

Pick Interpolation for free holomorphic functions

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August 6, 2013

1 Introduction

A holomorphic function in d (scalar) variables behaves locally like a polynomial. Given such a function ϕ , one can evaluate it also on d -tuples of commuting matrices whose joint spectrum lies in the domain of ϕ . A free holomorphic function is a generalization of this notion, where the matrices are no longer required to commute. We replace polynomials by free polynomials, *i.e.* polynomials in non-commuting variables, and consider functions that are locally limits of free polynomials.

To make this precise, let us first define \mathbb{M}_n^d to be the set of all d -tuples of n -by- n complex matrices, and $\mathbb{M}^{[d]} = \cup_{n=1}^{\infty} \mathbb{M}_n^d$. Let \mathbb{P}^d denote the algebra of free polynomials in d variables. A *graded function* defined on a subset of $\mathbb{M}^{[d]}$ is a function ϕ with the property that if $x \in \mathbb{M}_n^d$, then $\phi(x) \in \mathbb{M}_n$.

Definition 1.1. An *nc-function* is a graded function ϕ defined on a set $D \subseteq \mathbb{M}^{[d]}$ such that

- i) If $x, y, x \oplus y \in D$, then $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$.
- ii) If $s \in \mathbb{M}_n$ is invertible and $x, s^{-1}xs \in D \cap \mathbb{M}_n^d$ then $\phi(s^{-1}xs) = s^{-1}\phi(x)s$.

*Partially supported by National Science Foundation Grant DMS 1068830

†Partially supported by National Science Foundation Grant DMS 1300280

Nc-functions have been studied for a variety of reasons: by Taylor [25], in the context of the functional calculus for non-commuting operators; Voiculescu [26, 27], in the context of free probability; Popescu [19, 20, 21, 22], in the context of extending classical function theory to d -tuples of bounded operators; Ball, Groenewald and Malakorn [7], in the context of extending realization formulas from functions of commuting operators to functions of non-commuting operators; Alpay and Kalyuzhnyi-Verbovetskii [4] in the context of realization formulas for rational functions that are J -unitary on the boundary of the domain; Helton [10] in proving positive matrix-valued functions are sums of squares; and Helton, Klep and McCullough [11, 12] and Helton and McCullough [13] in the context of developing a descriptive theory of the domains on which LMI and semi-definite programming apply. Recently, Kaliuzhnyi-Verbovetskyi and Vinnikov have written a monograph [16] that gives a panoramic view of the developments in the field to date, and establishes their Taylor-Taylor formula for nc-functions.

There are two topologies that we wish to consider on $\mathbb{M}^{[d]}$. The first is called the *disjoint union topology*: a set U is open in the disjoint union topology if and only if $U \cap \mathbb{M}_n^d$ is open for every n . This topology is too fine for some purposes; for example, compact sets must have a bound on the size of the matrices that they contain. The other topology we wish to consider is the *free topology*, which is most conveniently defined by giving a basis. A basic free open set in $\mathbb{M}^{[d]}$ is a set of the form

$$G_\delta = \{x \in \mathbb{M}^{[d]} : \|\delta(x)\| < 1\},$$

where δ is a J -by- J matrix with entries in \mathbb{P}^d . We define the free topology to be the topology on $\mathbb{M}^{[d]}$ which has as a basis all the sets G_δ , as J ranges over the positive integers, and the entries of δ range over all polynomials in \mathbb{P}^d . (Notice that $G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}$, so these sets do form the basis of a topology). The free topology is a natural topology when considering semi-algebraic sets.

Definition 1.2. A free holomorphic function ϕ on a free open set U in $\mathbb{M}^{[d]}$ is a function that is locally a bounded nc-function, *i.e.* for every x in U there is a basic free open set $G_\delta \subseteq U$ that contains x and such that $\phi|_{G_\delta}$ is a bounded nc-function.

It is a principal result of [1] that free holomorphic functions are locally approximable by free polynomials (see Theorem 2.1 below).

The main result of this note is a criterion for solving a Pick interpolation problem on a basic open set, Theorem 1.3 below, and its generalization to extending bounded free holomorphic functions off free varieties, Theorem 1.5.

Let $H^\infty(U)$ denote the bounded free holomorphic functions on a free open set U with the supremum norm, and let $H_1^\infty(U)$ denote the closed unit ball of $H^\infty(U)$. For $1 \leq i \leq N$, let $\lambda_i \in G_\delta \cap \mathbb{M}_n^d$ and let $w_i \in \mathbb{M}_{n_i}$. The Pick problem is to determine whether or not there is a function in $H_1^\infty(U)$ that maps each λ_i to the corresponding w_i .

Note first that if U is closed under direct sums, then by letting $\Lambda = \bigoplus_{i=1}^N \lambda_i$ and $W = \bigoplus_{i=1}^N w_i$, the original N point problem is the same as solving the one point Pick problem of mapping Λ to W . Secondly, unlike in the scalar case, one cannot always solve the Pick problem if one drops the norm constraint. For example, no holomorphic function maps

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

To state the theorem, let us make the following definitions for Λ in $\mathbb{M}^{[d]}$. Define

$$\mathcal{I}_\Lambda = \{p \in \mathbb{P}^d : p(\Lambda) = 0\}$$

and

$$V_\Lambda = \{x \in \mathbb{M}^{[d]} : p(x) = 0 \text{ whenever } p \in \mathcal{I}_\Lambda\}.$$

Let

$$\mathbb{M}_\Lambda = \{p(\Lambda) : p \in \mathbb{P}^d\}.$$

Note that since \mathbb{M}_Λ is a finite dimensional vector space, it is closed.

Theorem 1.3. Let $\Lambda \in G_\delta \cap \mathbb{M}_n^d$ and $W \in \mathbb{M}_n$. There exists a function ϕ in the closed unit ball of $H^\infty(G_\delta)$ such that $\phi(\Lambda) = W$ if and only if

- (i) $W \in \mathbb{M}_\Lambda$, so there exists $p_0 \in \mathbb{P}^d$ such that $p_0(\Lambda) = W$.
- (ii) $\sup\{\|p_0(x)\| : x \in V_\Lambda \cap G_\delta\} \leq 1$.

We prove this theorem in Section 3. Note that when $d = 1$, the question of whether p_0 can be found satisfying $p_0(\Lambda) = W$ can be resolved by looking at the Jordan canonical form of Λ . In this basis, the algebra \mathbb{M}_Λ has a straightforward description. When $d > 1$, the determination of \mathbb{M}_Λ is more delicate; generically¹, however, the algebra \mathbb{M}_Λ will be all of \mathbb{M}_n .

¹For example, if Λ^1 has n distinct eigenvalues and Λ^2 has no non-zero entry when the matrix is expressed in the basis given by the eigenvectors of Λ^1 .

In Section 4, we give a description in Theorem 4.6 of all the solutions of a (solvable) Pick problem — this is called the Nevanlinna problem. Our approach is indebted to the solution in the scalar case by J. Ball, T. Trent and V. Vinnikov [6].

Theorem 1.3 has a remarkable corollary. Suppose \mathfrak{A} is an algebra in \mathbb{P}^d , and let $\mathfrak{V} = \text{Var}(\mathfrak{A})$ be given by

$$\mathfrak{V} = \{x \in \mathbb{M}^d : p(x) = 0 \ \forall p \in \mathfrak{A}\}. \quad (1.4)$$

If Λ is in \mathfrak{V} , then $\mathfrak{A} \subseteq \mathcal{I}_\Lambda$, and $V_\Lambda \subseteq \mathfrak{V}$. Let U be a free open set in $\mathbb{M}^{[d]}$; we shall say that a function f defined on $\mathfrak{V} \cap U$ is free holomorphic if, for every point x in $\mathfrak{V} \cap U$ there is a basic free open set $G_\delta \subseteq U$ containing x and a free holomorphic function ψ defined on G_δ such that $\psi|_{\mathfrak{V} \cap G_\delta} = f|_{\mathfrak{V} \cap G_\delta}$.

In the scalar case, every holomorphic function defined on an analytic variety inside a domain of holomorphy extends to a holomorphic function on the whole domain, by a celebrated theorem of H. Cartan [8]. The geometric conditions that guarantee that all bounded holomorphic functions extend to be bounded on the whole domain have been investigated by Henkin and Polyakov [15] and Knese [17]; however, even when bounded extensions exist, the extension is almost never isometric [3]. But in the matrix case, any bounded free holomorphic function on $\mathfrak{V} \cap G_\delta$ does extend to a free holomorphic function on G_δ with the same norm.

Theorem 1.5. Let \mathfrak{V} be as in (1.4), and let δ be a matrix of free polynomials such that $\mathfrak{V} \cap G_\delta$ is non-empty. Let f be a bounded free holomorphic function defined on $G_\delta \cap \mathfrak{V}$. Then there is a free holomorphic function ϕ on G_δ that extends f and such that

$$\|\phi\|_{H^\infty(G_\delta)} = \sup_{x \in \mathfrak{V} \cap G_\delta} \|f(x)\| \quad (1.6)$$

We prove this in Section 5. In Section 6 we give some applications.

The definition (1.4) naturally leads one to ask what the ideal of \mathfrak{V} , the set

$$I_{\mathfrak{V}} = \{p \in \mathbb{P}^d : p(x) = 0 \ \forall x \in \mathfrak{V}\},$$

is. In the complex case, the answer is simpler than in the scalar case, at least if \mathfrak{A} is finitely generated. In [14], Bergman, Helton and McCullough proved that $I_{\mathfrak{V}}$ is the smallest ideal containing \mathfrak{A} , provided this ideal is finitely generated. The real (self-adjoint) case is more subtle — see *e.g.* [9].

2 Background material

We shall need some results from [1]. The first we have already referenced:

Theorem 2.1. Let D be a free domain and let ϕ be a graded function defined on D . Then ϕ is a free holomorphic function if and only if ϕ is locally approximable by polynomials.

The second, [1, Thm 8.1], says that a function is in $H_1^\infty(G_\delta)$ if and only if it has a free δ -realization.

Definition 2.2. Let ϕ be a graded function on G_δ , where δ is a J -by- J matrix of free polynomials. A free δ -realization of ϕ is a Hilbert space \mathcal{L} , an isometry $V : \mathbb{C} \oplus (\mathbb{C}^J \otimes \mathcal{L}) \rightarrow \mathbb{C} \oplus (\mathbb{C}^J \otimes \mathcal{L})$ that can be written

$$V = \begin{pmatrix} \mathbb{C} & \mathbb{C}^J \otimes \mathcal{L} \\ \mathbb{C}^J \otimes \mathcal{L} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{pmatrix},$$

and such that

$$\begin{aligned} \phi(x) &= \text{id}_{\mathbb{C}^n} \otimes A + \\ &(\text{id}_{\mathbb{C}^n} \otimes B)(\delta(x) \otimes \text{id}_{\mathcal{L}})[\text{id}_{\mathbb{C}^n} \otimes \text{id}_{\mathbb{C}^J \otimes \mathcal{L}} - (\text{id}_{\mathbb{C}^n} \otimes D)(\delta(x) \otimes \text{id}_{\mathcal{L}})]^{-1}\text{id}_{\mathbb{C}^n} \otimes C \end{aligned}$$

for all $x \in G_\delta \cap \mathbb{M}_n^d$.

We call ϕ the transfer function of V (where δ is understood).

Theorem 2.3. Let ϕ be a graded function on G_δ . Then ϕ is in $H_1^\infty(G_\delta)$ if and only if ϕ has a free δ -realization.

The third is a Montel theorem.

Theorem 2.4. Let $(\phi_i)_{i=1}^\infty$ be a sequence in $H_1^\infty(U)$. Then there is a subsequence $(\phi_{i_j})_{j=1}^\infty$ and a function ϕ in $H_1^\infty(U)$ such that $(\phi_{i_j})_{j=1}^\infty$ converges to ϕ locally uniformly on U in the disjoint union topology.

3 Proof of Theorem 1.3

Let $E = V_\Lambda \cap G_\delta$, and let

$$E^{[2]} = \{(x, y) : x, y \in V_\Lambda \cap G_\delta \cap \mathbb{M}_m^d, \text{ for some } m\}.$$

Let us start with some lemmata.

Lemma 3.1. Let $\Lambda, x \in \mathbb{M}^{[d]}$. The following are equivalent:

- (i) $x \in V_\Lambda$.
- (ii) There is a homomorphism $\alpha : \mathbb{M}_\Lambda \rightarrow \mathbb{M}_x$ such that $\alpha(\Lambda^r) = x^r$ for $r = 1, \dots, d$.
- (iii) The map $p(\Lambda) \mapsto p(x)$ is a well-defined map from \mathbb{M}_Λ to \mathbb{M}_x .
- (iv) The map $p(\Lambda) \mapsto p(x)$ is a completely bounded homomorphism.

PROOF: The equivalence of (i) - (iii) is by definition. That (iii) is equivalent to (iv) is because every bounded homomorphism defined on a finite dimensional space is automatically completely bounded [18]. \square

Lemma 3.2. Let ϕ be in $H^\infty(G_\delta)$. Then there exists a polynomial $p_0 \in \mathbb{P}^d$ so that

$$\phi(x) = p_0(x) \quad \forall x \in V_\Lambda \cap G_\delta. \quad (3.3)$$

PROOF: By Theorem 2.1, the free function ϕ can be uniformly approximated on a free neighborhood of Λ by free polynomials. In particular, since \mathbb{M}_Λ is closed, there is a polynomial p_0 such that $\phi(\Lambda) = p_0(\Lambda)$.

Fix $x \in V_\Lambda \cap G_\delta$. By another application of the same theorem, there is a free polynomial p_1 such that $\phi(\Lambda \oplus x) = p_1(\Lambda \oplus x)$. Therefore $p_0(\Lambda) = p_1(\Lambda)$, so by the definition of V_Λ , we also have $p_0(x) = p_1(x)$. Therefore (3.3) holds, as desired. \square

We let \mathcal{V} denote the vector space of nc-polynomials on E , where we identify polynomials that agree on E ; and we let $\mathcal{V}_{\mathcal{L}(\mathcal{H}, \mathcal{M})}$ denote the vector space of $\mathcal{L}(\mathcal{H}, \mathcal{M})$ -valued nc-polynomials on E . As any such polynomial on E is uniquely determined by its values on Λ , the space of such functions is finite dimensional, if \mathcal{H} and \mathcal{M} are finite dimensional.

Consider the following vector spaces of functions on $E^{[2]}$, where all sums are over a finite set of indices:

$$\begin{aligned} H(E) &= \{h(y, x) = \sum g_i(y)^* f_i(x) : f_i, g_i \in \mathbb{P}^d\} \\ R(E) &= \{h \in H(E) : h(x, y) = h(y, x)^*\} \\ C(E) &= \{h(y, x) = \sum u_i(y)^* [\text{id} - \delta(y)^* \delta(x)] u_i(x) : \\ &\quad u_i \text{ is } \mathcal{L}(\mathbb{C}, \mathbb{C}^J) \text{-valued nc polynomial}\} \\ P(E) &= \{h(y, x) = \sum f_i(y)^* f_i(x) : f_i \in \mathbb{P}^d\} \end{aligned}$$

We topologize $H(E)$ with the norm

$$\|h(y, x)\| = \|h(\Lambda, \Lambda)\|.$$

Lemma 3.4. Let \mathcal{H}, \mathcal{M} be finite dimensional Hilbert spaces, and let $F(y, x)$ be an arbitrary graded $\mathcal{L}(\mathcal{M})$ -valued function on $E^{[2]}$. Let $N_0 = \dim(\mathcal{V}_{\mathcal{L}(\mathcal{H}, \mathcal{M})})$. Then if G can be represented in the form

$$G(y, x) = \sum_{i=1}^m g_i(y)^* F(y, x) g_i(x), \quad (x, y) \in E^{[2]}$$

where $m \in \mathbb{N}$ and $g_i \in \mathcal{V}_{\mathcal{L}(\mathcal{H}, \mathcal{M})}$ for $i = 1, \dots, m$, then G can be represented in the form

$$G(y, x) = \sum_{i=1}^{N_0} f_i(y)^* F(y, x) f_i(x), \quad (x, y) \in E^{[2]} \quad (3.5)$$

where $f_i \in \mathcal{V}_{\mathcal{L}(\mathcal{H}, \mathcal{M})}$ for $i = 1, \dots, N_0$.

PROOF: Let $\langle e_l(x) \rangle_{l=1}^{N_0}$ be a basis of $\mathcal{V}_{\mathcal{L}(\mathcal{H}, \mathcal{M})}$. For each $i = 1, \dots, m$, let

$$g_i(x) = \sum_{l=1}^{N_0} c_{il} e_l(x).$$

Form the $m \times N_0$ matrix $C = [c_{il}]$. As C^*C is an $N_0 \times N_0$ positive semidefinite matrix, there exists an $N_0 \times N_0$ matrix $A = [a_{kl}]$ such that $C^*C = A^*A$. This leads to the formula,

$$\sum_{i=1}^m \bar{c}_{il_1} c_{il_2} = \sum_{k=1}^{N_0} \bar{a}_{kl_1} a_{kl_2},$$

valid for all $l_1, l_2 = 1, \dots, N_0$. If $(x, y) \in E^{[2]}$, then

$$\begin{aligned}
G(y, x) &= \sum_{i=1}^m g_i(y)^* F(y, x) g_i(x) \\
&= \sum_{i=1}^m \left(\sum_{l=1}^{N_0} c_{il} e_l(y) \right)^* F(y, x) \left(\sum_{l=1}^{N_0} c_{il} e_l(x) \right) \\
&= \sum_{l_1, l_2=1}^{N_0} \left(\sum_{i=1}^m \bar{c}_{il_1} c_{il_2} \right) e_{l_1}(y)^* F(y, x) e_{l_2}(x) \\
&= \sum_{l_1, l_2=1}^{N_0} \left(\sum_{k=1}^{N_0} \bar{a}_{kl_1} a_{kl_2} \right) e_{l_1}(y)^* F(y, x) e_{l_2}(x) \\
&= \sum_{k=1}^{N_0} \left(\sum_{l=1}^{N_0} a_{kl} e_l(y) \right)^* F(y, x) \left(\sum_{l=1}^{N_0} a_{kl} e_l(x) \right).
\end{aligned}$$

This proves that (3.5) holds with $f_i = \sum_{l=1}^{N_0} a_{il} e_l$. \square

Lemma 3.6. $C(E)$ is closed.

PROOF: By Lemma 3.4, every element in $C(E)$ can be represented in the form

$$\sum_{i=1}^{N_0} u_i(y)^* [\text{id} - \delta(y)^* \delta(x)] u_i(x), \quad (3.7)$$

where $N_0 = \dim \mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}$. Suppose a sequence of elements of the form (3.7) approaches some h in $H(E)$ at the point (Λ, Λ) :

$$\sum_{i=1}^{N_0} u_i^{(k)}(\Lambda)^* [\text{id} - \delta(\Lambda)^* \delta(\Lambda)] u_i^{(k)}(\Lambda) \rightarrow h(\Lambda, \Lambda) \text{ as } k \rightarrow \infty.$$

Since $\Lambda \in G_\delta$, there is a constant M such that, for each i and k ,

$$\|u_i^{(k)}(\Lambda)\| \leq M.$$

Passing to a subsequence, one can assume that each $u_i^{(k)}(\Lambda)$ converges to some $u_i(\Lambda)$ (since $u_i^{(k)}$ is a graded $\mathcal{L}(\mathbb{C}, \mathbb{C}^J)$ valued function and $J < \infty$). By Lemma 3.1, we have

$$u_i^{(k)}(x) \rightarrow u_i(x) \quad \forall x \in E.$$

Therefore, for all $(x, y) \in E^{[2]}$, we have

$$\begin{aligned} \sum_{i=1}^{N_0} u_i^{(k)}(y)^* [\text{id} - \delta(y)^* \delta(x)] u_i^{(k)}(x) &\rightarrow \sum_{i=1}^{N_0} u_i(y)^* [\text{id} - \delta(y)^* \delta(x)] u_i(x) \\ &= h(y, x). \end{aligned}$$

□

Lemma 3.8. We have $P(E) \subseteq C(E)$.

PROOF: We have

$$f(y)^* f(x) - \sum_{k=0}^{m-1} f(y)^* \delta(y)^k [\text{id} - \delta(y) \delta(x)] \delta(x)^k f(x) = f(y)^* \delta(y)^m \delta(x)^m f(x). \quad (3.9)$$

As $m \rightarrow \infty$, the right-hand side of (3.9) goes to zero for every $(x, y) \in E^{[2]}$. Since $C(E)$ is closed by Lemma 3.6, this proves that $f(y)^* f(x) \in C(E)$, and hence so are finite sums of this form. □

Lemma 3.10. Suppose $\sup\{\|p_0(x)\| : x \in E\} \leq 1$. Then the function

$$h(y, x) = \text{id} - p_0(y)^* p_0(x)$$

is in $C(E)$.

PROOF: This will follow from the Hahn-Banach theorem [23, Thm. 3.3.4] if we can show that $L(h(y, x)) \geq 0$ whenever

$$L \in R(E)^* \quad \text{and} \quad L(g) \geq 0 \quad \forall g \in C(E). \quad (3.11)$$

Assume (3.11) holds, and define $L^\sharp \in H(E)^*$ by the formula

$$L^\sharp(h(y, x)) = L\left(\frac{h(y, x) + h(x, y)^*}{2}\right) + iL\left(\frac{h(y, x) - h(x, y)^*}{2i}\right),$$

and then define sesquilinear forms on \mathcal{V} and $\mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}$ by the formulas

$$\begin{aligned} \langle f, g \rangle_{L_1} &= L^\sharp(g(y)^* f(x)), \quad f, g \in \mathcal{V} \\ \langle F, G \rangle_{L_2} &= L^\sharp(G(y)^* F(x)), \quad F, G \in \mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}. \end{aligned}$$

Observe that Lemma 3.8 implies that $f(y)^* f(x) \in C(E)$ whenever $f \in \mathcal{V}$ or $\mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}$. Hence, (3.11) implies that $\langle f, f \rangle_{L_1} \geq 0$ for all $f \in \mathcal{V}$, and

$\langle F, F \rangle_{L_2} \geq 0$ for all $F \in \mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}$, i.e., $\langle \cdot, \cdot \rangle_{L_1}$ and $\langle \cdot, \cdot \rangle_{L_2}$ are pre-inner products on \mathcal{V} and $\mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}$ respectively.

To make them into inner products, choose $\varepsilon > 0$ and define

$$\langle f, g \rangle_1 = L^\sharp(g(y)^* f(x)) + \varepsilon \operatorname{tr}(g(\Lambda)^* f(\Lambda)), \quad f, g \in \mathcal{V} \quad (3.12)$$

$$\langle F, G \rangle_2 = L^\sharp(G(y)^* F(x)) + \varepsilon \operatorname{tr}(G(\Lambda)^* F(\Lambda)), \quad F, G \in \mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)} \quad (3.13)$$

We let $H_{L_1}^2$ and $H_{L_2}^2$ denote the Hilbert spaces \mathcal{V} and $\mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}$ equipped with the inner products (3.12) and (3.13).

The fact that L is non-negative on $C(E)$ means that

$$\langle F, F \rangle_{L_2} \geq \langle \delta F, \delta F \rangle_{L_2} \quad (3.14)$$

for all F in $\mathcal{V}_{\mathcal{L}(\mathbb{C}, \mathbb{C}^J)}$. Since $\|\delta(\Lambda)\| < 1$, we also have

$$\operatorname{tr}(F(\Lambda)^* F(\Lambda)) > \operatorname{tr}(F(\Lambda)^* \delta(\Lambda)^* \delta(\Lambda) F(\Lambda)), \quad (3.15)$$

if $F \neq 0$, and combining (3.14) and (3.15) we get that multiplication by δ is a strict contraction on $H_{L_2}^2$.

Let M denote the d -tuple of multiplication by the coordinate functions x^r on $H_{L_1}^2$. We have just shown that $\|\delta(M)\| < 1$, so M is in G_δ . As M is also in V_Λ , we have that M is in E . Therefore $\|p_0(M)\| \leq 1$, by hypothesis. Therefore

$$\operatorname{id} - p_0(M)^* p_0(M) \geq 0,$$

and so for all f in \mathcal{V} we have

$$\begin{aligned} L^\sharp(f(y)^* f(x)) + \varepsilon \operatorname{tr}(f(\Lambda)^* f(\Lambda)) &\geq \\ L^\sharp(f(y)^* p_0(y)^* p_0(x) f(x)) + \varepsilon \operatorname{tr}(f(\Lambda)^* p_0(\Lambda)^* p_0(\Lambda) f(\Lambda)). \end{aligned}$$

Letting f be the function 1 and letting $\varepsilon \rightarrow 0$, we get

$$L(\operatorname{id} - p_0(y)^* p_0(x)) \geq 0,$$

as desired. \square

We can now prove the theorem.

PROOF OF THEOREM 1.3: (Necessity). Condition (i) follows from Lemma 3.2. Condition (ii) follows because $p_0(x) = f(x)$ for $x \in V_\Lambda \cap G_\delta$, and f is in the unit ball of $H^\infty(G_\delta)$, so $\|f(x)\| \leq 1$ for every x in G_δ .

(Sufficiency). Suppose (i) and (ii) hold. By Lemma 3.10, the function

$$h(y, x) = \text{id} - p_0(y)^* p_0(x)$$

is in $C(E)$. By Lemma 3.4, there is some positive integer $N \leq \dim(\mathcal{V}_{\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)})$ and an $\mathcal{L}(\mathbb{C}, \mathbb{C}^{JN})$ -valued nc polynomial u such that, for $x, y \in E \cap \mathbb{M}_n^d$,

$$\begin{aligned} h(y, x) &= \text{id}_{\mathbb{C}^n} - p_0(y)^* p_0(x) \\ &= u(y)^* [\text{id}_{\mathbb{C}^{JN}} - (\delta(y)^* \otimes \text{id}_{\mathbb{C}^N}) (\delta(x) \otimes \text{id}_{\mathbb{C}^N})] u(x). \end{aligned} \quad (3.16)$$

Replace x in (3.16) with sxs^{-1} where s is invertible in \mathbb{M}_n and sxs^{-1} is in G_δ to get

$$\begin{aligned} s - p_0(y)^* sp_0(x) &= \\ u(y)^* [s \otimes \text{id}_{\mathbb{C}^{JN}} - (\delta(y)^* \otimes \text{id}_{\mathbb{C}^N}) s \otimes \text{id}_{\mathbb{C}^{JN}} (\delta(x) \otimes \text{id}_{\mathbb{C}^N})] u(x). \end{aligned} \quad (3.17)$$

Equation (3.17) is true for all s in a neighborhood of the identity, and as linear combinations of such elements span \mathbb{M}_n , we get that (3.17) actually holds for all s in \mathbb{M}_n . For $k = 1, \dots, n$, define $\pi_k : \mathbb{C}^n \rightarrow \mathbb{C}$ by the formula

$$\pi_k(v) = v_k, \quad v = (v_1, \dots, v_n) \in \mathbb{C}^n.$$

Letting $s = \pi_l^* \pi_k$ in (3.17) and applying to v and taking the inner product with w , where v and w are in \mathbb{C}^n , leads to

$$\begin{aligned} \langle [\pi_l^* \pi_k - p_0(y)^* \pi_l^* \pi_k p_0(x)] v, w \rangle &= \\ \langle [\pi_l^* \pi_k \otimes \text{id} - (\delta(y)^* \otimes \text{id})(\pi_l^* \pi_k \otimes \text{id})(\delta(x) \otimes \text{id})] u(x)v, u(y)w \rangle. \end{aligned} \quad (3.18)$$

For each $v \in \mathbb{C}^n$ define vectors p_v and q_v in $\mathbb{C}^{n(1+NJ)}$ by

$$\begin{aligned} p_v &= \begin{bmatrix} \text{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \text{id}_{\mathbb{C}^N}] u(\Lambda) \end{bmatrix} v \\ q_v &= \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix} v. \end{aligned}$$

For each $1 \leq k \leq n$, define vectors $p_{k,v}$ and $q_{k,v}$ in \mathbb{C}^{1+NJ} by

$$\begin{aligned} p_{k,v} &= [\pi_k \otimes \text{id}_{\mathbb{C}^{1+NJ}}] p_v \\ q_{k,v} &= [\pi_k \otimes \text{id}_{\mathbb{C}^{1+NJ}}] q_v. \end{aligned}$$

Then (3.18), with Λ in place of both x and y , becomes

$$\langle p_{k,v}, p_{l,w} \rangle = \langle q_{k,v}, q_{l,w} \rangle \quad \forall v, w \in \mathbb{C}^n, \forall 1 \leq k, l \leq n. \quad (3.19)$$

So by (3.19), there is an isometry V that maps each $p_{k,v}$ to $q_{k,v}$. If the span of the vectors $\{p_{k,v}\}$ is not all of \mathbb{C}^{1+NJ} , we can extend V to the orthocomplement so that it becomes an isometry (indeed, a unitary) from all of \mathbb{C}^{1+NJ} to \mathbb{C}^{1+NJ} .

With respect to the decomposition $\mathbb{C} \oplus \mathbb{C}^{JN}$, write

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

We have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} [\pi_k \otimes \text{id}_{\mathbb{C}^{1+NJ}}] \begin{bmatrix} \text{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \text{id}_{\mathbb{C}^N}] u(\Lambda) \end{bmatrix} = [\pi_k \otimes \text{id}_{\mathbb{C}^{1+NJ}}] \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix}. \quad (3.20)$$

Since (3.20) holds for each k , we get that

$$\begin{bmatrix} \text{id}_{\mathbb{C}^n} \otimes A & \text{id}_{\mathbb{C}^n} \otimes B \\ \text{id}_{\mathbb{C}^n} \otimes C & \text{id}_{\mathbb{C}^n} \otimes D \end{bmatrix} \begin{bmatrix} \text{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \text{id}_{\mathbb{C}^N}] u(\Lambda) \end{bmatrix} = \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix}. \quad (3.21)$$

For x in $G_\delta \cap \mathbb{M}_n^d$, define

$$\begin{aligned} \phi(x) &= \text{id}_{\mathbb{C}^n} \otimes A + [(\text{id}_{\mathbb{C}^n} \otimes B)(\delta(x) \otimes \text{id}_{\mathbb{C}^N})] \\ &\quad [\text{id}_{\mathbb{C}^n} \otimes \text{id}_{\mathbb{C}^{JN}} - (\text{id}_{\mathbb{C}^n} \otimes D)(\delta(x) \otimes \text{id}_{\mathbb{C}^N})]^{-1} \text{id}_{\mathbb{C}^n} \otimes C. \end{aligned}$$

Then ϕ is in the unit ball of $H^\infty(G_\delta)$ by Theorem 2.3. Moreover, by (3.21),

$$\phi(\Lambda) = p_0(\Lambda),$$

as desired. \square

4 The Nevanlinna Problem

There are two sources of non-uniqueness in the solution of the Pick interpolation problem. The first is the choice of u in (3.16); the second is in the extension of V . This problem has been analyzed in the scalar case by J. Ball, T. Trent and V. Vinnikov [6]; their ideas extend to our situation.

Let us suppose throughout this section that

$$\Lambda \mapsto p_0(\Lambda) \quad (4.1)$$

is a solvable Pick problem, and we have found a finite-dimensional space \mathcal{L} , an $\mathcal{L}(\mathbb{C}, \mathbb{C}^J \otimes \mathcal{L})$ -valued nc polynomial u satisfying

$$\text{id}_{\mathbb{C}^n} - p_0(\Lambda)^* p_0(\Lambda) = u(\Lambda)^* [\text{id}_{\mathbb{C}^{nJ} \otimes \mathcal{L}} - (\delta(\Lambda)^* \delta(\Lambda) \otimes \text{id}_{\mathcal{L}})] u(\Lambda), \quad (4.2)$$

and V satisfying (3.20):

$$V[\pi_k \otimes \text{id}_{\mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}}] \begin{bmatrix} \text{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \text{id}_{\mathcal{L}}] u(\Lambda) \end{bmatrix} = [\pi_k \otimes \text{id}_{\mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}}] \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix}. \quad (4.3)$$

Let $\mathcal{L}_0 = \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$, and

$$\mathcal{N}_2 := \vee_{k=1}^n \vee_{v \in \mathbb{C}^n} \left[\frac{\pi_k v}{[(\pi_k \otimes \text{id}_{\mathbb{C}^J \otimes \mathcal{L}})(\delta(\Lambda) \otimes \text{id}_{\mathbb{C}^N})] u(\Lambda) v} \right] \subseteq \mathcal{L}_0.$$

Let

$$\mathcal{N}_1 := \vee_{k=1}^n \vee_{v \in \mathbb{C}^n} \left[\frac{\pi_k p_0(\Lambda) v}{(\pi_k \otimes \text{id}_{\mathbb{C}^J \otimes \mathcal{L}}) u(\Lambda) v} \right] \subseteq \mathcal{L}_0,$$

and define $\mathcal{M}_2 = \mathcal{L}_0 \ominus \mathcal{N}_2$ and $\mathcal{M}_1 = \mathcal{L}_0 \ominus \mathcal{N}_1$. Then V is an isometry from \mathcal{N}_2 onto \mathcal{N}_1 . Define a unitary

$$\begin{aligned} U : \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{N}_2 &\rightarrow \mathcal{M}_2 \oplus \mathcal{M}_1 \oplus \mathcal{N}_1 \\ \begin{bmatrix} m_1 \\ m_2 \\ n_2 \end{bmatrix} &\mapsto \begin{bmatrix} m_2 \\ m_1 \\ Vn_2 \end{bmatrix}. \end{aligned} \quad (4.4)$$

By identifying $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{N}_2 \cong \mathcal{M}_1 \oplus \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$ and $\mathcal{M}_2 \oplus \mathcal{M}_1 \oplus \mathcal{N}_1 \cong \mathcal{M}_2 \oplus \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$, we can think of U as a unitary from $\mathbb{C} \oplus \mathcal{M}_1 \oplus \mathbb{C}^J \otimes \mathcal{L}$ to $\mathbb{C} \oplus \mathcal{M}_2 \oplus \mathbb{C}^J \otimes \mathcal{L}$, and it has a corresponding transfer function G that is a free $\mathcal{L}(\mathbb{C} \oplus \mathcal{M}_1, \mathbb{C} \oplus \mathcal{M}_2)$ -valued rational function (since all the spaces are finite dimensional). Write this G as

$$G = \begin{pmatrix} \mathbb{C} & \mathcal{M}_1 \\ \mathcal{M}_2 & \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \end{pmatrix}. \quad (4.5)$$

Theorem 4.6. The function ϕ in $\text{ball}(H^\infty(G_\delta))$ satisfies $\phi(\Lambda) = p_0(\Lambda)$ if and only if, for some u satisfying (4.2) and G the transfer function of U in (4.4), there is a function Θ in $\text{ball}(H_{\mathcal{L}(\mathcal{M}_1, \mathcal{M}_2)}^\infty(G_\delta))$ such that

$$\phi = G_{11} + G_{12}\Theta(I_{\mathcal{M}_1} - G_{22}\Theta)^{-1}G_{21}. \quad (4.7)$$

PROOF: (\Leftarrow) This is a straightforward calculation.

(\Rightarrow) By Theorem 2.3, ϕ has a free δ -realization, and by Lemma 3.4, we can assume that $\{u(x) : x \in V_\Lambda\}$ lie in a finite dimensional space that we can embed in $\mathbb{C}^J \otimes \mathcal{L}$. So we can assume that ϕ is the transfer function of some unitary $X : \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{K} \rightarrow \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{K}$, and that $\mathcal{L} \subseteq \mathcal{K}$. For $x \in G_\delta \cap \mathbb{M}_m^d$ we have

$$[\text{id}_{\mathbb{C}^m} \otimes X] \begin{bmatrix} \text{id}_{\mathbb{C}^m} \\ (\delta(x) \otimes \text{id}_{\mathcal{K}})\xi(x) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \xi(x) \end{bmatrix}. \quad (4.8)$$

Let $\mathcal{K}' = \mathcal{K} \ominus \mathcal{L}$. Then

$$X = \frac{\mathcal{N}_1}{\mathcal{M}_1 \oplus \mathbb{C}^J \otimes \mathcal{K}'} \begin{pmatrix} \mathcal{N}_2 & \mathcal{M}_2 \oplus \mathbb{C}^J \otimes \mathcal{K}' \\ V & 0 \\ 0 & Y \end{pmatrix}. \quad (4.9)$$

Let Θ be the transfer function of Y . Then we claim that (4.7) holds.

Let $x \in G_\delta \cap \mathbb{M}_m^d$ and $v \in \mathbb{C}^m$ be fixed for now. Let

$$\begin{aligned} p &= v \oplus (\delta(x) \otimes \text{id}_{\mathcal{K}})\xi(x)v = n_2 \oplus m_2 \oplus h_2 \\ q &= \phi(x)v \oplus \xi(x)v = n_1 \oplus m_1 \oplus h_1 \end{aligned} \quad (4.10)$$

where $n_2 \in \mathbb{C}^m \otimes \mathcal{N}_2$, $m_2 \in \mathbb{C}^m \otimes \mathcal{M}_2$, $n_1 \in \mathbb{C}^m \otimes \mathcal{N}_1$, $m_1 \in \mathbb{C}^m \otimes \mathcal{M}_1$ and $h_2, h_1 \in \mathbb{C}^m \otimes \mathbb{C}^J \otimes \mathcal{K}'$. Note from (4.8) that

$$[\text{id}_{\mathbb{C}^m} \otimes X]p = q. \quad (4.11)$$

Let P' be the projection from $\mathbb{C}^J \otimes \mathcal{K}$ to $\mathbb{C}^J \otimes \mathcal{K}'$. As $\delta(x) \otimes \text{id}_{\mathcal{K}}$ commutes with $\text{id}_{\mathbb{C}^m} \otimes P'$, we get from (4.10) that

$$[\delta(x) \otimes \text{id}_{\mathcal{K}'}]h_1 = h_2.$$

Therefore

$$[\text{id}_{\mathbb{C}^m} \otimes Y](m_2 \oplus [\delta(x) \otimes \text{id}_{\mathcal{K}'}]h_1) = m_1 \oplus h_1. \quad (4.12)$$

As Θ is the transfer function of Y , (4.12) implies that

$$\Theta(x)m_2 = m_1. \quad (4.13)$$

Let P be the projection from $\mathbb{C}^m \otimes \mathbb{C}^J \otimes \mathcal{K}$ onto $\mathbb{C}^m \otimes \mathbb{C}^J \otimes \mathcal{L}$, and let $\eta = P\xi(x)v$. Then under the identifications of $\mathcal{N}_1 \oplus \mathcal{M}_1$ and $\mathcal{N}_2 \oplus \mathcal{M}_2$ with $\mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$, we get

$$\begin{aligned} n_1 \oplus m_1 &= \phi(x)v \oplus \eta \\ n_2 \oplus m_2 &= v \oplus (\delta(x) \otimes \text{id}_{\mathcal{L}})\eta. \end{aligned}$$

Then from (4.4)

$$U : v \oplus m_1 \oplus (\delta(x) \otimes \text{id}_{\mathcal{L}})\eta \mapsto \phi(x)v \oplus m_2 \oplus \eta.$$

By (4.13) this gives

$$\begin{pmatrix} G_{11}(x) & G_{12}(x) \\ G_{21}(x) & G_{22}(x) \end{pmatrix} \begin{pmatrix} v \\ \Theta(x)m_2 \end{pmatrix} = \begin{pmatrix} \phi(x)v \\ m_2 \end{pmatrix}. \quad (4.14)$$

As (4.14) holds for all choices of x and v , we get (4.7), as desired. \square

5 Extending functions defined on varieties

PROOF OF THEOREM 1.5: Without loss of generality, assume that

$$\sup_{x \in \mathfrak{V} \cap G_\delta} \|f(x)\| = 1. \quad (5.1)$$

Choose a sequence $(\lambda_j)_{j=1}^\infty$ in $G_\delta \cap \mathfrak{V}$ that is dense in the disjoint union topology, so for all $\varepsilon > 0$, for all $x \in G_\delta \cap \mathfrak{V}$, there exists some λ_j such that $\max_{1 \leq r \leq d} \|\lambda_j^r - x^r\| < \varepsilon$.

Let $\Lambda_n = \bigoplus_{j=1}^n \lambda_j$. By Theorem 2.1, f is locally approximable by polynomials, and so has the property that

$$\forall x \in \mathfrak{V} \cap G_\delta, f(x) \in \mathbb{M}_x.$$

Therefore there is some polynomial $p_n \in \mathbb{P}^d$ such that

$$p_n(\Lambda_n) = f(\Lambda_n). \quad (5.2)$$

Moreover, if $x \in \mathfrak{V} \cap G_\delta$, then by Theorem 2.1 again, one can approximate f at $x \oplus \Lambda_n$ by a sequence of free polynomials, and so by Lemma 3.1

$$\forall x \in \mathfrak{V} \cap G_\delta, f(x) = p_n(x). \quad (5.3)$$

As $V_\Lambda \subseteq \mathfrak{V}$, putting (5.2), (5.3) and (5.1) together, the hypotheses of Theorem 1.3 are satisfied, so there exists ϕ_n in $H_1^\infty(G_\delta)$ such that

$$\phi_n(\Lambda_n) = f(\Lambda_n).$$

By Theorem 2.4, some subsequence of ϕ_n converges locally uniformly (in the disjoint union topology) to a function ϕ in $H_1^\infty(G_\delta)$. Moreover, for each j , $\phi(\lambda_j) = f(\lambda_j)$, so by continuity, ϕ is an extension of f . \square

6 Examples

Example 6.1 Let q_1, \dots, q_m be polynomials in d commuting variables, and let $V = \{z \in \mathbb{C}^d : q_i(z) = 0, i = 1, \dots, m\}$. Let f be a (scalar-valued) holomorphic function defined on $V \cap \mathbb{D}^d$.

Let T be a d -tuple of commuting matrices that are strict contractions, and such that $q_i(T) = 0$ for $i = 1, \dots, m$. If they are simultaneously diagonalizable, then their joint eigenvalues lie in $V \cap \mathbb{D}^d$, and it makes sense to define $f(T)$ by applying f to the diagonal entries, in the basis of joint eigenvectors. If the matrices are not simultaneously diagonalizable, then one can still define $f(T)$, either by the Taylor functional calculus [24], or, more constructively, as in [2].

Let us write \mathcal{F} for the set of all $T = (T^1, \dots, T^d)$ of commuting matrices such that $q_i(T) = 0, i = 1, \dots, m$, and such that $\|T\| < 1$. Note that $\mathcal{F} = \mathfrak{V} \cap G_\delta$, where \mathfrak{A} is the algebra generated by q_1, \dots, q_m and the polynomials $\{x^i x^j - x^j x^i : 1 \leq i < j \leq d\}$, $\mathfrak{V} = \text{Var}(\mathfrak{A})$, and $\delta(x)$ is the diagonal matrix with entries x^1, x^2, \dots, x^d . Define a norm on holomorphic functions on $V \cap \mathbb{D}^d$ by

$$\|f\|_{\mathfrak{V} \cap G_\delta} = \sup\{\|f(T)\| : T \in \mathcal{F}\}.$$

To apply Theorem 1.5, we need to know that f is a free holomorphic function on $\mathfrak{V} \cap G_\delta$, in other words that locally in G_δ it extends to a free holomorphic function (*i.e.* it can be applied to non-commuting matrices). This is true, and is proved in [2]. Then Theorem 1.5 asserts that there is a bounded extension ϕ of f , defined on the set $\{R \in \mathbb{M}^d : \|R\| < 1\}$, if and only if $\|f\|_{\mathfrak{V} \cap G_\delta}$ is finite. Moreover, if this quantity is finite, then ϕ can be found with exactly this norm. In particular, an extension to the non-commuting ball G_δ can always be found with the same norm as is attained by evaluating on commuting matrices in the variety.

Example 6.2 Specializing the previous example to the case $d = 2$, and using Andô's inequality [5], we conclude the following: if we wish to extend a polynomial p_0 off $V \cap \mathbb{D}^2$, where V is the joint zero set of the q_i 's, then the minimum norm of the extension ϕ is the same when calculated as a scalar-valued function in $H^\infty(\mathbb{D}^2)$, as a function on pairs of commuting contractive matrices, or as a function on pairs of contractive matrices. The norm is attained, and is given by

$$\sup_{n \in \mathbb{N}} \sup\{\|p_0(T)\| : T \in \mathbb{M}_n^2, \|T^1\| < 1, \|T^2\| < 1, \\ T^1 T^2 = T^2 T^1, q_i(T) = 0 \forall 1 \leq i \leq m\}. \quad (6.3)$$

Unless $V \cap \mathbb{D}^2$ is a retract of \mathbb{D}^2 , one can by [3] always find some p_0 so that (6.3) is strictly greater than

$$\sup\{|p_0(z)| : z \in \mathbb{D}^2 \cap V\}.$$

Example 6.4 Suppose $\delta(x)$ has first column x^1, \dots, x^d and its other entries zero, so

$$G_\delta = \{T : T^{1*} T^1 + \dots + T^{d*} T^d < 1\}.$$

(This is called the row ball). Suppose $\Lambda, H \in G_\delta \cap \mathbb{M}_n^d$ and one wishes to solve the interpolation problem

$$\begin{aligned} \phi(\Lambda) &= W \\ D\phi(\Lambda)[H] &= X, \end{aligned} \quad (6.5)$$

where $D\phi(\Lambda)[H]$, the derivative of ϕ at Λ in the direction H , is defined by

$$D\phi(\Lambda)[H] = \lim_{t \rightarrow 0} \frac{\phi(\Lambda + tH) - \phi(\Lambda)}{t}.$$

A necessary condition to find a function $\phi \in H^\infty(G_\delta)$ solving this problem is that there is some free polynomial p_0 with $p_0(\Lambda) = W$ and $Dp_0(\Lambda)[H] = X$. The minimum norm of a solution can be found from Theorem 1.5 by letting

$$\mathfrak{U} = \{p \in \mathbb{P}^d : p(\Lambda) = 0, Dp(\Lambda)[H] = 0\},$$

$\mathfrak{V} = \text{Var}(\mathfrak{U})$, and calculating

$$\sup_{x \in \mathfrak{V} \cap G_\delta} \|p_0(x)\|.$$

The problem can also be solved using Theorem 1.3, as (6.5) is the same as solving the one point problem

$$\begin{pmatrix} \Lambda & H \\ 0 & \Lambda \end{pmatrix} \mapsto \begin{pmatrix} W & X \\ 0 & W \end{pmatrix},$$

since by [12, Prop 2.5], for any continuous nc-function f , one has

$$f \begin{pmatrix} \Lambda & H \\ 0 & \Lambda \end{pmatrix} = \begin{pmatrix} f(\Lambda) & Df(\Lambda)[H] \\ 0 & f(\Lambda) \end{pmatrix}.$$

References

- [1] J. Agler and J.E. McCarthy. Global holomorphic functions in several non-commuting variables. To appear.
- [2] J. Agler and J.E. McCarthy. Operator theory and the Oka extension theorem. To appear.
- [3] J. Agler and J.E. McCarthy. Norm preserving extensions of holomorphic functions from subvarieties of the bidisk. *Ann. of Math.*, 157(1):289–312, 2003.
- [4] D. Alpay and D. S. Kalyuzhnyi-Verbovetskii. Matrix- J -unitary non-commutative rational formal power series. In *The state space method generalizations and applications*, volume 161 of *Oper. Theory Adv. Appl.*, pages 49–113. Birkhäuser, Basel, 2006.
- [5] T. Andô. On a pair of commutative contractions. *Acta Sci. Math. (Szeged)*, 24:88–90, 1963.
- [6] J.A. Ball, T.T. Trent, and V. Vinnikov. Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces. In *Operator Theory and Analysis*, pages 89–138. Birkhäuser, Basel, 2001.
- [7] Joseph A. Ball, Gilbert Groenewald, and Tanit Malakorn. Conservative structured noncommutative multidimensional linear systems. In *The state space method generalizations and applications*, volume 161 of *Oper. Theory Adv. Appl.*, pages 179–223. Birkhäuser, Basel, 2006.

- [8] H. Cartan. *Séminaire Henri Cartan 1951/2*. W.A. Benjamin, New York, 1967.
- [9] Jakob Cimprič, J. William Helton, Scott McCullough, and Christopher Nelson. Real nullstellensatze and $*$ -ideals in $*$ -algebras. <http://arxiv.org/pdf/1302.4722v2.pdf>.
- [10] J. William Helton. “Positive” noncommutative polynomials are sums of squares. *Ann. of Math. (2)*, 156(2):675–694, 2002.
- [11] J. William Helton, Igor Klep, and Scott McCullough. Analytic mappings between noncommutative pencil balls. *J. Math. Anal. Appl.*, 376(2):407–428, 2011.
- [12] J. William Helton, Igor Klep, and Scott McCullough. Proper analytic free maps. *J. Funct. Anal.*, 260(5):1476–1490, 2011.
- [13] J. William Helton and Scott McCullough. Every convex free basic semi-algebraic set has an LMI representation. *Ann. of Math. (2)*, 176(2):979–1013, 2012.
- [14] J. William Helton and Scott A. McCullough. A Positivstellensatz for non-commutative polynomials. *Trans. Amer. Math. Soc.*, 356(9):3721–3737 (electronic), 2004.
- [15] G.M. Henkin and P.L. Polyakov. Prolongement des fonctions holomorphes bornes d’une sous-varieté du polydisque. *Comptes Rendus Acad. Sci. Paris Sér. I Math.*, 298(10):221–224, 1984.
- [16] Dmitry S. Kaluzhnyi-Verbovetskyi and Victor Vinnikov. Foundations of non-commutative function theory. arXiv:1212.6345.
- [17] Greg Knese. Polynomials defining distinguished varieties. *Trans. Amer. Math. Soc.*, 362(11):5635–5655, 2010.
- [18] V.I. Paulsen. *Completely bounded maps and operator algebras*. Cambridge University Press, Cambridge, 2002.
- [19] Gelu Popescu. Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$. *J. Funct. Anal.*, 241(1):268–333, 2006.

- [20] Gelu Popescu. Free holomorphic functions and interpolation. *Math. Ann.*, 342(1):1–30, 2008.
- [21] Gelu Popescu. Free holomorphic automorphisms of the unit ball of $B(\mathcal{H})^n$. *J. Reine Angew. Math.*, 638:119–168, 2010.
- [22] Gelu Popescu. Free biholomorphic classification of noncommutative domains. *Int. Math. Res. Not. IMRN*, (4):784–850, 2011.
- [23] W. Rudin. *Functional Analysis*. McGraw-Hill, New York, 1991.
- [24] J.L. Taylor. The analytic functional calculus for several commuting operators. *Acta Math.*, 125:1–38, 1970.
- [25] J.L. Taylor. Functions of several non-commuting variables. *Bull. Amer. Math. Soc.*, 79:1–34, 1973.
- [26] Dan Voiculescu. Free analysis questions. I. Duality transform for the coalgebra of ∂_{X^*B} . *Int. Math. Res. Not.*, (16):793–822, 2004.
- [27] Dan-Virgil Voiculescu. Free analysis questions II: the Grassmannian completion and the series expansions at the origin. *J. Reine Angew. Math.*, 645:155–236, 2010.