Open Problem presented at MFO: the Lindelöf Hypothesis and $\mathcal{B}MO$

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Problem: Can one use functional analysis, specifically a BMO theory, to prove the Lindelöf hypothesis?

MOTIVATION. For a finite Dirichlet series $f(s)$, define a norm for $1 \leq p < \infty$ by

$$\|f\|^p_{\mathcal{H}^p} \;=\; \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(it)|^p dt; \tag{1}$$

and let $\mathcal{H}^p$ be the completion of the finite Dirichlet series in this norm. We shall call the expression on the right-hand side of (1) a limit in the mean of order p. Let $\zeta_1(s) = \sum(-1)^n n^{-s}$ be the alternating zeta function. The Lindelöf hypothesis can be rephrased[1] [2, Thm. 13.2] as the following assertion:

$$\forall \frac{1}{2} < \sigma < 1, \forall p < \infty, \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\zeta_1(\sigma+it)|^p dt < \infty.$$

It is known that $\zeta_1(\sigma + it)$ is unbounded on the imaginary axis, so in some sense you want to show that an unbounded function is in every $\mathcal{H}^p$ space for p finite. Can you do this by developing a BMO-type space, $\mathcal{B}MO$ say, which

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[1]The advantage of working with ζ_1 rather than ζ is that it is analytic up to the imaginary axis; but as $\zeta_1(s) = (2^{1-s} - 1)\zeta(s)$, they have the same order of growth on any vertical line except $\sigma = 1$.

has the properties that

$$\forall f \in \mathcal{B}MO, \quad \forall \quad \sigma > 0, \forall p < \infty, \ \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma + it)|^p dt < \infty. \quad (2)$$

$$\forall \quad \sigma > \frac{1}{2}, \ \zeta_1(\sigma + it) \in \mathcal{B}MO. \quad (3)$$

Discussion:

There are two obvious candidates, $\mathcal{B}MO_1$ and $\mathcal{B}MO_2$. The space $\mathcal{B}MO_1$ is the dual of $\mathcal{H}^1$; the space $\mathcal{B}MO_2$ is the dual of

$$\mathcal{H}^2 \odot \mathcal{H}^2 = \{\sum_{n=1}^{\infty} f_n g_n : f_n, g_n \in \mathcal{H}^2, \sum \|f_n\|^2 < \infty, \sum \|g_n\|^2 < \infty\}.$$

Ortega-Cerda and Seip have shown that $\mathcal{B}MO_1 \subsetneq \mathcal{B}MO_2$ [1].

The space $\mathcal{B}MO_2$ is interesting in its own right — it is the space of symbols of bounded Hankel forms on $\mathcal{H}^2$. To prove that a function h is in $\mathcal{B}MO_2$, it is enough to show that for any $f, g \in \mathcal{H}^2$, we have that $\langle fs, h\rangle$ is bounded by $\|f\|\|g\|$. It is easy to see that if $h(s) = \zeta(\sigma + s)$ and one does the formal calculation on the Dirichlet coefficients, then

$$\sum \widehat{fg}(k)\hat{h}(k) \ = \ f(\sigma)g(\sigma),$$

and the right-hand side is bounded by $\|f\|\|g\|$ if $\sigma > 1/2$. But $h(1 + s)$ does not have limits in the mean for any $p \geq 1$, because of the pole.

The same argument shows that $\zeta(\sigma + s)$ is in $\mathcal{B}MO_1$ for every $\sigma > 1/2$.

So neither $\mathcal{B}MO_1$ nor $\mathcal{B}MO_2$ are the right space for the purpose of proving Lindelöf. Is there another, smaller, space sandwiched between them and $\mathcal{H}^\infty$ that satisfies both (2) (which $\mathcal{H}^\infty$ obviously does) and (3)?

References

[1] Joaquim Ortega-Cerdà and Kristian Seip. A lower bound in Nehari's theorem on the polydisc. *J. Anal. Math.*, 118(1):339–342, 2012.

[2] E.C. Titchmarsh. *The theory of functions*. Oxford University Press, London, 1932.