203: Homework 7     Due: 13/October

1. Prove the squeeze theorem: Let $I$ be an open interval in $\mathbb{R}$, and $a \in I$. Assume that

$$f(x) \;\leq\; g(x) \;\leq\; h(x) \qquad \forall x \;\in I \setminus \{a\},$$

and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x). \tag{1.1}$$

Prove that $\lim_{x \to a} g(x)$ exists and equals the limit in (1.1).

2. Suppose $(a_n)$ and $(b_n)$ are convergent sequences. Prove that

$$\lim_{n \to \infty} a_n b_n \;=\; (\lim_{n \to \infty} a_n)\,(\lim_{n \to \infty} b_n).$$

3. Prove or disprove: if the sequence $(a_n + b_n)$ converges, then both $(a_n)$ and $(b_n)$ converge.

4. Suppose $I$ is an interval in $\mathbb{R}$, $(x_n)$ is a sequence in $I$ that converges to $c \in I$. Suppose $f : I \to \mathbb{R}$. Prove that if $f$ is continuous at $c$, then $\lim_{n \to \infty} f(x_n) = f(c)$.

5. Give an example to show that the converse to the assertion in (4.) is not true.

6. Give an example of a continuous bounded function on a closed interval that does not attain its maximum.