Beurling's theorem for the Hardy operator on $L^2[0,1]$

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Abstract

We prove that the invariant subspaces of the Hardy operator on $L^2[0,1]$ are the spaces that are limits of sequences of finite dimensional spaces spanned by monomial functions.

1 Introduction

The space $L^2[0,1]$ is a cornerstone of analysis. One way to analyze it is to use the exponential functions e^{itx}, which have the advantage of being eigenfunctions for differentiation. Another way is to use the monomial functions x^s. The Müntz-Szász theorem gives necessary and sufficient conditions for a collection of monomial functions to span $L^2[0,1]$. Monomials are eigenfunctions for the Hardy operator H, defined by

$$Hf(x) \ = \ \frac{1}{x}\int_0^x f(t)dt.$$

Conversely, if T is a bounded linear operator on $L^2[0,1]$ that has x^s as an eigenvector whenever x^s is in $L^2[0,1]$, then T is a function of H; specifically, it is of the form $\phi(H)$ for some function ϕ that is bounded and analytic on the disk $\mathbb{D}(1,1) = \{z \in \mathbb{C} : |z-1| < 1\}$ [?].

We shall use L^2 to denote $L^2[0,1]$ throughout. Hardy proved in [?] that H is bounded on L^2 (and indeed on L^p for all $p > 1$). For a treatment of H consult the book [?]. What are its invariant subspaces?

Let $\mathbb{S}$ denote the half plane $\{s \in \mathbb{C} : \mathrm{Re}\ (s) > -\frac{1}{2}\}$. Then if $s \in \mathbb{S}$, the monomial function x^s is in L^2, and $Hx^s \ = \ \frac{1}{s+1}x^s$; moreover the monomials constitute all the eigenvectors of

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H. Any space that is the linear span of finitely many monomial functions is invariant for H. We shall call such a space a finite monomial space. It is the object of this note to prove that every invariant subspace of H is a limit of finite monomial spaces.

The Hardy operator is unitarily equivalent to $1 - S^*$, where S is the unilateral shift [?]. Its invariant subspaces are therefore described by the celebrated theorem of Beurling [?] which described the invariant subspaces of the shift using the beautiful theory of Hardy spaces of holomorphic functions. Using this theory, Theorem ?? below is well-known. It is proved as the Theorem on Finite Dimensional Approximation [?, p.37]. However, the point of this note is to describe the invariant subspaces of H without using any Hardy space theory, just using L^2 techniques and functional analysis. Our hope is that this approach will not only illuminate L^2 with a new light, but may also generalize to related spaces, such as L^p or weighted L^p spaces.

Definition 1.1. *For S a finite subset of $\mathbb{S}$ we let $\mathcal{M}(S)$ denote the span in L^2 of the monomials whose exponents lie in S, i.e.,*

$$\mathcal{M}(S) = \{\sum_{s \in S} a(s) x^s \mid a : S \to \mathbb{C}\}.$$

We refer to sets in L^2 that have the form $\mathcal{M}(S)$ for some finite subset S of $\mathbb{S}$ as finite monomial spaces*.*

Definition 1.2. *If $\mathcal{M}$ is a subspace of a Hilbert space $\mathcal{H}$ and $\{\mathcal{M}_n\}$ is a sequence of closed subspaces, we say that $\{\mathcal{M}_n\}$* tends to $\mathcal{M}$ *and write*

$$\mathcal{M}_n \to \mathcal{M} \quad as \quad n \to \infty$$

if

$$\mathcal{M} = \{f \in \mathcal{H} \mid \lim_{n \to \infty} \operatorname{dist}(f, \mathcal{M}_n) = 0\}.$$

Definition 1.3. *We say that a subspace $\mathcal{M}$ of L^2 is a* monomial space *if there exists a sequence $\{\mathcal{M}_n\}$ of finite monomial spaces such that $\mathcal{M}_n \to \mathcal{M}$.*

Equipped with these definitions, we can now state our main theorem.

Theorem 1.4. *Let $\mathcal{M}$ be a closed non-zero subspace of L^2. Then $\mathcal{M}$ is invariant for H if and only if $\mathcal{M}$ is a monomial space.*

One way to construct a monomial space is to take the closed linear span of an infinite set of monomial functions,

$$\mathcal{M} \;=\; \vee\{x^{s_k} : k \in \mathbb{N}\}. \tag{1.5}$$

The Müntz-Szász theorem (proved in [?, ?] for integer exponents, and in [?] for general real exponents) characterizes when such a space is a proper subspace of L^2. See [?] for a thorough treatment.

Theorem 1.6. *(Müntz-Szász)*

$$\vee\{x^{s_k} : k \in \mathbb{N}\} \;=\; L^2 \quad \textit{if and only if} \quad \sum_k \frac{2\mathrm{Re}\ s_k + 1}{|s_k + 1|^2} = \infty.$$

Not every monomial space looks like (??). It is easy to see that for any $0 < s < 1$, the space $\{f \in L^2 : f = 0 \text{ a.e. on } [0,s]\}$ is invariant for H, and hence is a monomial space. (For an explicit construction of finite monomial spaces that converge to this subspace, see [?].)

Our goal is to give a real analysis proof of Theorem ??. To do this, we first need some preliminary results. In Section ?? we state two theorems about Hilbert spaces that we will use. The first, due to von Neumann in 1929, describes isometries on a Hilbert space. The second, due to Quiggin in 1993, gives a sufficient condition to extend partially defined multipliers of a reproducing kernel Hilbert space without increasing the norm. We apply Quiggin's theorem to the commutant of the Hardy operator in Section ??.

In Section ?? we describe the Laguerre basis for L^2, the basis obtained by evaluating the Laguerre polynomials on $\log \frac{1}{x}$, which are also the functions obtained by applying $(1 - H^*)^n$ to the constant function 1. In section ?? we deal with multiplicity; this corresponds to generalizing the notion of finite monomial space to allow not just monomials x^s, but also functions of the form $(\log x)^m x^s$. In Section ?? we prove that certain rational functions are cyclic for H^*. Finally in Section ?? we prove Theorem ??. Our strategy to prove that an invariant subspace $\mathcal{M}$ of H is a monomial space is to look at the projection η of the constant function 1 onto $\mathcal{M}^\perp$, and show that the function η uniquely characterizes $\mathcal{M}$. We then approximate η by functions that arise in a similar way from finite monomial spaces, and show that this proves that the finite monomial spaces coverge to $\mathcal{M}$ in the sense of Definition ??.

2 Some results from operator theory

An operator V defined on a Hilbert space $\mathcal{H}$ is called an isometry if it preserves norms; a co-isometry is the adjoint of an isometry. An isometry V is called pure if $\cap_{n=0}^\infty \mathrm{ran}(V^*)^n = 0$. The von Neumann-Wold decomposition describes the structure of isometeries [?, ?]. We state it not in its most general form, but in a way that will be useful below.

Theorem 2.1. *(von Neumann - Wold)*

(i) Every isometry is the direct sum of a unitary operator and a pure isometry.

(ii) If V is a pure isometry on the space $\mathcal{H}$, and $\mathcal{M} = \ker V^*$, then $\mathcal{H} = \vee\{V^j m : m \in \mathcal{M}\}$. The dimension of $\mathcal{M}$ is called the multiplicity of V.

(iii) The spaces $V^j\mathcal{M}$ form an orthogonal decomposition of $\mathcal{H}$.

(iv) If V is a pure isometry of multiplicity 1 and f is any non-zero vector in $\mathcal{H}$ then

$$\mathcal{H} \;=\; \vee\{V^i f, (V^*)^j f : i, j \geq 0\}.$$

We shall also need a result on extending the adjoints of multiplication operators, due to Quiggin [?]. We say that a sesquilinear form $\ell(x,y)$ has one positive square if for any finite set of points $\{x_1, \ldots, x_N\}$, the self-adjoint N-by-N matrix $\ell(x_i, x_j)$ has one positive eigenvalue.

Theorem 2.2. *(Quiggin): Let $(\mathcal{H}, k)$ be a reproducing kernel Hilbert space on a set X. A sufficient condition that every bounded operator T defined on $\vee\{k_x : x \in X_0\}$ for some subset $X_0 \subseteq X$ that has the form*

$$Tk_x \;=\; \alpha(x) k_x, \qquad x \in X_0$$

extend to a bounded operator $\widetilde{T} : \mathcal{H} \to \mathcal{H}$ that has the form

$$\widetilde{T} k_x \;=\; \widetilde{\alpha}(x) k_x, \qquad x \in X_0$$

and satisfies $\|\widetilde{T}\| = \|T\|$ is that the reciprocal $\frac{1}{k(x,y)}$ has exactly one positive square.

In the form stated, the converse to Quiggin's theorem is not true. However, if one requires norm-preserving extensions in the vector-valued case too, then the condition that $\frac{1}{k(x,y)}$ has one positive square is both necessary and sufficient. This was proved by McCullough [**?**] in a different context, and put in a unified context in [**?**]. See also the paper by Knese [**?**] for an elegant proof of necessity, and [**?**] for a discussion in a book.

3 The Laguerre basis for L^2

The following identity is a special case of one in [**?**]. In our case, it is easily proved by checking on polynomials; see e.g. [**?**].

Lemma 3.1. *Let $f \in L^2$. Then*

$$\|f\|^2 = \|(1-H)f\|^2 + |\int_0^1 f(x)dx|^2$$

Consequently, $1 - H$ is a co-isometry with a one dimensional kernel, which consists of the constant functions. As

$$(1-H)^k x^n \;=\; \left(\frac{n}{n+1}\right)^k x^n,$$

we see that $1 - H^*$ is a pure isometry of multiplicity 1. Let us state this for future use.

Proposition 3.2. *(Brown, Halmos, Shields) The operator $(1 - H^*)$ is a pure isometry of multiplicity one.*

Proposition **??** was first proved in [**?**]. If we apply powers of $(1 - H^*)$ to the constant function 1, we get a useful orthonormal basis. This was first found explicitly in [**?**], and developed further in [**?**].

Lemma 3.3.

$$(H^*)^j \; 1 = (-1)^j \frac{(\log x)^j}{j!} \tag{3.4}$$

Proof. We proceed by induction. Clearly, (**??**) holds when $j = 0$. Assume $j \geq 0$ and (**??**)

holds. Then

$$
\begin{aligned}
(H^*)^{j+1}\ 1 &= H^*((H^*)^j\ 1) \\
&= \frac{(-1)^j}{j!} H^*(\log x)^j \\
&= \frac{(-1)^j}{j!} \int_x^1 \frac{(\log t)^j}{t} dt \\
&= \frac{(-1)^j}{j!} \int_{\log x}^0 u^j du \\
&= (-1)^{j+1} \frac{(\log x)^{j+1}}{(j+1)!}
\end{aligned}
$$

□

Lemma 3.5.

$$
(1 - H^*)^n\ 1 \ = \ \sum_{j=0}^{n} \binom{n}{j} \frac{(\log x)^j}{j!}
$$

Proof. By Lemma **??**,

$$
\begin{aligned}
(1 - H^*)^n\ 1 &= \sum_{j=0}^{n} (-1)^j \binom{n}{j}\ (H^*)^j\ 1 \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} ((-1)^j \frac{(\log x)^j}{j!}) \\
&= \sum_{j=0}^{n} \binom{n}{j} \frac{(\log x)^j}{j!}.
\end{aligned}
$$

□

From Lemma **??** and part (iii) of the von Neumann-Wold theorem, it follows that the functions

$$
e_n(x) \ = \ \sum_{j=0}^{n} \binom{n}{j} \frac{(\log x)^j}{j!} \tag{3.6}
$$

are orthonormal. To see that they are complete, note that their closed linear span $\mathcal{M}$ is invariant under H^* and contains the function 1. Since the constant functions are the kernel of the pure co-isometry $(1 - H)$, this means $\mathcal{M} = L^2$ by the von Neumann-Wold Theorem **??**. So we have proved the following result, which was first proved in [**?**] and [**?**].

Theorem 3.7. *(Brown, Halmos, Shields) The functions e_n defined by (**??**) form an orthonormal basis for L^2.*

The Laguerre polynomials are the polynomials

$$p_n(x) \;=\; \sum_{j=0}^{n}(-1)^j \binom{n}{j}\frac{(x)^j}{j!}.$$

These are orthogonal polynomials for $L^2[0,\infty)$ with the weight function e^{-x}. As $e_n(x) = p_n(\log \frac{1}{x})$, the change of variables $t = \log \frac{1}{x}$ is an alternative way to prove that e_n are orthonormal.

The functions e_n are generalized eigenvectors of H at 1. Later we shall need the following.

Proposition 3.8. *Let $s \in \mathbb{S}$. The $(n+1)^{\text{st}}$ generalized eigenvector of H with eigenvalue $\frac{1}{s+1}$ is in the linear span of $\{x^s, (\log x)x^s, \dots (\log x)^n x^s\}$.*

Proof: We want to prove

$$\operatorname{Ker}(H - \frac{1}{s+1})^{n+1} \;=\; \vee\{x^s, (\log x)x^s, \dots (\log x)^n x^s\}. \tag{3.9}$$

This is true when $n = 0$, since

$$Hx^s \;=\; \frac{1}{s+1}x^s. \tag{3.10}$$

Differentiate both sides of (**??**) with respect to s. We get

$$H(\log x)x^s \;=\; \frac{1}{s+1}(\log x)x^s - \frac{1}{(s+1)^2}x^s. \tag{3.11}$$

Now we proceed by induction. The inductive hypothesis is that

$$H(\log x)^n x^s \;=\; \frac{1}{s+1}(\log x)^n x^s + \sum_{j=0}^{n-1} c_j(s)(\log x)^j x^s \tag{3.12}$$

for some functions c_j. We have proved (**??**) for $n = 0$ and 1. (The $n = 1$ case we proved just for expositional clarity). Assume the hypothesis holds up to n. Differentiate (**??**) with respect to s and we get

$$H(\log x)^{n+1}x^s \;=\; \frac{1}{s+1}(\log x)^{n+1}x^s - \frac{1}{(s+1)^2}(\log x)^n x^s + \sum_{j=0}^{n-1} c_j'(s)(\log x)^j x^s + c_j(s)(\log x)^{j+1}x^s.$$

Thus by induction, (**??**) holds for all n, and hence so does (**??**). $\square$

4 Commutant Lifting for the Hardy operator

Suppose $T : L^2 \to L^2$ commutes with H. Then it must have the same eigenvectors, and so be a monomial operator of the form

$$T : x^s \;\mapsto\; \alpha(s) x^s. \tag{4.1}$$

When is such an operator bounded?

Theorem 4.2. *The operator T commutes with H and has norm at most M if and only if T is of the form (**??**) and, for any finite set $\{s_i\}_{i=1}^N \subset \mathbb{S}$, the matrix*

$$\left(\frac{M^2 - \overline{\alpha(s_i)}\alpha(s_j)}{1 + \overline{s_i} + s_j} \right)_{i,j=1}^{N} \tag{4.3}$$

is positive semidefinite.

T may be defined by (**??**) just on some subspace of L^2. The positivity of (**??**) on this set is necessary and sufficient to lift T from the span of $\{x^{s_i}\}$ to an operator on all of L^2 that commutes with T and has the same norm. Without loss of generality we can take $M = 1$.

Theorem 4.4. *Suppose that for some subset $\mathbb{S}_0 \subseteq \mathbb{S}$ there is an operator*

$$\begin{aligned} T : \vee\{x^s : s \in \mathbb{S}_0\} \quad &\to \quad \vee\{x^s : s \in \mathbb{S}_0\} \\ T : x^s \quad &\mapsto \quad \alpha(s) x^s. \end{aligned}$$

A necessary and sufficient condition for T to extend to an operator from L^2 to L^2 that commutes with H and has norm at most one is that for every finite set $\{s_i\} \subseteq \mathbb{S}_0$, we have

$$\left(\frac{1 - \overline{\alpha(s_i)}\alpha(s_j)}{1 + \overline{s_i} + s_j} \right) \;\geq\; 0.$$

Notice that Theorem **??** is a special case of Theorem **??**, so we shall just prove the latter theorem.

PROOF: *(of Theorem **??**.)* Necessity: We have that $1 - T^*T \geq 0$. Therefore

$$\langle (1 - T^*T) x^{s_j}, x^{s_i} \rangle \;=\; \left(\frac{1 - \overline{\alpha(s_i)}\alpha(s_j)}{1 + \overline{s_i} + s_j} \right) \tag{4.5}$$

is a positive semi-definite matrix for any subset of $\mathbb{S}_0$.

Sufficiency: Suppose that (**??**) is positive semi-definite for every finite subset of $\mathbb{S}_0$. Then T commutes with $H|_{\vee\{x^s : s \in \mathbb{S}_0\}}$. Let us define a kernel on $\mathbb{S}$ by

$$\begin{aligned} k(s,t) \quad &= \quad \int_0^1 x^t \overline{x^s} dx \\ &= \quad \frac{1}{1 + t + \overline{s}}. \end{aligned}$$

The reciprocal of k is the sesquilinear form

$$\begin{aligned}\ell(s,t) &= (\frac{1}{2}+t)+\overline{(\frac{1}{2}+s)} \\ &= \frac{1}{2}(\frac{3}{2}+\bar{s})(\frac{3}{2}+t)-\frac{1}{2}(\frac{1}{2}-\bar{s})(\frac{1}{2}-t).\end{aligned}$$

So for any $N \geq 2$ the matrix $[\ell(s_i,s_j)]_{i,j=1}^N$ is a rank 2 symmetric matrix, with one positive and one negative eigenvalue. By Theorem **??**, T extends to an operator of norm 1 on all of L^2 that has each x^s as an eigenvector, and hence commutes with H. □

5 Monomial spaces with multiplicity

If one takes the two dimensional monomial spaces $\mathcal{M}(s,s+h)$ and lets $h \to 0$, the spaces converge to the two-dimensional space spanned by x^s and $\frac{\partial}{\partial s}x^s = (\log x)x^s$. So if we have a multi-set $S = \{s_1,\ldots,s_1,s_2,\ldots,s_2,\ldots,s_n\}$, where each s_j appears m_j times, we will define

$$\mathcal{M}(S) = \vee\{x^{s_1},(\log x)x^{s_1},\ldots,(\log x)^{m_1-1}x^{s_1},\ldots,x^{s_n},(\log x)x^{s_n},\ldots,(\log x)^{m_n-1}x^{s_n}\}. \tag{5.1}$$

We shall call a set of the form (**??**) a generalized finite monomial space.

Proposition 5.2. *Every generalized finite monomial space is a limit of finite monomial spaces.*

PROOF: Fix $m \geq 2$. Let

$$\mathcal{M}_1 = \vee\{x^s,(\log x)x^s,\ldots,(\log x)^{m-1}x^s\}.$$

Let ω be a primitive m^{th} root of unity, and let h be a small positive number. Let

$$\mathcal{M}_2 = \vee\{x^{s+\omega^j h} : 0 \leq j \leq m-1\}.$$

We shall prove that there is a constant C, which depends on s and m but not h, so that

$$f \in \mathcal{M}_1 \quad \Rightarrow \quad \operatorname{dist}(f,\mathcal{M}_2) \leq C\|f\|h^m \tag{5.3}$$
$$f \in \mathcal{M}_2 \quad \Rightarrow \quad \operatorname{dist}(f,\mathcal{M}_1) \leq C\|f\|h. \tag{5.4}$$

As every generalized monomial space of the form (**??**) is the sum of finitely many spaces of the form $\mathcal{M}_1$, this will prove the proposition.

In the proof we shall use C for a constant that depends on m but not h, and which may change from one line to the next.

Proof of (**??**). (i) First take $s=0$. By Taylor's theorem, for any unimodular number τ and any $x > 0$ we have

$$|x^{\tau h} - \sum_{n=0}^{m-1}\frac{(\tau h)^n}{n!}(\log x)^{n-1}| \leq \frac{h^m}{m!}(\log x)^m x^{-h}. \tag{5.5}$$

Consider the function $f(x) = (\log x)^n$, for some $n \leq m-1$. We shall approximate this by the function $g \in \mathcal{M}_2$ given by

$$g(x) \;=\; \frac{1}{m}\frac{n!}{h^n}\sum_{j=0}^{m-1} \bar{\omega}^{nj} x^{\omega^j h}.$$

The choice of arguments for the coefficients means that if one adds together the Taylor series for each $x^{\omega^j h}$, all the terms cancel except for the ones that are n mod m one, so

$$|g(x) - (\log x)^n| \;\leq\; Ch^m (\log x)^{m+n} x^{-h} \tag{5.6}$$

where C is independent of x. Integrating the square of (**??**) we get that $\operatorname{dist}((\log x)^n, \mathcal{M}_2) \leq Ch^m$. As the functions $(\log x)^n$ form a basis for $\mathcal{M}_1$, we deduce that (**??**) holds.

(ii) For general s, the above argument shows that for each function $x^s(\log x)^n$ there is a function g in $\mathcal{M}_2$ that satisfies the pointwise estimate

$$|g(x) - x^s(\log x)^n| \;\leq\; Ch^m(\log x)^{m+n}x^{\mathrm{Re}\ s-h}.$$

As long as h is small enough that Re $s - h > -\frac{1}{2}$, we again can deduce (**??**).

Proof of (**??**). (i) First take $s = 0$. From (**??**), we get that $\operatorname{dist}(x^{\omega^j h}, \mathcal{M}_1) \leq Ch^m$. So the result will follow if we prove that whenever $\sum c_j x^{\omega^j h}$ is in the unit ball of $\mathcal{M}_2$, then $c_j = O(\frac{1}{h^{m-1}})$. This in turn will follow if we can show that

$$\operatorname{dist}(x^{\omega^\ell h}, \vee\{x^{\omega^i h} : 0 \leq i \leq m-1,\ i \neq \ell\}) \;\geq\; Ch^{m-1} \tag{5.7}$$

for some non-zero C, as this proves that the functions $x^{\omega^i h}$ are not too colinear. For definiteness, we will prove (**??**) for $\ell = 0$. Let $G(i,j)$ denote the Gram matrix with (i,j) entry $\langle x^{\omega^i h}, x^{\omega^j h}\rangle = \frac{1}{1+\omega^i h + \bar{\omega}^j h}$. Then

$$\operatorname{dist}(x^h, \vee_{1\leq i\leq m-1}\{x^{\omega^i h}\})^2 \;=\; \det G(i,j)_{i,j=0}^{m-1} / \det G(i,j)_{i,j=1}^{m-1}. \tag{5.8}$$

By Cauchy's formula for determinants

$$\det\left(\frac{1}{1+\omega^i h + \bar{\omega}^j h}\right) \;=\; \frac{\prod_{j<i} |\omega^i h - \bar{\omega}^j h|^2}{\prod_{i,j}(1+\omega^i h + \bar{\omega}^j h)}.$$

Putting this into (**??**), we get

$$\operatorname{dist}(x^h, \vee_{1\leq i\leq m-1}\{x^{\omega^i h}\})^2 \;=\; \frac{h^{2m-2}\prod_{i=1}^{m-1}|\omega^i - 1|^2}{(1+2h)\prod_{i=1}^{m-1}|1+(1+\omega^i)h|^2}.$$

This equation yields (**??**) for $\ell = 0$, and by symmetry for all ℓ.

(ii) For general $s \in \mathbb{S}$, a similar argument gives $\operatorname{dist}(x^{s+\omega^\ell h}, \mathcal{M}_1) \leq Ch^m$, and

$$\operatorname{dist}(x^{s+h}, \vee_{1\leq i\leq m-1}\{x^{s+\omega^i h}\})^2 \;=\; \frac{h^{2m-2}\prod_{i=1}^{m-1}|\omega^i - 1|^2}{(1+2\mathrm{Re}\ s+2h)\prod_{i=1}^{m-1}|1+2\mathrm{Re}\ s+(1+\omega^i)h|^2}.$$

$\square$

With more work, one can improve (**??**) to $O(h^m)$, but we do not need a sharper estimate.

Corollary 5.9. *Any space that is a limit of generalized finite monomial spaces is a monomial space.*

6 Some cyclic vectors for H^*

We know from Proposition **??** that the spectrum of H is $\overline{\mathbb{D}(1,1)}$, and for $\lambda \in \mathbb{D}(1,1)$ that $H - \lambda$ is Fredholm with index 1. It follows that $1 + sH$ and $1 + sH^*$ are invertible if and only if $s \in \mathbb{S}$.

Lemma 6.1. *If $s \in \mathbb{S}$, then*
$$x^s \ = \ (1 + sH^*)^{-1}1.$$

PROOF: We have

$$\begin{aligned} \langle (1 + sH^*)x^s, x^t \rangle \quad &= \quad \langle x^s, (1 + \bar{s}\frac{1}{t+1}x^t \rangle \\ &= \quad \frac{1}{\bar{t}+1} \\ &= \quad \langle 1, x^t \rangle. \end{aligned}$$

□

Lemma 6.2. *Suppose $f(x) = \sum_{j=0}^{N} c_j x^{s_j}$, where each $s_j \in \mathbb{S}$. If f is not orthogonal to any monomial x^t for $t \in \mathbb{S}$, then f is cyclic for H^*.*

PROOF: By Lemma **??**, we have

$$f(x) \ = \ \sum_{j=0}^{N} c_j (1 + s_j H^*)^{-1} 1.$$

Define a rational function $r(z)$ by

$$r(z) \ = \ \sum_{j=0}^{N} c_j \frac{1}{1 + s_j z},$$

and let p, q be polynomials with no common factors and $r = p/q$. The zeroes of q are at the points $\{-\frac{1}{s_j} : 1 \leq j \leq N\}$. We have

$$f \ = \ p(H^*)q(H^*)^{-1}1. \tag{6.3}$$

Claim: p has no roots in $\mathbb{D}(1,1)$.

Indeed, suppose $p(z_0) = 0$ for some $z_0 \in \mathbb{D}(1,1)$. Let $t_0 = \frac{1-\bar{z}_0}{\bar{z}_0} \in \mathbb{S}$. Factor p as $p(z) = (z - z_0)\tilde{p}(z)$. Then

$$\begin{aligned} \langle f, x^{t_0} \rangle \quad &= \quad \langle (H^* - z_0)\tilde{p}(H^*)q(H^*)^{-1}1, x^{t_0} \rangle \\ &= \quad \langle \tilde{p}(H^*)q(H^*)^{-1}1, (\frac{1}{t_0+1} - \bar{z}_0)x^{t_0} \rangle \\ &= \quad 0. \end{aligned}$$

This would contradict the assumption that $\langle f, x^t \rangle \neq 0$ for all $t \in \mathbb{S}$.

Since $q(H^*)$ is invertible, f is cyclic if and only if $p(H^*)1$ is cyclic. We now factor $p(z) = c\prod(z - z_j)$. If $z_j \notin \overline{\mathbb{D}(1,1)}$, then $(H^* - z_j)$ is invertible. If $z_j \in \partial\mathbb{D}(1,1)$, then $(H^* - z_j)$ has dense range, since H has no eigenvectors on $\partial\mathbb{D}(1,1)$. Therefore $p(H^*)$ has dense range, and in particular takes cyclic vectors to cyclic vectors. □

If $\langle f, x^t\rangle = 0$, then f is in the range of $H^* - \frac{1}{1+\bar{t}}$ (which is closed since the operator is Fredholm). We shall say that $\langle f, x^t\rangle$ vanishes to order m if f is orthogonal to $\{x^t, (\log x)x^t, \ldots, (\log x)^{m-1}x^t\}$.

Lemma 6.4. *Suppose $f(x) = \sum_{j=0}^N c_j x^{s_j}$, where each $s_j \in \mathbb{S}$, and $f \neq 0$. Let*

$$\mathcal{T} \ := \ \{t \in \mathbb{S} : \langle f, x^t\rangle = 0\}.$$

Then $\mathcal{T}$ is finite, and we write it as

$$\mathcal{T} \ = \ \{t_1, \ldots, t_m\},$$

counted with multiplicity. Let $z_i = \frac{1}{1+\bar{t}_i}$ for $1 \leq i \leq m$. Then

$$f \ = \ \prod_{i=1}^m (H^* - z_i) g, \tag{6.5}$$

where g is cyclic for H^*.

Proof: Write $f = p(H^*)q(H^*)^{-1}1$ as in (**??**). Let $p^\cup(z) := \overline{p(\bar{z})}$. Then

$$\begin{aligned}
\langle f, x^t\rangle &= \langle p(H^*)q(H^*)^{-1}1, x^t\rangle \\
&= \langle 1, p^\cup(H)q^\cup(H)^{-1}1\rangle \\
&= \langle 1, \frac{p^\cup(\frac{1}{t+1})}{q^\cup(\frac{1}{t+1})} x^t\rangle \\
&= \frac{p(\frac{1}{\bar{t}+1})}{q(\frac{1}{\bar{t}+1})}\langle 1, x^t\rangle.
\end{aligned}$$

So $t \in \mathcal{T}$ if and only if $\frac{1}{\bar{t}+1}$ is a root of p in $\mathbb{D}(1,1)$, proving that $\mathcal{T}$ is finite, and that the roots of p that lie in $\mathbb{D}(1,1)$ are exactly the points $\{z_i : 1 \leq i \leq m\}$. (Multiplicity is handled by Proposition **??**). Factor p as $p(z) = \prod_{i=1}^m (z - z_i)\tilde{p}(z)$, where $\tilde{p}$ has no roots in $\mathbb{D}(1,1)$. Let $g = \tilde{p}(H^*)q(H^*)^{-1}1$. Then g is cyclic, and (**??**) holds. □

Later we will need the next lemma.

Lemma 6.6. *Let $z \in \mathbb{D}(1,1)$. Then*

$$(H^* - z)[(\bar{z} - 1)H^* - \bar{z}]^{-1}$$

is an isometry.

Proof: This follows by calculation, using the fact that $1 - H^*$ is isometric. □

7 Proof of Theorem ??

Sufficiency is obvious. For necessity, let $\mathcal{M}$ be a proper closed subspace of L^2 that is invariant for H. We must show that it is a monomial space.

Lemma 7.1. *Let $\mathcal{M}$ be a finite dimensional subspace of L^2, of dimension $n+1$, that is invariant for H. Then $\mathcal{M}$ is a generalized finite monomial space, i.e. there exist $n+1$ points $s_0, \ldots, s_n$, with multiplicity allowed, so that $\mathcal{M} = \vee\{x^{s_i} : 0 \leq i \leq n\}$.*

PROOF: Consider $H|_{\mathcal{M}}$, which leaves $\mathcal{M}$ invariant. The space $\mathcal{M}$ is spanned by the eigenvectors and generalized eigenvectors of H that lie in $\mathcal{M}$. Suppose the corresponding eigenvalues are s_j, with multiplicity m_j. By Proposition **??**, the generalized eigenvectors are of the form $x^{s_j}, (\log x)x^{s_j}, \ldots, (\log x)^{m_j-1}x^{s_j}$. Therefore $\mathcal{M}$ is the generalized finite monomial space corresponding to the exponents s_j with multiplicity m_j. $\square$

To prove the full theorem, we use the idea of wandering subspace, due to Halmos [**?**][1]. Let

$$k_0 \ := \ \min\{k : e_k \notin \mathcal{M}\}.$$

Write $\mathcal{N}$ for $\mathcal{M}^\perp$. Write $e_{k_0} = \xi + \eta$, where $\xi \in \mathcal{M}$ and $\eta \in \mathcal{N}$. The assumption that $e_{k_0} \notin \mathcal{M}$ means $\eta \neq 0$. Let $u = \frac{\eta}{\|\eta\|}$.

Lemma 7.2. *We have $u \perp (1 - H^*)\mathcal{N}$.*

PROOF: Let $f \in \mathcal{N}$. Then

$$\begin{aligned} \langle u, (1-H^*)f \rangle &= \langle \|\eta\| \, e_{k_0}, (1-H^*)f \rangle \\ &= \|\eta\| \langle (1-H)e_{k_0}, f \rangle. \end{aligned}$$

If $k_0 = 0$, then $(1-H)e_{k_0} = 0$. If $k_0 > 0$, then $(1-H)e_{k_0} = e_{k_0 - 1} \in \mathcal{M}$. Either way, the inner product with f is 0. $\square$

Define an operator $R : L^2 \to L^2$ in terms of the orthonormal basis e_n from (**??**) by

$$R : e_n \ \mapsto \ (1-H^*)^n u. \tag{7.3}$$

Lemma 7.4. *The operator R defined by (**??**) is an isometry from L^2 onto $\mathcal{N}$.*

PROOF: The functions $\{(1-H^*)^n u : n \geq 0\}$ form an orthonormal set. Indeed, by Proposition **??** and Lemma **??**, if $m \geq n$ then

$$\begin{aligned} \langle (1-H^*)^m u, (1-H^*)^n u \rangle &= \langle (1-H^*)^{m-n} u, u \rangle \\ &= \delta_{m,n}. \end{aligned}$$

As R maps an orthonormal basis to an orthonormal set, it must be an isometry onto its range.

[1] A wandering subspace for an operator T on $\mathcal{H}$ is a space $\mathcal{M} \subset \mathcal{H}$ such that $\{T^j\mathcal{M} : j \geq 0\}$ are orthogonal and span $\mathcal{H}$. In particular, by the von Neumann Wold theorem, the kernel of the adjoint forms a wandering subspace for any isometry.

We know that the range of R is contained in $\mathcal{N}$. To see that it is all of $\mathcal{N}$, observe that by Lemma **??**, we have that

$$\vee\{(1-H)^m u : m \geq 1\}$$

is contained in $\mathcal{N}^\perp = \mathcal{M}$. As $1-H$ is a pure isometry of multiplicity 1, by Theorem **??** for any non-zero vector f the vectors

$$\{(1-H)^m f, (1-H^*)^n f : m, n \geq 0\}$$

span L^2. Therefore in particular, $\vee\{(H^*)^n u : n \geq 0\}$ and $\vee\{H^m u : m \geq 1\}$ span L^2, so

$$\begin{aligned} \mathcal{N} &= \vee\{(H^*)^n u : n \geq 0\} \\ \mathcal{M} &= \vee\{H^m u : m \geq 1\}. \end{aligned}$$

□

Let us calculate $T = R^*$, the adjoint of R.

Lemma 7.5. *The adjoint of R is given by the operator*

$$T : x^s \mapsto (1+s)\langle x^s, u\rangle x^s. \tag{7.6}$$

Proof: We have

$$\begin{aligned} \langle R^* x^s, e_n\rangle &= \langle x^s, (1-H^*)^n u\rangle \\ &= \langle (1-H)^n x^s, u\rangle \\ &= \left(\frac{s}{s+1}\right)^n \langle x^s, u\rangle. \end{aligned}$$

We also have by Lemma **??**

$$\begin{aligned} \langle T x^s, e_n\rangle &= (1+s)\langle x^s, u\rangle\langle x^s, (1-H^*)^n 1\rangle \\ &= (1+s)\langle x^s, u\rangle\langle (1-H)^n x^s, 1\rangle \\ &= \left(\frac{s}{s+1}\right)^n \langle x^s, u\rangle. \end{aligned}$$

Therefore $T = R^*$. □

We want to approximate $\mathcal{M}$ by monomial spaces. We shall do this by approximating u by linear combinations of monomials. Since R is an isometry, T is a co-isometry, and since each eigenvector of H is an eigenvector of T, it follows that T commutes with H. This means by Theorem **??** that for each N, the matrix

$$\left(\frac{1-(i+1)(j+1)\langle u, x^i\rangle\langle x^j, u\rangle}{1+i+j}\right)_{i,j=0}^N \geq 0.$$

We shall assume for the remainder of this section that N is large enough that $\langle u, x^i\rangle \neq 0$ for some $i \leq N$. Let $C_N \geq 1$ be the largest number C so that

$$\left(\frac{1-C^2(i+1)(j+1)\langle u, x^i\rangle\langle x^j, u\rangle}{1+i+j}\right)_{i,j=0}^N \geq 0.$$

The hypothesis on N means C_N is finite, and $\lim_{N\to\infty} C_N = 1$. Define $\tilde{T}_N$ by

$$\tilde{T}_N : x^i \ \mapsto \ C_N(i+1)\langle x^i, u\rangle x^i, \qquad 0 \leq i \leq N.$$

By Theorem **??**, this extends to an operator T_N that maps L^2 to L^2, commutes with H, and has norm equal to 1. So T_N is of the form

$$T_N : x^s \ \mapsto \ \alpha_N(s) x^s. \tag{7.9}$$

Lemma 7.10. *The function $\alpha_N(s)$ is a rational function of degree at most N, and maps $\mathbb{S}$ to $\mathbb{D}$.*

Proof: We know that

$$\alpha_N(i) = C_N(i+1)\langle u, x^i\rangle, \qquad 0 \leq i \leq N, \tag{7.11}$$

Let γ be a non-zero vector in the kernel of

$$\left(\frac{1 - C_N^2(i+1)(j+1)\langle u, x^i\rangle\langle x^j, u\rangle}{1+i+j} \right)_{i,j=0}^{N} \ \geq \ 0.$$

By Theorem **??**, the matrix (**??**) has to be positive semidefinite when we augment the set $\{0, \dots, N\}$ by any other point s. This means by Lemma **??** that the first $N+1$ entries in the last column of the extended $(N+2)$-by-$(N+2)$ matrix must be orthogonal to γ, so

$$\sum_{i=0}^{N} \frac{1 - \overline{\alpha_N(i)}\alpha_N(s)}{1+i+s}\gamma_i \ = \ 0.$$

This equation yields

$$\left(\sum_{i=0}^{N} \frac{\overline{\alpha_N(i)}\gamma_i}{1+i+s} \right) \alpha_N(s) \ = \ \sum_{i=0}^{N} \frac{\gamma_i}{1+i+s} \tag{7.13}$$

Let $R(s)$ denote the right-hand side of (**??**), and $L(s)$ denote the coefficient of $\alpha_N(s)$ on the left. Both R and L are rational functions, vanishing at infinity, with simple poles exactly in the set

$$\{-1-i : \gamma_i \neq 0\}.$$

Their ratio $\alpha_N = R/L$, therefore, is a rational function with poles at the zero set of L, and zeroes on the zero set of R. The degree will be at most N, since they both have zeroes at infinity.

As $\|Tx^s\| = |\alpha_N(s)|\|x^s\| \leq \|x^s\|$, we have $\alpha_N : \mathbb{S} \to \mathbb{D}$. □

We used the following lemma, whose proof is elementary linear algebra.

Lemma 7.14. *Suppose A is a positive semi-definite matrix, and γ is a non-zero vector in the kernel of A. If there is a vector β and a constant c so that*

$$\begin{pmatrix} A & \beta \\ \beta^* & c \end{pmatrix} \ \geq \ 0,$$

then $\langle \beta, \gamma\rangle = 0$.

Lemma 7.15. *Let α be a rational function of degree N with all its poles in the set $\{s : \text{Re}\,(s) < -\frac{1}{2}\}$, and with no pole at ∞.*

(i) If $\alpha(-1) \neq 0$, then there exists a sequence $\{s_0, \ldots, s_N\}$, with multiplicity allowed, and a function u_N in the generalized finite monomial space $\mathcal{M}(\{s_0, s_1, \ldots, s_N\})$, so that

$$\alpha(s) \;=\; (1+s)\langle x^s, u_N\rangle \qquad \forall s \in \mathbb{S}. \tag{7.16}$$

Moreover we can take $s_0 = 0$.

(ii) If $\alpha(-1) = 0$, then there exists a sequence $\{s_1, \ldots, s_N\}$, with multiplicity allowed, and a function u_N in the generalized finite monomial space $\mathcal{M}(\{s_1, \ldots, s_N\})$, so that

$$\alpha(s) \;=\; (1+s)\langle x^s, u_N\rangle \qquad \forall s \in \mathbb{S}. \tag{7.17}$$

PROOF: Expand $\alpha(s)/(s+1)$ by partial fractions to get

$$\frac{\alpha(s)}{s+1} \;=\; \sum_{j=1}^{p}\sum_{r=1}^{m_j} \frac{(r-1)!c_j^r}{(s-\lambda_j)^r}.$$

There is no constant term, since the left-hand side vanishes at ∞. We can assume that $c_j^{m_j} \neq 0$ for each j. In case (i), there is a pole, which we denote λ_1, at -1, and $\sum_{j=1}^{p} m_j = N+1$. In case (ii) there is no pole at -1, and $\sum_{j=1}^{p} m_j = N$.

The inverse Laplace transform of $\alpha(s)/(s+1)$ is

$$F(t) \;=\; \sum_{j=1}^{p}\sum_{r=1}^{m_j} c_j^r t^{r-1} e^{\lambda_j t}.$$

Define

$$\begin{aligned} u_N(x) &= \overline{\frac{1}{x}F(\log\frac{1}{x})} \\ &= \sum_{j=1}^{p}\sum_{r=1}^{m_j} \bar{c}_j^r (\log\frac{1}{x})^{r-1} x^{-\bar{\lambda}_j - 1}. \end{aligned} \tag{7.18}$$

Then, making the substitution $e^{-t} = x$, we get

$$\begin{aligned} \langle x^s, u_N\rangle &= \int_0^1 x^s \overline{u_N(x)}dx \\ &= \int_0^\infty e^{-st}F(t)dt \\ &= (\mathcal{L}F)(s) \\ &= \frac{\alpha(s)}{s+1}. \end{aligned}$$

Notice that each point $-1-\bar{\lambda}_j$ is in $\mathbb{S}$. We now define the multiset $\{s_0, s_1, \ldots, s_N\}$ (respectively, $\{s_1, \ldots, s_N\}$) by taking m_j copies of the point $-\bar{\lambda}_j - 1$ for each j. $\Box$

We shall prove in Lemma **??** that case (ii) cannot occur for α_N. In the next two lemmas we use the fact that if $\|A\| \leq 1$, then $h \in \ker(1 - AA^*)$ if and only if $\|A^*h\| = \|h\|$.

Lemma 7.18. *Let $\mathcal{K} = \ker(1 - T_N T_N^*)$. Then $\mathcal{K}$ is H^* invariant.*

Proof: As H commutes with T_N and $(1-H)(1-H^*) = 1$ by Lemma **??**, we have

$$T_N \;=\; (1-H)T_N(1-H^*).$$

So if $g \in \mathcal{K}$ then

$$\begin{aligned}\|g\|^2 &= \|T_N^* g\|^2 \\ &= \langle (1-H)T_N^*(1-H^*)g, T_N^* g\rangle.\end{aligned}$$

As $\|1-H^*\|$ and $\|T_N^*\|$ are both equal to 1, we have

$$\begin{aligned}\|T_N^*(1-H^*)g\| &= \|(1-H^*)g\| \\ &= \|g\|.\end{aligned}$$

Therefore $(1-H^*)g$ is also in $\mathcal{K}$, and hence $\mathcal{K}$ is H^* invariant. □

Lemma 7.19. *The operator T_N is a co-isometry.*

Proof: Let γ be as in the proof of Lemma **??**. Let $f(x) = \sum_{j=0}^{N} \gamma_j x^j$. Then $(1 - T_N^* T_N)f = 0$, so T_N attains its norm on f. Let

$$g \;=\; T_N f \;=\; \sum_{j=0}^{N} \gamma_j \alpha(j) x^j.$$

As $f = T_N^* T_N f$, we have $g = T_N T_N^* g$.

To prove T_N is a co-isometry, we must show that

$$\mathcal{K} \;=\; \ker(1 - T_N T_N^*)$$

is all of L^2. By Lemma **??**, we know that $\mathcal{K}$ is H^* invariant, and it contains the polynomial g. If g were not orthogonal to any x^t, then we would be done by Lemma **??**.

As $\langle x^t, g\rangle$ is a non-zero rational function of t, it can only have finitely many zeroes in $\mathbb{S}$; label these $\{t_1, \ldots, t_m\}$, counting multiplicity. By Lemma **??** we have

$$g \;=\; \prod_{i=1}^{m} (H^* - z_i) h_1,$$

where h_1 is cyclic for H^* and $z_i = \frac{1}{1+\bar{t}_i}$. Let

$$h_2 \;=\; \prod_{i=1}^{m} [(\bar{z}_i - 1)H^* - \bar{z}_i]\; h_1.$$

Then h_2 is cyclic since it is an invertible operator applied to h_1. Let

$$r(z) \;=\; \prod_{i=1}^{m} \frac{z - z_i}{(\bar{z}_i - 1)z - \bar{z}_i}.$$

By Lemma **??**, $r(H^*)$ is an isometry, and we have $r(H^*)h_2 = g$. Therefore

$$\begin{aligned}
\|h_2\| &= \|r(H^*)h_2\| \\
&= \|T_N T_N^* r(H^*)h_2\| \\
&\leq \|T_N^* r(H^*)h_2\| \\
&= \|r(H^*)T_N^* h_2\| \\
&= \|T_N^* h_2\| \\
&\leq \|h_2\|.
\end{aligned}$$

Therefore $h_2 \in \mathcal{K}$, and since $\mathcal{K}$ is H^* invariant and h_2 is cyclic, we get that $\mathcal{K}$ is all of L^2 and hence T_N^* is an isometry. □

Let $R_N = T_N^*$. A similar calculation to the proof of Lemma **??** yields:

Lemma 7.21. *The operator R_N maps e_n to $(1 - H^*)^n u_N$.*

We are now ready to define $\mathcal{M}_N$. Let α_N be as in (**??**). Apply Lemma **??** to α_N to get, in case (i), a space $\mathcal{M}(\{s_0, s_1, \ldots, s_N\})$ that contains u_N given by (**??**) and satisfies (**??**), and in case (ii) a space $\mathcal{M}(\{s_1, \ldots, s_N\})$ that contains u_N given by (**??**) and satisfies (**??**).

We show that Case (ii) of Lemma **??** cannot occur.

Lemma 7.22. *We have $\alpha_N(-1) \neq 0$.*

Proof: Let us assume that we are in Case (ii) of Lemma **??**. Let $t_j = -\bar{\lambda}_j - 1$. In the sequence $\{s_1, \ldots, s_N\}$ each t_j appears with multiplicity m_j, and no t_j is 0. We have

$$u_N = \sum_{j=1}^{p} \sum_{r=1}^{m_j} \bar{c}_j^r (-\log x)^{r-1} x^{t_j}.$$

Since R_N is isometric by Lemma **??**, we have u_N is orthogonal to $(1 - H^*)^k u_N$ for every $k \geq 1$, and hence u_N is also orthogonal to $(1 - H)^k u_N$ for every $k \geq 1$. For each $j \geq 1$, let p_j^k be a polynomial that vanishes at 0, vanishes at t_i to order m_i if $i \neq j$, and vanishes at t_j to order k. Since each such polynomial vanishes at zero, we have

$$\langle u_N, p_j^k(1 - H)u_N \rangle \;=\; 0. \tag{7.23}$$

Consider

$$p_j^{m_j}(1 - H)u_N \;=\; \bar{c}_j^{m_j} x^{t_j}.$$

By (**??**), we conlcude that $u_N \perp x^{t_j}$. Similarly $p_j^{m_j - 1}(1 - H)u_N$ equals $\bar{c}_j^{m_j}(\log x)x^{t_j}$ plus some multiple of x^{t_j}. Therefore we conclude that u_N is also orthogonal to $(\log x)x^{t_j}$. Continuing in this way, we conclude that u_N is orthogonal to every function in $\mathcal{M}(\{s_1, \ldots, s_N\})$. Since u_N itself is in this space, we conclude that $u_N = 0$, a contradiction. □

Let $\mathcal{M}_N = \mathcal{M}(\{s_1, \ldots, s_N\})$, in other words the space $\mathcal{M}(\{s_0, s_1, \ldots, s_N\})$ with the multiplicity at 0 reduced by 1. Here is the final step.

Lemma 7.24. *The sequence $\mathcal{M}_N$ tends to $\mathcal{M}$.*

Proof: Let $t_j = -\bar{\lambda}_j - 1$, with $t_1 = 0$. We have

$$u_N = \sum_{j=1}^{p} \sum_{r=1}^{m_j} \bar{c}_j^r (-\log x)^{r-1} x^{t_j}.$$

As in the proof of Lemma **??**, we conclude that u_N is orthogonal to $p(1-H)\mathcal{M}(\{s_0, s_1, \ldots, s_N\})$ for every polynomial p that vanishes at 0. So u_N is a constant multiple of the projection of e_{m_1-1} onto $\mathcal{M}_n^{\perp}$.

By Lemma **??**, T_N^* is an isometry from L^2 onto $\mathcal{M}_N^{\perp}$. Therefore the projection $P_{\mathcal{M}_N}$ onto $\mathcal{M}_N$ is given by $1 - T_N^* T_N$. We have

$$\begin{aligned} T_N : x^i &\mapsto (i+1)\langle x^i, u_N \rangle x^i, \qquad 0 \le i \le N \\ &= C_N (i+1)\langle x^i, u \rangle x^i, \qquad 0 \le i \le N. \end{aligned}$$

As $N \to \infty$, we have $u_N \to u$ weakly and so $T_N \to T$ in SOT. Therefore $P_{\mathcal{M}_N} \to P_{\mathcal{M}} = 1 - T^*T$ in WOT and hence also SOT (since a sequence of projections converges in the SOT if and only if it converges WOT).

If $f \in \mathcal{M}$, then $f_n = P_{\mathcal{M}_N} f$ is a sequence of functions tending to f, so $\mathcal{M}$ is contained in the limit of the subspaces. But as each $\mathcal{M}_N \subset \mathcal{M}$, the limit must be exactly $\mathcal{M}$. □

8 Open Question

Let $1 < p < \infty$, and $p \neq 2$. The Hardy operator is bounded on $L^p[0,1]$, and has x^s as an eigenvector whenever $s \in \mathbb{S}_p = \{s \in \mathbb{C} : \text{Re } (s) > -\frac{1}{p}\}$. Any space that is the limit of finite monomial spaces (with powers in $\mathbb{S}_p$) is therefore invariant for H. Is every closed subspace of $L^p[0,1]$ that is invariant for H of this form?

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