Reflexivity of Isometries

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Abstract

We prove that any set of commuting isometries on a separable Hilbert space is reflexive.

Let $\mathcal{C}$ be a set of bounded operators on a Hilbert space $\mathcal{H}$, $Lat(\mathcal{C})$ be the lattice of all closed subspaces of $\mathcal{H}$ left invariant by every element of $\mathcal{C}$, and $AlgLat(\mathcal{C})$ be the algebra of all bounded operators that leave invariant every element of $Lat(\mathcal{C})$. Any polynomial in elements of $\mathcal{C}$ will be in $AlgLat(\mathcal{C})$, and so will anything in the closure, in the weak operator topology, of this set of polynomials: denote this set by $W(\mathcal{C})$. If $W(\mathcal{C})$ is all of $AlgLat(\mathcal{C})$, then $\mathcal{C}$ is called *reflexive*. In this note we prove that any set of commuting isometries on a separable space is reflexive.

The idea of reflexivity was first introduced by D. Sarason, who proved that any set of commuting normal operators is reflexive, and that so is any set of analytic Toeplitz operators [10]. In [4], J. Deddens proved that every isometry is reflexive, and later it was shown that a pair of commuting isometries T_1 and T_2 with the additional property that T_1 commuted with $T_2^\star$ is reflexive [7] and [8, 9]. The first author and H. Bercovici have shown that a pair of commuting isometries, one of which is a shift of finite multiplicity, is reflexive [2]. Recently, K. Horák and V. Müller showed that if $\mathcal{C}$ is any set of commuting isometries, then $AlgLat(\mathcal{C})$ is contained in both the commutant of $\mathcal{C}$ and in its double commutant [6], and we use their result to prove that if $\mathcal{C}$ is any collection of commuting isometries on a

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separable Hilbert space, then it is reflexive. Independently, H. Bercovici has also shown that any collection of commuting isometries is reflexive [1].

Theorem 1 *Let $\mathcal{C} = \{T_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of commuting isometries on a separable Hilbert space. Then $\mathcal{C}$ is reflexive.*

PROOF: The isometries $\{T_\alpha\}$ have commuting unitary extensions $\{U_\alpha\}$, and so are a (possibly infinite) subnormal tuple. (A subnormal tuple is, by definition, a set of commuting subnormal operators that have simultaneous extensions to commuting normal operators; commuting subnormal operators need not, in general, have commuting normal extensions, but commuting isometries do. See [7] for basic facts about subnormal tuples). By Lemma 2, we can assume that $\{U_\alpha\}$ acts on a separable space.

It was observed in [7, Section 2] for finite subnormal tuples, but is true for all subnormal tuples with separably acting minimal normal extensions, that they are reflexive if and only if their restriction to every cyclic subspace is reflexive (the proof is an easy modification of the proof for a single subnormal operator given in the book [3, Theorem VII.8.5]). Moreover, a cyclic tuple of isometries $\{T_\alpha\}$ can be represented as the tuple $\{M_{z_\alpha}\}$ on some space $P^2(\mu)$, where μ is a regular Borel measure on a torus, a product of unit circles, $P^2(\mu)$ denotes the closure in $L^2(\mu)$ of the polynomials, and M_{z_α} denotes multiplication by the α^{th} coordinate function.

By a result of Horák and Müller [6], any operator in $AlgLat(\mathcal{C})$ is in the commutant of $\mathcal{C}$. The commutant of $\{M_{z_\alpha}\}$ is the set of all multiplication operators M_ϕ, where ϕ is in $P^2(\mu) \cap L^\infty(\mu)$, and the weak-star topology on the set of multiplication operators, as a subset of the bounded linear operators on $P^2(\mu)$, coincides with the weak-star topology on $P^2(\mu) \cap L^\infty(\mu)$ considered as a subset of $L^\infty(\mu)$. We claim (Lemma 3 below) that the algebra $P^2(\mu) \cap L^\infty(\mu)$ is *elementary* (also known as having property $\mathbb{A}_1$), which means that for every weak-star continuous linear functional Λ on $\{M_\phi : \phi \in P^2(\mu) \cap L^\infty(\mu)\}$, there exist vectors f and g in $P^2(\mu)$, such that

$$\Lambda(M_\phi) = (M_\phi f, g) = \int \phi f \bar{g} d\mu.$$

Given the claim, it follows that any operator M_ϕ in $AlgLat(\mathcal{C})$ cannot be separated from the weak-star closure of the polynomials in M_{z_α} by a weak-star continuous linear functional. Indeed, if there were a Λ that vanished on the polynomials of $M_{z_1}, M_{z_2}, \ldots$, the vector g would have to be orthogonal to the invariant subspace generated by f, and this invariant subspace would be left invariant by M_ϕ, and so Λ would vanish on M_ϕ also (this argument first appeared in [5]). Therefore $AlgLat(\mathcal{C})$ coincides with the weak-star closed algebra generated by $T_1, T_2, \ldots$, and so C is reflexive. $\Box$

Lemma 2 *Let $[S]$ be a subnormal tuple on a separable space $\mathcal{H}$. Then the minimal normal extension of $[S]$ acts also on a separable space.*

PROOF: As $\mathcal{H}$ is the closure of a countable union of cyclic subspaces, it is sufficient to prove the lemma for cyclic tuples. Assume, therefore, that $\mathcal{H} = P^2(\mu)$ for some finite Borel measure μ. As $P^2(\mu)$ is separable, the set of monomials $\{z_{\alpha_1}^{i_1} z_{\alpha_2}^{i_2} \ldots z_{\alpha_k}^{i_k}\}$ has a countable dense subset $\{z^{\beta_1}, z^{\beta_2}, \ldots\}$, for some multi-indices $\beta_1, \beta_2, \ldots$.

The set of all rational linear combinations of $\{z^{\beta_m}\}$ and $\{\bar{z}^{\beta_n}\}$ is then a countable dense subset of $L^2(\mu)$. $\Box$

Lemma 3 *Let μ be a finite Borel measure on a torus. For every weak-star continuous linear functional Λ on $P^2(\mu) \cap L^\infty(\mu)$, and every $\epsilon > 0$, there exist $f, g \in P^2(\mu)$ such that*

$$\Lambda(\phi) = \int \phi f \bar{g} d\mu$$

for every ϕ in $P^2(\mu) \cap L^\infty(\mu)$, and $\|f\|, \|g\| \leq (1+\epsilon)\|\Lambda\|^{1/2}$.

PROOF: First, let us assume that μ is a finite measure on $\mathbb{T}^2$. Let Λ be any weak-star continuous linear functional on $P^2(\mu) \cap L^\infty(\mu)$. We will assume that Λ is non-zero. Let δ be a small positive number that we shall specify later. Fix a function F in $L^1(\mu)$ so that

$$\Lambda(\phi) = \int \phi F d\mu, \quad \textit{and} \quad \|F\|_{L^1(\mu)} \leq (1+\delta)\|\Lambda\|.$$

We wish to write F as $f\bar{g}$ for some f, g in $P^2(\mu)$.

(i) First we do it approximately. Observe that F can be approximated within $\delta' = \delta^2/12$ in $L^1(\mu)$ by a finite linear combination of characteristic functions of disjoint sets that are the product of open intervals in the unit circle:

$$\int |F - \sum_{k=1}^{n} a_k \chi_{I_k \times J_k}| d\mu < \delta'.$$

Moreover, this can be done in such a way that for any open rectangle $I_k \times J_k$, there are larger intervals $I_k' \supset \bar{I}_k$ and $J_k' \supset \bar{J}_k$ such that

$$\mu(I_k' \times J_k' \setminus I_k \times J_k) < \frac{\delta'}{nM},$$

where $M = \|\mu\|(1 + \max\{|a_k| : 1 \leq k \leq n\})$. Let ξ_k (resp. η_k) be functions in the disk algebra (the closure of the polynomials in the uniform norm on the unit disk) that have norm 1, modulus 1 on I_k (resp. J_k) and modulus δ'/nM off I'_k (resp. J'_k). As ξ_k and η_k are approximable in sup-norm by polynomials, the map $(z_1, z_2) \to \xi_k(z_1)\eta_k(z_2)$ is in $P^2(\mu)$. Let

$$\begin{aligned} f_0(z_1, z_2) &= \sum_{k=1}^{n} \frac{a_k}{\sqrt{|a_k|}} \xi_k(z_1)\eta_k(z_2), \\ g_0(z_1, z_2) &= \sum_{k=1}^{n} \sqrt{|a_k|}\xi_k(z_1)\eta_k(z_2). \end{aligned}$$

Both f_0 and g_0 are in $P^2(\mu)$, $\|f_0\|, \|g_0\| \leq (\|F\|_{L^1(\mu)} + 3\delta')^{1/2}$, and

$$\int |F - f_0\bar{g}_0| d\mu < 3\delta' = \delta^2/4.$$

(ii) Now we try to improve our approximation in the standard way. Suppose we are given f_1 in $P^2(\mu)$, g_1 in $L^2(\mu)$, $\epsilon_1, \epsilon_2 > 0$, such that

$$\int |F - f_1\bar{g}_1| d\mu < \epsilon_1, \quad \textit{and} \quad \|f_1\|, \|g_1\| \leq (\|F\|_{L^1(\mu)} + \epsilon_1)^{1/2}.$$

We claim we can find f_2 in $P^2(\mu)$, g_2 in $L^2(\mu)$, such that

$$\|f_1 - f_2\| \leq 2\sqrt{\epsilon_1}, \quad \|g_2\| \leq \|g_1\| + 2\sqrt{\epsilon_1} \tag{4}$$

and

$$\int |F - f_2\bar{g}_2| d\mu < \epsilon_2. \tag{5}$$

To prove the claim, fix a very small $\epsilon' > 0$. By part (i), we can find ξ in $P^2(\mu)$, η in $L^2(\mu)$, with

$$\|\xi\|^2 = \|\eta\|^2 \leq (1 + \epsilon')\|F - f_1\bar{g}_1\|_{L^1(\mu)},$$

and

$$\|F - f_1\bar{g}_1 - \xi\bar{\eta}\|_{L^1(\mu)} \leq \epsilon'.$$

Let $E = \{z \in \mathbb{T}^2 : |f_1(z)| < |\xi(z)|\}$. We can find two open sets B_1 and B_2 in $\mathbb{T}^2$, such that $E \subset B_2$, $\bar{B}_1 \subset B_2$, $\mu(B_1 \setminus E) + \mu(E \setminus B_1) < \epsilon'$, $\mu(B_2 \setminus B_1) < \epsilon'$. There exists u in the poly-disk algebra of norm 2, $|u(z)| = 2$ if $z \in \bar{B}_1$, and $|u(z)| = \epsilon'$ if $z \notin B_2$. Now define $f_2 = f_1 + u\xi$ and

$$g_2(z) = \begin{cases} \frac{\bar{f}_1 g_1 + \bar{\xi}\eta}{\bar{f}_1 + \bar{u}\bar{\xi}} & \text{z} \in (B_1 \cap E) \cup (\mathbb{T}^2 \setminus B_2) \\ 0 & \text{z} \in B_2 \setminus B_1. \end{cases}$$

Thus f_2, g_2 will satisfy (4) and (5) if ϵ' is sufficiently small (to get (5), note that $\|g_2\| \leq \frac{1}{1-\epsilon'}(\|g_1\| + \|\eta\|)$).

One can now continue inductively, to get f_n in $P^2(\mu)$ and g_n in $L^2(\mu)$ with

$$\begin{aligned}\int |F - f_n\bar{g}_n| d\mu &\leq 4^{-n-1}\delta^2 \\ \|f_n - f_{n+1}\| &\leq 2^{-n}\delta \\ \|g_{n+1}\| &\leq \|g_n\| + 2^{-n}\delta\end{aligned}$$

Let f be the norm limit of $\{f_n\}$, and h be any weak cluster-point of $\{g_n\}$. Finally let g be the projection of h from $L^2(\mu)$ onto $P^2(\mu)$. Then for any ϕ in $P^2(\mu) \cap L^\infty(\mu)$,

$$\Lambda(\phi) = \int \phi F d\mu = \int \phi f \bar{h} d\mu = \int \phi f \bar{g} d\mu,$$

$$\|f\| \leq \|f_0\| + \delta \leq ((\|F\|_{L^1(\mu)} + 3\delta')^{1/2} + \delta) \leq (\|\Lambda\| + \delta)^{1/2} + \delta,$$

and

$$\|g\| \leq \|g_0\| + \delta \leq (\|\Lambda\| + \delta)^{1/2} + \delta.$$

Thus f and g are as required if δ is small enough.

The modifications for finitely many isometries are obvious. If there are an infinite number, we use the fact that any finite Borel measure on a torus is regular, and again approximate $L^1(\mu)$ functions by linear combinations of characteristic functions of products of intervals, where all but finitely many of the factors are the whole circle. $\square$

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