\phi(\lambda)} k_\lambda \rangle, \end{aligned}$$

so

$$M_\phi^* k_\lambda \ = \ \overline{\phi(\lambda)} k_\lambda. \qquad (2.1)$$

Suppose that ϕ is in the closed unit ball of H^∞, so $\|M_\phi\| \leq 1$. One can write this norm inequality as the operator inequality

$$I - M_\phi M_\phi^* \ \geq \ 0. \tag{2.2}$$

Let $\mathcal{M}$ be the N-dimensional subspace of H^2 spanned by $\{k_{\lambda_i} : 1 \leq 1 \leq N\}$, and let P be the orthogonal projection from H^2 onto $\mathcal{M}$. If (2.2) holds on all of H^2, then

$$P - PM_\phi M_\phi^* P \ \geq \ 0,$$

so the N-by-N matrix

$$\left[\langle (P - PM_\phi M_\phi^* P) k_{\lambda_j}, k_{\lambda_i} \rangle\right]_{i,j=1}^N \tag{2.3}$$

is positive semi-definite. Using (2.1), one gets that (2.3) equals

$$\left[(1 - \phi(\lambda_i)\overline{\phi(\lambda_j)}) \langle k_{\lambda_j}, k_{\lambda_i} \rangle\right]_{i,j=1}^N . \tag{2.4}$$

But if ϕ is any function satisfying (1.5), then (2.4) equals the Pick matrix (1.6). This proves the necessity of Pick's condition: the Pick matrix must be positive semi-definite if the interpolation problem has a solution.

For sufficiency, observe that the backward shift $S^* : H^2 \to H^2$, which is the adjoint of multiplication by the independent variable, leaves $\mathcal{M}$ invariant (this is a special case of (2.1), with $\phi(z) = z$). Define an operator $T : \mathcal{M} \to \mathcal{M}$ by

$$T(\sum_{i=1}^N c_i k_{\lambda_i}) \ = \ \sum_{i=1}^N c_i \bar{w}_i k_{\lambda_i}.$$

Then T commutes wtih $S^*|_{\mathcal{M}}$, and saying that (1.6) is positive is the same as saying that $I_{\mathcal{M}} - T^*T \geq 0$, in other words that T is a contraction on $\mathcal{M}$. Sarason proved that if $\mathcal{N}$ is any S^*-invariant subspace of H^2, and $R : \mathcal{N} \to \mathcal{N}$ any operator that commutes with $S^*|_{\mathcal{N}}$, then R has an extension to an operator $\tilde{R}$ on H^2 that commutes with S^* and has the same norm as R. As the contractions that commute with S^* are exactly the adjoints of multiplication operators by functions in the closed ball of H^∞, this means that T has an extension to an operator of the form M_ϕ^* for some ϕ in the ball of H^∞. As $M_\phi^* k_{\lambda_i} = T k_{\lambda_i} = \bar{w}_i k_{\lambda_i}$, this means that $\phi(\lambda_i) = w_i$, and so ϕ solves the interpolation problem.

Sarason's theorem was generalized by B. Sz.-Nagy and C. Foiaş to the Commutant Lifting theorem [54, 55], which provides a framework to treat matrix-valued interpolation problems on the disk. An appealing feature of Sarason's approach is that it naturally unifies the Pick interpolation theorem with Carathéodory's, where you are given not N values of the function, but the first N Taylor coefficients at 0: just take $\mathcal{M}$ in this case to be the span $\vee\{1, z, \dots, z^{N-1}\}$.

3 The Pick property

Saying $\mathcal{H}$ is a reproducing kernel Hilbert space on a set X, with kernel function k, means that every element of $\mathcal{H}$ can be thought of as a function on the set X, and evaluation at each point λ of X is a continuous functional, given by inner product with the function $k_\lambda(\cdot) := k(\cdot, \lambda)$. Let $\mathrm{Mult}(\mathcal{H})$ denote the multiplier algebra of $\mathcal{H}$, equipped with the operator norm.

The necessity argument in Section 2 holds in any reproducing kernel Hilbert space.

Theorem 3.1. *Suppose $\mathcal{H}$ is a reproducing kernel Hilbert space on a set X, with kernel function k. Let $\lambda_1, \dots, \lambda_N$ be points of X, and $w_1, \dots, w_N \in \mathbb{C}$. A necessary condition to solve the interpolation problem*

$$\phi : \lambda_i \mapsto w_i, \quad 1 \le i \le N,$$

with a function ϕ in the closed unit ball of $\mathrm{Mult}(\mathcal{H})$ *is that the matrix*

$$[(1 - w_i \bar{w}_j) k(\lambda_i, \lambda_j)]_{i,j=1}^N \tag{3.2}$$

be positive semi-definite.

Pick's theorem asserts that for H^2, the condition in Theorem 3.1 is also sufficient. For the Bergman space, however, which also has H^∞ as its multiplier algebra, the condition is not sufficient. The Pick interpolation problem in $\mathrm{Mult}(\mathcal{H})$ is determining when an interpolation problem has a solution in the closed unit ball of $\mathrm{Mult}(\mathcal{H})$.

Question 3.3. *When is the positivity of* (3.2) *a sufficient condition to solve the Pick interpolation problem?*

There is a matrix-valued version of Pick interpolation. Fix some positive integer s. The space $\mathcal{H} \otimes \mathbb{C}^s$ can be thought of as vector-valued functions on X, and the multiplier algebra will consist of the

s-by-s matrices with entries from Mult($\mathcal{H}$). The matrix Pick problem is to determine, given points $\lambda_1, \ldots, \lambda_N$ in X and s-by-s matrices $W_1, \ldots, W_N$, whether there exists a function Φ in the closed unit ball of Mult($\mathcal{H} \otimes \mathbb{C}^s$) such that $\Phi(\lambda_i) = W_i, 1 \leq i \leq N$. By essentially the same argument as before, a necessary condition to solve the problem in any reproducing kernel Hilbert space is that the Ns-by-Ns matrix

$$\left[k(\lambda_i, \lambda_j) \otimes (I - W_i W_j^*)\right] \ \geq \ 0. \tag{3.4}$$

Say that k has the $M_{s \times s}$ Pick property if this condition is sufficient. When does this happen? There is an operator theoretic answer, at least in the case where there is analytic structure. For simplicity, assume that X is an open set in $\mathbb{C}$, and that M_z, multiplication by the independent variable, is bounded on $\mathcal{H}$.

First some notation. If $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ is a finite set of distinct points in $\mathbb{C}$, let I_Λ denote the ideal of polynomials that vanish on Λ, and let V_Λ be the set of operators T with the property that $p(T) = 0$ whenever p is a polynomial in I_Λ. Let $\mathcal{A}(T)$ denote the weak-star closure of the polynomials in T (if $T \in V_\Lambda$ for some finite Λ, then $\mathcal{A}(T)$ will be finite dimensional).

Let $\mathcal{M}_\Lambda$ be the subspace of $\mathcal{H}$ spanned by the kernel functions from Λ:

$$\mathcal{M}_\Lambda \ = \ \vee\{k_{\lambda_i} \ : \ 1 \leq i \leq N\}.$$

Say $\mathcal{H}$ is *regular* if the following additional assumptions hold:

(i) $\sigma(M_z) \subseteq \mathrm{cl}(X)$.

(ii) $\sigma_e(M_z) \subseteq \partial X$.

(iii) For every finite set Λ in X,

$$\mathcal{M}_\Lambda^\perp \ = \ \vee \ \{pf \ : \ f \ \in \mathcal{H}, \ p \in I_\Lambda\}.$$

For $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ a finite set in X, let P_Λ be orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}_\Lambda$. Let $\mathbf{S}_\Lambda$ denote the compression of M_z to $\mathcal{M}_\Lambda$, *i.e.*

$$\mathbf{S}_\Lambda \ := \ P_\Lambda \, M_z \, |_{\mathcal{M}_\Lambda}.$$

The map

$$\begin{array}{rcccl} \rho \ : \ Mult(\mathcal{H}) & \to & \mathcal{A}(\mathbf{S}_\Lambda) & & \\ \phi & \mapsto & P_\Lambda \phi(M_z) \, |_{\mathcal{M}_\Lambda} & = & \phi(\mathbf{S}_\Lambda) \end{array}$$

is a complete contraction. (A map is an s-contraction if the extension of the map to s-by-s matrices is a contraction; it is a complete contraction if it is an s-contraction for all $s \in \mathbb{N}$. See *e.g.* [48]).

If $T \in V_\Lambda$, then by the spectral mapping theorem $\sigma(T) \subseteq \Lambda$, so the map

$$\begin{array}{rcl} \rho_T \ : \ Mult(\mathcal{H}) & \to & \mathcal{A}(T) \\ \phi & \mapsto & \phi(T) \end{array}$$

is a surjective homomorphism. One can define a map

$$\begin{array}{rcl} \kappa_T \ : \ \mathcal{A}(\mathbf{S}_\Lambda) & \to & \mathcal{A}(T) \\ \phi(\mathbf{S}_\Lambda) & \mapsto & \phi(T) \end{array}$$

so that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{A}(\mathbf{S}_\Lambda) \arrow[dr, dotted, "\kappa_T"] & \\ Mult(\mathcal{H}) \arrow[ur, "\rho"] \arrow[rr, "\rho_T"] & & \mathcal{A}(T) \end{tikzcd}$$

Figure 3.5

Definition: The kernel k has the *s-contractive localization property* if, whenever Λ is a finite subset of X and T is an operator in V_Λ for which ρ_T is an s-contraction, then κ_T is an s-contraction.

The following theorem is proved in [8]; with minor modifications it works if X is an open set in $\mathbb{C}^m$ (one replaces M_z with the m-tuple $(M_{z^1}, \dots)$ and uses the Taylor spectrum in lieu of the spectrum).

Theorem 3.6. *Let $\mathcal{H}$ be a regular holomorphic Hilbert space with kernel k. Then k has the $M_{s \times s}$ Pick property if and only if k has the s-contractive localization property.*

4 The McCullough-Quiggin Theorem

If one asks for a characterization of what kernels have the $M_{s\times s}$ Pick property for every s, then there is a very elegant answer which does not require regularity.

Call a kernel k a *complete Pick kernel* if condition (3.4) is always sufficient to solve the matrix Pick problem. The Szegő kernel for H^2 is a complete Pick kernel. So is the kernel for the Dirichlet space, the set of analytic functions on $\mathbb{D}$ whose derivatives are in L^2 of area measure [3], and the Sobolev space $W_1^2[0,1]$ [4].

An irreducible kernel on X is one for which there is no non-trivial partition of $X = X_1 \sqcup X_2$ such that $k(x_1, x_2) = 0$ whenever $x_1 \in X_1$ and $x_2 \in X_2$. The following theorem was proved in [42, 43, 51]. For compactness, let k_{ij} denote $k(\lambda_i, \lambda_j)$.

Theorem 4.1. [McCullough-Quiggin] *A necessary and sufficient condition for an irreducible kernel k to be a complete Pick kernel is that, for any finite set $\{\lambda_1, \dots, \lambda_N\}$ of $N \geq 2$ distinct elements of X, the $(N-1)$-by-$(N-1)$ matrix*

$$\left(1 - \frac{k_{iN}k_{Nj}}{k_{ij}k_{NN}}\right)_{i,j=1}^{N-1}$$

is positive semi-definite.

For each cardinal $m \geq 1$, let $\mathbb{B}_m$ be the open unit ball in an m-dimensional Hilbert space. On $\mathbb{B}_m$, define a kernel a_m by

$$a_m(\zeta, \lambda) \;=\; \frac{1}{1 - \langle \zeta, \lambda \rangle}.$$

Let H_m^2 be the holomorphic Hilbert space on $\mathbb{B}_m$ that has a_m as its kernel. It follows from Theorem 4.1 that a_m has the complete Pick property; this was also proved directly in [16, 28, 44, 50] and as a consequence of Theorem 3.6 in [8]. For finite $m \geq 2$, the space H_m^2 was first studied by A. Lubin [41] and S. Drury [30]. Because of the influential article [17] by W. Arveson, it is now often called the Drury-Arveson space. It gives a universal complete Pick space when $m = \aleph_0$ [7]:

Theorem 4.2. *Let $\mathcal{H}$ be a separable Hilbert function space on X with an irreducible kernel k. Then k has the complete Pick property if and* *only if there is an injection $b : X \to \mathbb{B}_{\aleph_0}$ and a nowhere-vanishing function δ on X such that*

$$k(\zeta,\lambda) \;=\; \overline{\delta(\zeta)}\delta(\lambda)a_{\aleph_0}(b(\zeta),b(\lambda)).$$

5 Multiple Kernels

If one replaces the domain in Theorem 1.4 by the annulus, one gets the Pick problem for the annulus. M.B. Abrahamse [1] showed that the Pick problem for the annulus can be solved if an infinite number of Pick matrices are positive semi-definite.

Let Ω be a finitely connected smoothly bounded domain in $\mathbb{C}$, of connectivity $p+1$. Choose one point $z_i, 1 \leq i \leq p$, in the interior of each bounded component of $\mathbb{C} \setminus \Omega$. Let ω be harmonic measure on $\partial\Omega$. For each p-tuple of real numbers $\alpha = (\alpha_1,\ldots,\alpha_p)$, let

$$d\mu^\alpha(z) \;=\; |z-z_1|^{\alpha_1}|z-z_2|^{\alpha_2}\ldots|z-z_p|^{\alpha_p}d\omega(z).$$

Let $A^2(\mu^\alpha)$ be the closure of $H^\infty(\Omega)$ in $L^2(\mu^\alpha)$.

Theorem 5.1. [Abrahamse] *Let $\{\lambda_i : 1 \leq i \leq N\} \subset \Omega$, and $\{w_i : 1 \leq i \leq N\} \subset \mathbb{C}$. The Pick problem*

$$\exists\ \phi \in H^\infty(\Omega),\ \phi(\lambda_i) = w_i\ \forall\ i,\ \|\phi\| \leq 1\ ? \tag{5.2}$$

has an affirmative solution if and only if all the Pick matrices

$$\left[(1-w_i\bar{w}_j)k^{\mu^\alpha}(\lambda_i,\lambda_j)\right] \qquad \alpha \in [0,1)^p \tag{5.3}$$

are positive semi-definite. Moreover the problem is extremal if and only if one of the Pick matrices is singular. In this event, the solution is unique and has modulus 1 ω-a.e. on X.

The matrix-valued version of Theorem 5.1 is also true; this was proved by J. Ball [18]. Ball and K. Clancey [19] showed that it is not sufficient to check a finite number of kernels in (5.3), even when $p=2$ and $N=2$.

Checking a (large) family of Pick matrices for necessary and sufficient conditions to solve the Pick interpolation problem in H^∞ of a domain (or even other algebras) is often successful; see *e.g.* [14,26,57]. For an investigation into when this works in general, see [29, 38]. Roughly speaking, you start with every kernel that has the desired algebra as its multiplier algebra. If this collection is sufficient, you then try and whittle down to a smaller one.

This approach will work in $H^\infty(\mathbb{D}^2)$, the bounded analytic functions on the bidisk. Using Andô's inequality [15] the first author showed that there is a much more compact way of writing this [3]. Let (λ^1, λ^2) denote the coordinates of a point $\lambda \in \mathbb{D}^2$.

Theorem 5.4. *Let $\lambda_1, \ldots, \lambda_N$ be distinct points in $\mathbb{D}^2$, and $w_1, \ldots, w_N \in \mathbb{C}$. There is a function ϕ in the closed unit ball of $H^\infty(\mathbb{D}^2)$ that maps each λ_i to w_i if and only if there are positive semi-definite N-by-N matrices Γ^1 and Γ^2 such that*

$$(1 - w_i \bar{w}_j) \ = \ (1 - \lambda_i^1 \bar{\lambda}_j^1)\Gamma_{ij}^1 \ + \ (1 - \lambda_i^2 \bar{\lambda}_j^2)\Gamma_{ij}^2. \tag{5.5}$$

Theorem 5.4 is true in the matrix-valued case too [6, 22]. Uniqueness is not fully understood. In the non-extremal case (when the interpolation problem can be solved with a function of norm less than one), then the solution is clearly only unique on the original points $\{\lambda_i\}$. In the extremal case, there are two possibilities, even in the case $N = 2$: the uniqueness set can be the whole bidisk, or just a one-dimensional variety containing the points $\{\lambda_i\}$. For some information about the latter case, see [10, 11].

If one has $d > 2$ variables, it is natural to replace the two pieces on the right-hand side of (5.5) with d pieces. This works, but in the algebra for which the norm is defined by

$$\|\phi\| \ := \ \sup\{\|\phi(T)\| \ : \ T \text{ is a commuting } d\text{−tuple of contractions}\}.$$

Necessary conditions to solve the Pick interpolation problem in $H^\infty(\mathbb{D}^d)$ for $d \geq 3$ are given by [36] and [39].

6 Further Reading

There are many variants on the original Pick problem. If the Pick matrix (1.6) has ν negative eigenvalues, one can try to solve the interpolation problem with a function that is the ratio f/g of two Blaschke products, the denominator having degree ν. This was first studied by T. Takagi [56], and later by many other authors [2, 21, 24, 47]. Here is one result [20, Theorem 19.2.1]:

Theorem 6.1. *Suppose the matrix $\mathfrak{P}$ in (1.6) is invertible, and has π positive eigenvalues and ν negative eigenvalues. Then there exists a meromorphic function $\phi = f/g$ that satisfies*

$$\lim_{\lambda \to \lambda_i} \frac{f(\lambda)}{g(\lambda)} \; = \; w_i \qquad \forall \; 1 \leq i \leq N, \tag{6.2}$$

and is the quotient of a Blaschke product f of degree π by a Blaschke product g of degree ν.

If $\mathfrak{P}$ is not invertible, there is a subtle difference between the interpolation condition (6.2) and the condition

$$f(\lambda_i) \; = \; w_i g(\lambda) \qquad \forall \; 1 \leq i \leq N. \tag{6.3}$$

For solutions of (6.3), see the paper [24]. The solution of (6.2) in the degenerate case was first found by H. Woracek [58]; see also V. Bolotnikov's paper [23] and the paper [5].

Another variation is to look at limiting cases as the nodes λ_i tend to the boundary — this was first considered by Nevanlinna. See the papers [12, 25, 53] for recent results on the disk.

A small sample of monographs on Pick interpolation is [9, 13, 20, 32, 34].

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