203: Homework 8 Due: 20/October

1. Suppose I is an open interval in $\mathbb{R}$, that $f : I \to \mathbb{R}$, and $c \in I$. Suppose that for every sequence (x_n) in I that converges to c, $\lim_{n\to\infty} f(x_n)$ exists. Prove that all the limits are equal, and that $\lim_{x\to c} f(x)$ exists and equals the same number.

2. Prove that

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

3. Prove that the equation

$$x^2 = x \sin x + \cos x$$

has exactly two real solutions.

4. Prove that $|\sin x - \sin y| \leq |x - y|$, for all $x, y \in \mathbb{R}$.

5. A number c is called a *zero of multiplicity* m for the polynomial p if $p(x) = (x-c)^m q(x)$, where q is a polynomial and $q(c) \neq 0$. Suppose p has r zeros in the interval $[a, b]$, counting each zero as often as its multiplicity. Prove that p' has at least $r-1$ zeros in $[a, b]$.

6. On HW 4, you did exercise 4.7 in Structure and Proof. Use this to prove the binomial theorem, exercise 4.8.