AN H^p SCALE FOR COMPLETE PICK SPACES

ALEXANDRU ALEMAN, MICHAEL HARTZ, JOHN E. McCARTHY,
AND STEFAN RICHTER

Abstract. We define by interpolation a scale analogous to the Hardy H^p scale for complete Pick spaces, and establish some of the basic properties of the resulting spaces, which we call $\mathcal{H}^p$. In particular, we obtain an $\mathcal{H}^p - \mathcal{H}^q$ duality and establish sharp pointwise estimates for functions in $\mathcal{H}^p$.

1. Introduction

Let $\mathcal{M}$ be a reproducing kernel Hilbert space on a set X, with kernel function k. Let $\mathrm{Mult}(\mathcal{M})$ denote the multiplier algebra of $\mathcal{M}$. We shall make the following assumption throughout our paper:

$$(A) \qquad \mathrm{Mult}(\mathcal{M}) \text{ is densely contained in } \mathcal{M}.$$

We shall let $\mathcal{M} \odot \mathcal{M}$ denote the weak product of $\mathcal{M}$ with itself, which is

$$\mathcal{M} \odot \mathcal{M} \ := \ \Big\{ \sum_{n=1}^{\infty} f_n g_n \ : \ \sum_n \|f_n\|_{\mathcal{M}} \|g_n\|_{\mathcal{M}} < \infty \Big\}. \tag{1.1}$$

This is a Banach space, where the norm of a function h is the infimum of $\sum_n \|f_n\|_{\mathcal{M}} \|g_n\|_{\mathcal{M}}$ over all representations of h as $\sum_n f_n g_n$.

If we use the complex method of interpolation to interpolate between $\mathcal{M} \odot \mathcal{M}$ and its anti-dual (the space of bounded conjugate linear functionals) we get a scale of Banach function spaces, whose mid-point is the Hilbert space $\mathcal{M}$. By analogy with the case where $\mathcal{M}$ is the Hardy space H^2 on the unit disk, where the end-points become H^1 and BMOA and the intermediate spaces are H^p for $1 < p < \infty$, we shall define

$$\mathcal{H}^p \ := \ [\mathcal{M} \odot \mathcal{M}, (\mathcal{M} \odot \mathcal{M})^\dagger]_{[\theta]} \tag{1.2}$$

where $0 \le \theta \le 1$, we set $p = \frac{1}{1-\theta}$, and $A^\dagger$ denotes the anti-dual of A. See [8] for background on interpolation.

Date: November 11, 2020.

2010 *Mathematics Subject Classification.* Primary 46E22; Secondary 46B70, 47B35.

Key words and phrases. Reproducing kernel Hilbert space, Nevanlinna–Pick kernel, Drury-Arveson space, H^p space, interpolation space, Hankel operator.

M.H. was partially supported by a GIF grant.

J.M. was partially supported by National Science Foundation Grant DMS 1565243.

We consider $\mathcal{H}^p$ to play the rôle of the H^p scale for the space $\mathcal{M}$. We should note that even when k is the Szegő kernel, the spaces $\mathcal{H}^p$ are isomorphic but not isometric to H^p [12], so $\mathcal{H}^p$ can at best be considered a renormed version of H^p. In Section 2 we collect properties of the $\mathcal{H}^p$ spaces for general $\mathcal{M}$. In Section 3 we specialize to the case that $\mathcal{M}$ is a complete Pick space, and prove that several inequalities that hold in general become equivalences in complete Pick spaces.

Our main result is the following. We let δ_x denote the functional of evaluation at $x \in X$ and $k_x(y) = k(y,x)$ be the reproducing kernel of $\mathcal{M}$. We will explain what a normalized complete Pick kernel is in Section 3, and we will explain Han and Han_0, the dual and predual of $\mathcal{H}^1$, in Section 2.

Theorem 3.6. *Let k be a normalized complete Pick kernel on X. Then for all $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

(a) $\|\delta_x\|_{(\mathcal{H}^p)^*} = \|k_x\|_{\mathcal{H}^q} \approx k(x,x)^{1/p}$,
(b) $\|\delta_x\|_{(\mathcal{H}^1)^*} = \|k_x\|_{\mathrm{Han}} = k(x,x)$,
(c) $\|\delta_x\|_{\mathrm{Han}_0^*} = \|\delta_x\|_{\mathrm{Han}^*} = \|k_x\|_{\mathcal{H}^1} \lesssim 1 + \log(k(x,x))$,
(d) *for the Drury-Arveson kernel S, $\|\delta_x\|_{\mathrm{Han}_0^*} \approx 1 + \log(S(x,x))$,*

where the implied constants do not depend on k or x.

Here and in the sequel, if $f, g : X \to [0,\infty)$ are functions, we write $f(x) \lesssim g(x)$ to mean that there exists a constant C so that $f(x) \leq Cg(x)$ for all $x \in X$, and $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

We show in Examples 3.16 and 3.17 that the estimate in part (c) of the theorem may not be an equivalence. In Theorem 3.14 we show that when k is a normalized complete Pick kernel, the interpolating sequences for $\mathcal{H}^1$ and $\mathcal{H}^2$ coincide.

In Section 4 we close with some questions about $\mathcal{H}^p$ scales.

2. General Spaces

Let $\mathcal{M}$ be a reproducing kernel Hilbert space satisfying assumption (A). The space $(\mathcal{M} \odot \mathcal{M})^\dagger$ was described in [6]; let us recall that description. We let $\mathcal{M} \otimes_\pi \mathcal{M}$ denote the projective tensor product of $\mathcal{M}$ with itself. Its dual is $\mathcal{B}(\mathcal{M}, \overline{\mathcal{M}})$, where $\overline{\mathcal{M}}$ is the complex conjugate of $\mathcal{M}$. Let $\rho : \mathcal{M} \otimes_\pi \mathcal{M} \to \mathcal{M} \odot \mathcal{M}$ be defined by

$$\rho : \sum f_n \otimes g_n \mapsto \sum f_n(x) g_n(x).$$

Then $(\mathcal{M} \odot \mathcal{M})^*$ can be identified with $(\ker \rho)^\perp$. We can identify $(\mathcal{M} \odot \mathcal{M})^\dagger$ with

$$\mathrm{Han} \ := \ \{\overline{T1} : T \in (\ker \rho)^\perp\}.$$

If $b \in$ Han, which is a subset of $\mathcal{M}$, the corresponding conjugate linear functional on $\mathcal{M} \odot \mathcal{M}$ is given by

$$\Lambda_b : f \mapsto \langle b, f \rangle \quad \forall\ f \in \mathcal{M}.$$

We write H_b for the unique operator $H \in \mathcal{B}(\mathcal{M}, \overline{\mathcal{M}}) \cap (\ker \rho)^\perp$ that satisfies $H_b 1 = \overline{b}$. This operator is characterized by the identity

$$\langle H_b f, \overline{\phi} \rangle_{\overline{\mathcal{M}}} = \langle \phi f, b \rangle_{\mathcal{M}} \quad (\phi \in \mathrm{Mult}(\mathcal{M}), f \in \mathcal{M}).$$

We put a norm on Han by declaring $\|b\|$ equal to the operator norm of H_b. Let

$$\begin{aligned} \mathcal{X}(\mathcal{M}) \ :=\ \{ b \in \mathcal{M} :\ \exists\ C \geq 0 \quad & \text{s.t. } |\langle b, \phi f \rangle| \leq C \|\phi\|_{\mathcal{M}} \|f\|_{\mathcal{M}} \\ & \forall\ \phi \in \mathrm{Mult}(\mathcal{M}), f \in \mathcal{M} \}. \end{aligned} \tag{2.1}$$

Then under assumption (A) it is proved in [6, Thm 2.5] that

$$\mathrm{Han} \subseteq \mathcal{X}(\mathcal{M}).$$

In particular, Han $\subseteq \mathcal{M} \odot \mathcal{M}$ contractively, so $(\mathcal{M} \odot \mathcal{M}, \mathrm{Han})$ is a compatible couple of Banach spaces. For $1 \leq p < \infty$, we shall let $\mathcal{H}^p$ be defined by

$$\mathcal{H}^p = [\mathcal{M} \odot \mathcal{M}, \mathrm{Han}]_{[\theta]}$$

with $\theta = \frac{p-1}{p}$. Since Han is dense in $\mathcal{M} \odot \mathcal{M}$, we have $\mathcal{H}^1 = \mathcal{M} \odot \mathcal{M}$ and $[\mathcal{H}^1, \mathrm{Han}]_{[1]} = \mathrm{Han}$ with equality of norms; see [8, Thm 4.2.2]. Since we shall use it several times, let us state Calderón's reiteration theorem [8, Thm. 4.6.1].

Theorem 2.1. *Let X_0, X_1 be a compatible couple of complex Banach spaces with $X_1 \subseteq X_0$. For every $0 \leq \theta \leq 1$, and $0 \leq \theta_0 \leq \theta_1 \leq 1$, let $\eta = (1-\theta)\theta_0 + \theta\theta_1$. Then we have*

$$[\ [X_0, X_1]_{[\theta_0]}, [X_0, X_1]_{[\theta_1]}\ ]_{[\theta]} \ = \ [X_0, X_1]_{[\eta]}.$$

First, we remark that the spaces $\mathcal{H}^p$ are indeed function spaces.

Proposition 2.2. *The space $\mathcal{H}^p$ is a Banach function space on X for $1 \leq p < \infty$. Moreover, if $1 \leq p \leq q < \infty$, then*

$$\mathcal{M} \odot \mathcal{M} \supseteq \mathcal{H}^p \supseteq \mathcal{H}^q \supseteq \mathrm{Han}$$

with contractive inclusions.

Proof. Since Han is contractively contained in $\mathcal{M} \odot \mathcal{M}$, complex interpolation shows that

$$\mathcal{M} \odot \mathcal{M} \supseteq \mathcal{H}^p \supseteq \mathrm{Han}$$

with contractive inclusions for all $1 \leq p < \infty$. In particular, $\mathcal{H}^p$ consists of functions on X. Since point evaluations are continuous on $\mathcal{M} \odot \mathcal{M}$ and on

Han, they are continuous on $\mathcal{H}^p$ for all p. Thus, $\mathcal{H}^p$ is a Banach function space on X. Finally, the reiteration theorem 2.1 shows that interpolating between $\mathcal{H}^p$ and Han, we obtain $\mathcal{H}^q$ for $p \leq q < \infty$, hence $\mathcal{H}^p \supseteq \mathcal{H}^q \supseteq$ Han with contractive inclusions. $\square$

As one would expect, we recover the original Hilbert function space for $p = 2$.

Theorem 2.3. *We have $\mathcal{H}^2 = \mathcal{M}$ with equality of norms.*

Proof: Assumption (A) implies that $\mathcal{M}$ is dense in $\mathcal{M} \odot \mathcal{M}$. G. Pisier proved in [17] that if a Hilbert space $\mathcal{M}$ is densely and continuously contained in a Banach space A, and so $A^\dagger$ embeds in $\mathcal{M}$, then $[A, A^\dagger]_{[\frac{1}{2}]} = \mathcal{M}$, with equality of norms. His proof is in the context of operator spaces; a direct proof of the fact is given in [20]. See also [11] for another proof. $\square$

Next, we establish the expected duality between $\mathcal{H}^p$ spaces.

Theorem 2.4. *For $1 < p < \infty$, we have $(\mathcal{H}^p)^\dagger$ is isometrically isomorphic to $\mathcal{H}^q$, where q is the conjugate index to p. The action of $\mathcal{H}^q$ on $\mathcal{H}^p$ is given by the inner product of $\mathcal{H}$ on the common subspace* Han.

Proof: By the reiteration theorem 2.1, if we interpolate between $\mathcal{H}^1$ and $\mathcal{H}^2$ we get $\mathcal{H}^p$ for $1 < p < 2$, and if we interpolate between $\mathcal{H}^2$ and Han we get $\mathcal{H}^p$ for $2 < p < \infty$. Since $\mathcal{H}^2$ is reflexive, we have by the duality theorem [8, Cor. 4.5.2] and Theorem 2.3 that

$$[\mathcal{H}^1, \mathcal{H}^2]^\dagger_{[\theta]} \; = \; [\mathrm{Han}, \mathcal{H}^2]_{[\theta]}.$$

(The duality theorem also applies to anti-duals because $[\overline{A_0}, \overline{A_1}]_{[\theta]} = \overline{[A_0, A_1]_{[\theta]}}$ isometrically). It is part of the duality theorem that the action of an element of $[\mathrm{Han}, \mathcal{H}^2]_{[\theta]} \subset \mathcal{H}^2$ on an element of the subspace $\mathcal{H}^2 \subset [\mathcal{H}^1, \mathcal{H}^2]_{[\theta]}$ is given by the inner product of $\mathcal{H}^2$; see the discussion preceding [18, Theorem 2.7.4]. This proves the theorem for $1 < p \leq 2$.

In [9, 12.2], Calderon proved that if one end point space is reflexive, all the intermediate ones are too. So this proves the theorem for $2 < p < \infty$. $\square$

We define Han_0 by

$$\mathrm{Han}_0 \; := \; \{b \in \mathrm{Han} : H_b \text{ is compact}\}.$$

By [6, Thm. 2.5], the dual space of Han_0 is $\mathcal{M} \odot \mathcal{M}$. We think of Han_0 as the analogue of VMOA.

By [6, Thm. 2.1], point evaluations are in Han_0. Moreover, since $\mathcal{M}$ is dense in $\mathcal{M} \odot \mathcal{M}$, the point evaluations come from pairing with the kernel functions, so each kernel function is in Han_0, and by the Hahn-Banach theorem, the set of finite linear combinations of kernel functions is dense in Han_0.

Proposition 2.5. *For $0 < \theta < 1$, we have the isometric equality*

$$[\mathcal{M} \odot \mathcal{M}, \mathrm{Han}_0]_{[\theta]} \;=\; [\mathcal{M} \odot \mathcal{M}, \mathrm{Han}]_{[\theta]}.$$

PROOF: Since Han_0 and Han are contained in $\mathcal{M} \odot \mathcal{M}$, so are the interpolation spaces in the statement. By [11, Cor. 4.5], we have isometrically

$$[\mathcal{M} \odot \mathcal{M}, \mathrm{Han}_0]_{[\frac{1}{2}]} \;=\; \mathcal{M}.$$

Therefore the reiteration theorem and Theorem 2.3 prove the result for $0 < \theta \leq \frac{1}{2}$. It remains to prove that

$$[\mathcal{M}, \mathrm{Han}_0]_{[s]} \;=\; [\mathcal{M}, \mathrm{Han}]_{[s]}$$

for $0 < s < 1$. But applying the duality theorem twice we get an isometric isomorphism

$$[\mathcal{M}, \mathrm{Han}_0]^{**}_{[s]} \cong [\mathcal{M}, \mathrm{Han}]_{[s]},$$

and by Calderón's reflexivity theorem again, we have $[\mathcal{M}, \mathrm{Han}_0]_{[s]}$ is reflexive for $0 \leq s < 1$, so we are done. Using the fact that the inclusion $\mathrm{Han}_0 \subseteq \mathrm{Han} \cong (\mathrm{Han}_0)^{**}$ agrees with the canonical embedding into the bidual as well as the particular form of the duality in the duality theorem, one checks that the resulting isometric isomorphism $[\mathcal{M}, \mathrm{Han}]_{[s]} \cong [\mathcal{M}, \mathrm{Han}_0]_{[s]}$ is in fact the identity. $\square$

Corollary 2.6. *The linear span of kernel functions is dense in $\mathcal{H}^p$ for $1 \leq p < \infty$.*

Proof. It is a general result about complex interpolation that for a compatible couple of Banach spaces (A_0, A_1), the intersetion $A_0 \cap A_1$ is dense in the intermediate interpolation spaces; see [8, Theorem 4.2.2]. Thus, Proposition 2.5 implies that Han_0 is dense in $\mathcal{H}^p$ for $1 \leq p < \infty$, In turn, finite linear combinations of kernels are dense in Han_0 and the inclusion $\mathrm{Han}_0 \subset \mathcal{H}^p$ is continuous. $\square$

We can now show pointwise estimates that are valid in all reproducing kernel Hilbert spaces satisfying assumption (A).

Proposition 2.7. *Let $1 < p \leq 2$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $x \in X$,*

(a) $\|\delta_x\|_{(\mathcal{H}^1)^*} = \|k_x\|_{\mathrm{Han}} = k(x,x)$.
(b) $\|\delta_x\|_{(\mathcal{H}^p)^*} = \|k_x\|_{\mathcal{H}^q} \leq k(x,x)^{1/p}$,
(c) $\|\delta_x\|_{(\mathcal{H}^q)^*} = \|k_x\|_{\mathcal{H}^p} \geq k(x,x)^{1/q}$,

Proof. For each item, the first equality follows from duality; see Theorem 2.4 and the discussion at the beginning of the section.

(a) This follows from the fact that the Hankel operator with symbol k_x is the rank one operator given by $H_{k_x}(f) = \langle f, k_x \rangle \overline{k_x}$.

(b) Let $\theta = \frac{q-1}{q}$. By reiteration, $\mathcal{H}^q = [\mathcal{H}^1, \mathrm{Han}]_\theta = [\mathcal{H}^2, \mathrm{Han}]_s$, where $s = 2\theta - 1 = 1 - \frac{2}{q}$. Since $\|k_x\|_{\mathcal{H}^2} = k(x,x)^{1/2}$ and $\|k_x\|_{\mathrm{Han}} = k(x,x)$ by part (a), interpolation therefore yields

$$\|k_x\|_{\mathcal{H}^q} \le \|k_x\|_{\mathcal{H}^2}^{1-s} \|k_x\|_{\mathrm{Han}}^s = k(x,x)^{1-1/q}$$

for $2 \le q < \infty$.

(c) By part (b),

$$\|\delta_x\|_{(\mathcal{H}^q)^*} \ge \frac{k(x,x)}{\|k_x\|_{\mathcal{H}^q}} \ge k(x,x)^{1/q}. \qquad \square$$

In Section 3 we shall prove that the estimates are sharp (up to a constant) in complete Pick spaces.

3. Complete Pick Spaces

Pick's theorem [16] gives necessary and sufficient conditions to solve an interpolation problem in the multiplier algebra of H^2 (which is H^∞). It generalizes to matrix-valued functions. A Hilbert space in which this matrix-valued Pick theorem is true is called a complete Pick space (see the next paragraph for a formal definition). Examples include the Dirichlet space [15], the Sobolev space [1], and various Besov spaces on the ball [4]; see also [3].

If $\Lambda \subseteq X$, we define $\mathcal{M}_\Lambda$ to be the closed linear span of the kernel functions $\{k_\lambda : \lambda \in \Lambda\}$, and let P be the orthogonal projection from $\mathcal{M}$ onto $\mathcal{M}_\Lambda$. Define $\pi : \mathrm{Mult}(\mathcal{M}) \to B(\mathcal{M}_\Lambda)$ by $\pi(\phi) = PM_\phi P$, where M_ϕ is multiplication by ϕ. Then π is always a contractive homomorphism. If it is an exact quotient map (i.e. it maps the closed unit ball onto the closed unit ball), then $\mathcal{M}$ is said to have the Pick property. If it is a complete exact quotient map (the induced map on matrices is always an exact quotient map), then $\mathcal{M}$ is said to have the complete Pick property.

The Drury-Arveson space H^2_d is the Hilbert function space on the open unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ or $\ell^2(d)$ with kernel

$$S(z,w) \;=\; \frac{1}{1 - \langle z, w \rangle}.$$

We say a reproducing kernel Hilbert space $\mathcal{M}$ on X is *normalized* if for some choice of base-point x_0, we have $k(x_0, y) = 1$ for all y.

The Drury-Arveson space is a normalized space with the complete Pick property, and every normalized space with the complete Pick property can be embedded in it [2], in the sense that there is a function $b : X \to \mathbb{B}_d$ for some cardinal d so that

$$k(x,y) \;=\; S(b(x), b(y)).$$

Normalized complete Pick spaces always satisfy assumption (A), as the kernel functions are multipliers. We shall prove that for complete Pick spaces, the inequalities in Proposition 2.7 are equivalences. For a kernel k, let us write $\mathcal{H}^p(k)$ to denote the space in (1.2) corresponding to the reproducing kernel Hilbert space with kernel k (which will be called $\mathcal{H}^2(k)$ in this notation). We also write $\operatorname{Han}(k)$ in place of Han when we need to specify the kernel.

Remark 3.1 In general, for any complete Pick space $\mathcal{M}$, we have

$$\operatorname{Han}(\mathcal{M}) = \mathcal{X}(\mathcal{M}), \tag{3.1}$$

where $\mathcal{X}(\mathcal{M})$ is defined in Equation (2.1). Indeed, equality (3.1) was proved in [6, Thm 2.6] under the hypothesis that $\mathcal{M}$ has the column-row property, and recently in [13] it was shown that this property holds in all complete Pick spaces.

Lemma 3.2. *Let k, ℓ be reproducing kernels on X, Y respectively, and let $\varphi : Y \to X$ be a function. If the composition operator*

$$C_\varphi : \mathcal{H}^2(k) \to \mathcal{H}^2(\ell), \quad f \mapsto f \circ \varphi,$$

is well defined and bounded, then C_φ also maps $\mathcal{H}^p(k)$ to $\mathcal{H}^p(\ell)$ for $1 \le p \le 2$ and

$$||C_\varphi||_{\mathcal{H}^p(k)\to\mathcal{H}^p(\ell)} \le ||C_\varphi||^{2/p}_{\mathcal{H}^2(k)\to\mathcal{H}^2(\ell)}.$$

Proof. It suffices to show the statement for $p = 1$. The desired result then follows from complex interpolation (and the observation that a bounded operator between two Banach function spaces that acts by composition on a dense subset acts by composition everywhere).

To show the statement for $p = 1$, let $f \in \mathcal{H}^1(k)$ with $\|f\|_{\mathcal{H}^1(k)} < 1$. Then there exist $g_n, h_n \in \mathcal{H}^2(k)$ so that $f = \sum_n g_n h_n$ and

$$\sum_n ||g_n||_{\mathcal{H}^2(k)}\, ||h_n||_{\mathcal{H}^2(k)} < 1.$$

Hence

$$\begin{aligned} ||f \circ \varphi||_{\mathcal{H}^1(\ell)} &\leq \sum_n ||g_n \circ \varphi||_{\mathcal{H}^2(\ell)} \, ||h_n \circ \varphi||_{\mathcal{H}^2(\ell)} \\ &\leq ||C_\varphi||^2_{\mathcal{H}^2(k) \to \mathcal{H}^2(\ell)}, \end{aligned}$$

which completes the proof. □

The following lemma shows that the classical H^p spaces on the disc can be embedded into the $\mathcal{H}^p$ spaces corresponding to the Drury–Arveson space. In particular, this gives concrete examples of functions in these spaces on the ball. In the sequel, let s denote the Szegő kernel on the disc and let S be the Drury–Arveson kernel on $\mathbb{B}_d$.

Lemma 3.3. *Let d be a cardinal number, let $v \in \ell^2(d)$ be a unit vector and let*

$$Q : \ell^2(d) \to \mathbb{C}, \quad z \mapsto \langle z, v \rangle.$$

Then for every $1 \leq p < \infty$, the map

$$\mathcal{H}^p(s) \to \mathcal{H}^p(S), \quad f \mapsto f \circ Q,$$

is an isometry onto a complemented subspace of $\mathcal{H}^p(S)$. Similarly, the map

$$\operatorname{Han}(s) \to \operatorname{Han}(S), \quad f \mapsto f \circ Q,$$

is an isometry onto a complemented subspace of $\operatorname{Han}(S)$.

Proof. Since $V : f \mapsto f \circ Q$ is an isometry from H^2 into H^2_d, it follows from Lemma 3.2 that it is also a contraction from $\mathcal{H}^1(s)$ into $\mathcal{H}^1(S)$. Consider the embedding

$$i : \mathbb{D} \to \mathbb{B}_d, \quad \lambda \mapsto \lambda v.$$

Then $R : f \mapsto f \circ i$, being the adjoint of V, is a co-isometry from H^2_d onto H^2. Applying Lemma 3.2 again, we find that R is a contraction from $\mathcal{H}^1(S)$ into $\mathcal{H}^1(s)$.

Next, we use duality to prove the statement about Han. Let $h \in \operatorname{Han}(s)$ and let $f \in H^2_d \subset H^2_d \odot H^2_d$. Then

$$\begin{aligned} |\langle f, h \circ Q \rangle_{H^2_d}| &= |\langle f \circ i, h \rangle_{H^2}| \\ &\leq ||f \circ i||_{H^2 \odot H^2} \, ||h||_{\operatorname{Han}(s)} \\ &\leq ||f||_{H^2_d \odot H^2_d} \, ||h||_{\operatorname{Han}(s)}. \end{aligned}$$

Thus, $h \circ Q$ induces a bounded functional on $H^2_d \odot H^2_d$ of norm at most $||h||_{\operatorname{Han}(s)}$, so that $h \circ Q \in \operatorname{Han}(S)$ and the map $V : h \mapsto h \circ Q$ is a contraction from $\operatorname{Han}(s)$ to $\operatorname{Han}(S)$. Similarly, one checks that $R : f \mapsto f \circ i$ is a contraction from $\operatorname{Han}(S)$ to $\operatorname{Han}(s)$.

Interpolation therefore shows that $V : h \mapsto h \circ Q$ is a contraction from $\mathcal{H}^p(s)$ into $\mathcal{H}^p(S)$ for $1 \le p < \infty$ and that $R : h \mapsto h \circ i$ is a contraction from $\mathcal{H}^p(S)$ into $\mathcal{H}^p(s)$ for $1 \le p < \infty$. Clearly, $R \circ V$ is the identity on $\mathcal{H}^p(s)$ and on $\operatorname{Han}(s)$, hence V is an isometry from $\mathcal{H}^p(s)$ into $\mathcal{H}^p(S)$ and from $\operatorname{Han}(s)$ to $\operatorname{Han}(S)$. Moreover, $V \circ R$ is a projection onto the range of V, hence that space is complemented. □

Let $\mathbb{H}_{1/2}$ denote the half-plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > \frac{1}{2}\}$. It is the image of the unit disk under the map $\zeta \mapsto \frac{1}{1-\zeta}$.

Lemma 3.4. *Let $1 \le p < \infty$, let $w \in \mathbb{B}_d$ and let $r = ||w||$. If h is an analytic function on $\mathbb{H}_{1/2}$ such that $h \circ s_r \in \mathcal{H}^p(s)$, then $h \circ S_w \in \mathcal{H}^p(S)$ and*

$$||h \circ S_w||_{\mathcal{H}^p(S)} = ||h \circ s_r||_{\mathcal{H}^p(s)}.$$

Proof. Let v be a unit vector in $\ell^2(d)$ such that $rv = w$ (i.e. $v = w/r$ if $r \neq 0$ and v is an arbitrary unit vector if $r = 0$) and let $Q(z) = \langle z, v \rangle$. We apply the isometry of Lemma 3.3 to $f = h \circ s_r$. Since

$$(s_r \circ Q)(z) = \frac{1}{1 - r\langle z, v\rangle} = S_w(z),$$

we have $h \circ s_r \circ Q = h \circ S_w$, and the result follows from Lemma 3.3. □

Using universality of the Drury–Arveson space, we can also construct explicit examples of functions in $\mathcal{H}^p(k)$ for complete Pick kernels k, at least for $1 \le p \le 2$.

Proposition 3.5. *Let k be a normalized complete Pick kernel on X and let $1 \le p \le 2$. Let $x \in X$ and set*

$$r = \sqrt{1 - \frac{1}{k(x,x)}}.$$

If h is an analytic function on $\mathbb{H}_{1/2}$ such that $h \circ s_r \in \mathcal{H}^p(s)$, then $h \circ k_x \in \mathcal{H}^p(k)$ and

$$||h \circ k_x||_{\mathcal{H}^p(k)} \le ||h \circ s_r||_{\mathcal{H}^p(s)}.$$

In particular, $k_x^{2/p} \in \mathcal{H}^p(k)$ and $||k_x^{2/p}||_{\mathcal{H}^p(k)} \lesssim k(x,x)^{1/p}$ for all $x \in X$, where the implied constant depends only on p, not on k or x.

Proof. There exists a function $b : X \to \mathbb{B}_d$ for a suitable cardinal d such that $k(x,y) = S(b(x), b(y))$ and so that $f \mapsto f \circ b$ is a co-isometry from H^2_d onto $\mathcal{H}^2(k)$. Let $w = b(x)$ and note that $r = ||w||$. In this setting, Lemma 3.4 implies that $h \circ S_w \in \mathcal{H}^p(S)$ with $||h \circ S_w||_{\mathcal{H}^p(s)} = ||h \circ s_r||_{\mathcal{H}^p(s)}$. By

Lemma 3.2, the map $f \mapsto f \circ b$ is a contraction from $\mathcal{H}^p(S)$ into $\mathcal{H}^p(k)$, hence $h \circ S_w \circ b \in \mathcal{H}^p(k)$ with $||h \circ S_w \circ b||_{\mathcal{H}^p(k)} \le ||h \circ s_r||_{\mathcal{H}^p(s)}$. Since

$$(S_w \circ b)(y) = \frac{1}{1 - \langle b(y), w \rangle} = k_x(y)$$

for each $y \in X$, we have $S_w \circ b = k_x$, which completes the proof of the first statement.

To prove the additional statement, we let $h(\lambda) = \lambda^{2/p}$. Then it follows from the fact that $\mathcal{H}^p(s) = H^p$ isomorphically that

$$||s_r^{2/p}||_{\mathcal{H}^p(s)} \approx ||s_r||_{H^2}^{2/p} = s(r,r)^{1/p}, \tag{3.2}$$

so by the first paragraph, $k_x^{2/p} \in \mathcal{H}^p(k)$ and

$$||k_x^{2/p}||_{\mathcal{H}^p(k)} \lesssim s(r,r)^{1/p} = k(x,x)^{1/p}. \qquad \square$$

We can now give asymptotic bounds on $\|\delta_x\|$ for every $1 \le p \le \infty$. For $p = 1$, we already proved (b) in Proposition 2.7; we include it for completeness. For $p = \infty$ we have $\|\delta_x\|_{\mathrm{Mult}} = 1$. (Indeed, 1 is an upper bound because $M_\phi^* k_x = \overline{\phi(x)} k_x$, and it is attained since the constants are multipliers).

In [7], Arcozzi, Rochberg, Sawyer and Wick studied the weak product of the Dirichlet space with itself, and for that space proved (b) and (c) in Theorem 3.6, and moreover showed that $\|k_x\|_{\mathcal{H}^1} \approx 1 + \log(k(x,x))$.

Theorem 3.6. *Let k be a normalized complete Pick kernel on X. Then for all $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,*

(a) $\|\delta_x\|_{(\mathcal{H}^p)^*} = \|k_x\|_{\mathcal{H}^q} \approx k(x,x)^{1/p}$,
(b) $\|\delta_x\|_{(\mathcal{H}^1)^*} = \|k_x\|_{\mathrm{Han}} = k(x,x)$,
(c) $\|\delta_x\|_{\mathrm{Han}_0^*} = \|\delta_x\|_{\mathrm{Han}^*} = \|k_x\|_{\mathcal{H}^1} \lesssim 1 + \log(k(x,x))$,
(d) *for the Drury-Arveson kernel S,* $\|\delta_x\|_{\mathrm{Han}_0^*} \approx 1 + \log(S(x,x))$,

where the implied constants do not depend on k or x.

Proof. The equalities in (a) and (c) follow from the $\mathcal{H}^p$–$\mathcal{H}^q$ duality (Theorem 2.4) and the Han_0–$\mathcal{H}^1$ and the $\mathcal{H}^1$–Han dualities.

Suppose first that $1 \le q \le 2$. Then by Proposition 3.5 and the equality $H^p(s) = H^p$ with equivalent norms,

$$\|k_x\|_{\mathcal{H}^q(k)}^q \lesssim \|s_r\|_{H^q}^q = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1}{|1 - re^{it}|^q} \, dt, \tag{3.3}$$

where $r = (1 - k(x,x)^{-1})^{1/2}$. The integral above behaves like $(1-r^2)^{1-q}$ if $q > 1$ and like $1 + \log(1/(1-r^2))$ if $q = 1$; see [21, Theorem 1.12]. Hence

$$\|k_x\|_{\mathcal{H}^q} \lesssim k(x,x)^{(q-1)/q} = k(x,x)^{1/p} \tag{3.4}$$

if $1 < q \leq 2$ and

$$\|k_x\|_{\mathcal{H}^1} \lesssim 1 + \log(k(x,x)).$$

This proves (c) and also the inequality $\|k_x\|_{\mathcal{H}^q} \lesssim k(x,x)^{1/p}$ in (a) if $2 \leq p < \infty$. The reverse inequality was established in part (c) of Proposition 2.7, so (a) holds for $2 \leq p < \infty$.

Next, let $1 < p \leq 2$. Part (b) of Proposition 2.7 shows that $\|\delta_x\|_{(\mathcal{H}^p)^*} \leq k(x,x)^{1/p}$. On the other hand, by (3.4), we have

$$\|\delta_x\|_{(\mathcal{H}^p)^*} \geq \frac{k(x,x)}{\|k_x\|_{\mathcal{H}^p}} \gtrsim \frac{k(x,x)}{k(x,x)^{1-1/p}} = k(x,x)^{1/p},$$

so (a) also holds for $1 < p \leq 2$.

Finally, when $k = S$, Lemma 3.4 shows that the estimate in (3.3) is actually an equivalence, which gives (d). □

Corollary 3.7. *If k is a normalized complete Pick kernel on X and $\sup_x k(x,x) = \infty$, then for $1 \leq p < q < \infty$ the containment $\mathcal{H}^q(k) \subseteq \mathcal{H}^p(k)$ is strict.*

Remark 3.8 If f is in $\mathcal{H}^p$ for $1 \leq p < \infty$, then one has the pointwise estimate $|f(x)| \leq \|f\| \|\delta_x\|_{(\mathcal{H}^p)^*}$. But since linear combinations of kernel functions are dense in $\mathcal{H}^p$ by Corollary 2.6 and each individual kernel function is bounded, one can improve this, for each fixed f, to

$$|f(x)| \ = \ o(\|\delta_x\|_{(\mathcal{H}^p)^*}), \quad k(x,x) \to \infty.$$

Similarly, if $f \in \mathrm{Han}_0$ one gets

$$|f(x)| \ = \ o(\|k_x\|_{\mathcal{H}^1}), \quad \|k_x\|_{\mathcal{H}^1} \to \infty.$$

Below, we will provide an example to show that in general complete Pick spaces, the estimate in part (c) of Theorem 3.6 need not be an equivalence.

Recall that if $\mathcal{M}$ is a reproducing kernel Hilbert space on X, then a sequence (x_n) in X is said to be an interpolating sequence if the evaluation map

$$E : \varphi \mapsto (\varphi(x_n))$$

maps $\mathrm{Mult}(\mathcal{M})$ onto ℓ^∞.

If α_n is a sequence of positive numbers, and $p \geq 1$, we let $\ell^p(\alpha_n)$ denote the Banach sequence space with norm

$$\|(c_n)\| := \left(\sum_{n=1}^{\infty} |c_n|^p \alpha_n \right)^{1/p}.$$

Definition 3.9. The sequence (x_n) is an interpolating sequence for $\mathcal{H}^p$ if the evaluation map E maps $\mathcal{H}^p$ into and onto $\ell^p(1/\|\delta_{x_n}\|^p_{(\mathcal{H}^p)^*})$.

The closed graph theorem shows that if (x_n) is an interpolating sequence for $\mathcal{H}^p$, then E is a bounded map from $\mathcal{H}^p$ onto $\ell^p(1/\|\delta_{x_n}\|^p_{\mathcal{H}^{p*}})$, so by the open mapping theorem, the induced map $\mathcal{H}^p/\ker(E) \to \ell^p(1/\|\delta_{x_n}\|^p_{\mathcal{H}^{p*}})$ has a bounded inverse. The norm of the inverse is usually called the constant of interpolation.

Shapiro and Shields [19] showed that in the case of the Hardy space of the disc, the interpolating sequences for H^p are the same for $1 \le p \le \infty$. It was observed by Marshall and Sundberg that if $\mathcal{M}$ is a complete Pick space, the interpolating sequences for $\mathcal{M} = \mathcal{H}^2$ and $\mathrm{Mult}(\mathcal{M})$ are the same [15]. In [7], it was shown that for the Dirichlet space, the interpolating sequences for $\mathcal{H}^1$ and $\mathcal{H}^2$ are also the same. Their proof carries over to any complete Pick space. We first prove the easy implication, which is valid without the complete Pick assumption. If k is a kernel on X and $V \subset X$, we let $k\big|_V$ denote the restriction of k to $V \times V$. Thus, $\mathcal{H}^2(k|_V)$ is a space of functions on V.

Lemma 3.10. *Let $\mathcal{M}$ be a reproducing kernel Hilbert space on X with kernel k satisfying assumption (A) and let (x_n) be an interpolating sequence for $\mathcal{H}^2$.*

(a) *The sequence (x_n) is an interpolating sequence for $\mathcal{H}^1$.*
(b) *The evaluation map $E : \mathcal{H}^1 \to \ell^1(1/\|\delta_{x_n}\|_{\mathcal{H}^{1*}})$ has a bounded linear right-inverse.*
(c) *If $V = \{x_n : n \in \mathbb{N}\}$, then*

$$\|h\|_{\mathcal{H}^1(k|_V)} \approx \sum_n \frac{|h(x_n)|}{k(x_n, x_n)}.$$

for all $h \in \mathcal{H}^1(k|_V)$.

Proof. By Proposition 2.7 (a), we have $\|\delta_x\|_{(\mathcal{H}^1)^*} = k(x,x)$. Lemma 3.2 shows that the restriction map $R : \mathcal{H}^1 \to \mathcal{H}^1(k|_V)$ is contractive. Thus, the lemma will be proved once we show that

(1) the map $E_V : h \mapsto (h(x_n))$ maps $\mathcal{H}^1(k|_V)$ boundedly into $\ell^1(1/k(x_n, x_n))$, and
(2) there is a bounded linear operator $T : \ell^1(1/k(x_n, x_n)) \to \mathcal{H}^1$ so that $E \circ T$ is the identity.

Indeed, this follows from the commutative diagram

$$\begin{tikzcd} \mathcal{H}^1 \arrow[r, "R"] \arrow[dr, "E"'] & \mathcal{H}^1(k|_V) \arrow[d, "E_V"] \\ & \ell^1(\frac{1}{k(x_n,x_n)}) \end{tikzcd}$$

and injectivity of E_V.

To show (1), let $h \in \mathcal{H}^1(k|_V)$ with $\|h\|_{\mathcal{H}^1(k|_V)} < 1$. Then there exist $f_j, g_j \in \mathcal{M}$ so that $h = \sum_j f_j|_V g_j|_V$ and $\sum_j \|f_j\|_{\mathcal{M}} \|g_j\|_{\mathcal{M}} < 1$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_n \frac{|h(x_n)|}{k(x_n, x_n)} &\leq \sum_n \sum_j \frac{|f_j(x_n) g_j(x_n)|}{k(x_n, x_n)} \\ &\leq \sum_j \left[\sum_n \frac{|f_j(x_n)|^2}{k(x_n, x_n)} \sum_m \frac{|g_j(x_m)|^2}{k(x_m, x_m)} \right]^{1/2} \\ &\leq \|E\|^2_{\mathcal{H}^2 \to \ell^2(1/k(x_n, x_n))}.\end{aligned}$$

So E maps $\mathcal{H}^1$ boundedly into $\ell^1(1/k(x_n, x_n))$.

As for (2), observe that since (x_n) is an interpolating sequence for $\mathcal{H}^2$, the open mapping theorem implies that there exists a sequence (f_n) in $\mathcal{H}^2$ satisfying $f_n(x_k) = \delta_{nk}$ and $\|f_n\|^2_{\mathcal{H}^2} \lesssim 1/k(x_n, x_n)$, hence $\|f_n^2\|_{\mathcal{H}^1} \lesssim 1/k(x_n, x_n)$. Define

$$T : \ell^1(1/k(x_n, x_n)) \to \mathcal{H}^1, \quad (w_n) \mapsto \sum_n w_n f_n^2.$$

The series converges absolutely in $\mathcal{H}^1$, the operator T is bounded, and $T(w_n)(x_k) = w_k$, so $E \circ T$ is the identity. $\square$

Lemma 3.11. *Let u and v be unit vectors in a Hilbert space $\mathcal{M}$. Let τ and ω be complex numbers satisfying $|\tau| = |\omega|$ and*

$$\tau \langle v, u \rangle = \omega \langle u, v \rangle. \tag{3.12}$$

Then

$$\|\langle \cdot, u \rangle \overline{u} - \omega \langle \cdot, v \rangle \overline{v}\|_{\mathcal{M} \to \overline{\mathcal{M}}} = \|\langle \cdot, u \rangle u - \tau \langle \cdot, v \rangle v\|_{\mathcal{M} \to \mathcal{M}}.$$

Proof: Let $C : \overline{\mathcal{M}} \to \mathcal{M}$ be the anti-linear isometric operator $\bar{f} \mapsto f$. The norm squared of $C(\langle \cdot, u \rangle \overline{u} - \omega \langle \cdot, v \rangle \overline{v})$ is

$$\sup_{\|f\|_{\mathcal{M}} = 1} \|\langle u, f \rangle u - \bar{\omega} \langle v, f \rangle v\|^2. \tag{3.13}$$

The norm squared of $\langle \cdot, u \rangle u - \tau \langle \cdot, v \rangle v$ is

$$\sup_{\|f\|_{\mathcal{M}} = 1} \|\langle f, u \rangle u - \tau \langle f, v \rangle v\|^2. \tag{3.14}$$

Expanding (3.13) and (3.14) and using (3.12), we see that they are equal. $\square$

If k is a normalized kernel on X, the formula

$$d_k(x, y) = \sqrt{1 - \frac{|k(x, y)|^2}{k(x, x) k(y, y)}}$$

defines a pseudo-metric on X; see [3, Lemma 9.9].

Lemma 3.13. *Let k be a normalized kernel on X satisfying assumption (A). For $x \in X$ define $b_x = \frac{k_x}{k(x,x)}$. Let $x, y \in X$ and let ω be a unimodular complex number satisfying $\langle k_y, k_x \rangle = \omega \langle k_x, k_y \rangle$. Then $d_k(x,y) = \|b_x - \overline{\omega} b_y\|_{\mathrm{Han}}$.*

Proof. If $u_x = \frac{k_x}{\sqrt{k(x,x)}}$, then H_{b_x} is the rank one operator $f \mapsto \langle f, u_x \rangle \overline{u_x}$. Thus, if P_{u_x} denotes the orthogonal projection from $\mathcal{H}$ onto $\mathbb{C}u_x$, i.e. $P_{u_x} f = \langle f, u_x \rangle u_x$, then by Lemma 3.12, we have

$$\|b_x - \overline{\omega} b_y\|_{\mathrm{Han}} = \|H_{b_x} - \omega H_{b_y}\|_{\mathcal{H}^2 \to \overline{\mathcal{H}^2}} = \|P_{u_x} - P_{u_y}\|_{\mathcal{H}^2 \to \mathcal{H}^2}.$$

As u_x and u_y are unit vectors, we have

$$\|P_{u_x} - P_{u_y}\| = \sqrt{1 - |\langle u_x, u_y \rangle|^2} = d_k(x,y)$$

(this can be seen by observing that $P_{u_x} - P_{u_y}$ is a trace 0, rank 2 self-adjoint operator, with determinant $|\langle u_x, u_y \rangle|^2 - 1$ on the two-dimensional space spanned by u_x and u_y). ☐

We now establish the announced equality of interpolating sequences for $\mathcal{H}^1$ and $\mathcal{H}^2$ in the setting of complete Pick spaces.

Theorem 3.14. *Let k be a normalized complete Pick kernel on X. Then the interpolating sequences for $\mathcal{H}^1$ and $\mathcal{H}^2$ coincide.*

PROOF: In light of Lemma 3.10, it remains to show that every interpolating sequence (x_n) for $\mathcal{H}^1$ is interpolating for $\mathcal{H}^2 = \mathcal{M}$. To this end, by [5], we need to show that

(1) the sequence is weakly separated, which means there exists $\delta > 0$ so that for $m \neq n$ we have

$$d_k(x_m, x_n) \geq \delta,$$

and

(2) it satisfies the Carleson measure condition, namely there exists some constant C so that

$$\sum_{n=1}^{\infty} \frac{|f(x_n)|^2}{k(x_n, x_n)} \ \leq \ C\|f\|^2 \qquad \forall\, f \in \mathcal{M}.$$

As $f^2 \in \mathcal{H}^1$ with $\|f^2\|_{\mathcal{H}^1} \leq \|f\|^2_{\mathcal{M}}$, we get (2) immediately.

To see weak separation, we use the open mapping theorem to find a constant $c > 0$ so that for any distinct points x_m, x_n, there is a function h of norm at most 1 in $\mathcal{H}^1$ with $h(x_m) = ck(x_m, x_m)$ and $h(x_n) = -c\bar{\omega}k(x_n, x_n)$, where $\omega\langle k_{x_m}, k_{x_n}\rangle = \langle k_{x_n}, k_{x_m}\rangle$. This means

$$\Big\| \frac{1}{k(x_m, x_m)} \delta_{x_m} - \frac{\omega}{k(x_n, x_n)} \delta_{x_n} \Big\|_{(\mathcal{H}^1)^*} \geq 2c. \tag{3.15}$$

Write $b_x = k_x / k(x, x)$. Using the anti-linear $\mathcal{H}^1$–Han duality, and Lemma 3.13, the left-hand side of (3.15) is equal to

$$\|b_{x_m} - \overline{\omega} b_{x_n}\|_{\mathrm{Han}} = d_k(x_m, x_n),$$

hence (x_n) is weakly separated. □

We now give an example of a normalized complete Pick space with unbounded kernel in which every function in Han is bounded.

Example 3.16. Let (e_n) denote the standard basis of ℓ^2. Let (r_n) be a sequence in $[0, 1)$ tending to 1 with the properties that $r_0 = 0$, the sequence $x_n = r_n e_n$ is an interpolating sequence for $\mathrm{Mult}(H^2_\infty)$ and $\sum_n (1 - r_n^2) < \infty$ (this can be done for instance by [5, Proposition 5.1]; the last property in fact follows from being an interpolating sequence). Let $V = \{r_n e_n : n \in \mathbb{N}\}$ and let $\mathcal{M} = H^2_\infty\big|_V$, which is a normalized complete Pick space on V whose kernel k satisfies $\lim_n k(x_n, x_n) = \infty$. We claim that $\sup_x \|\delta_x\|_{\mathrm{Han}} < \infty$.

By Theorem 3.6, $\|\delta_x\|_{\mathrm{Han}} = \|k_x\|_{\mathcal{H}^1}$, so by Lemma 3.10 (c), it suffices to show that

$$\sup_j \sum_n \frac{|k(x_n, x_j)|}{k(x_n, x_n)} < \infty.$$

But $k(x_n, x_j) = \frac{1}{1 - r_n r_j \langle e_n, e_j\rangle} = 1$ if $n \neq j$, hence

$$\sum_n \frac{|k(x_n, x_j)|}{k(x_n, x_n)} = 1 + \sum_{n \neq j} \frac{1}{k(x_n, x_n)} \leq 1 + \sum_n (1 - r_n^2) < \infty,$$

which is independent of j.

The preceding example takes place on the ball in infinite dimensions. In fact, the log estimate in part (c) of Theorem 3.6 may not be an equivalence even on the disc, as the following example shows.

Example 3.17. Let (y_n) be a strictly increasing sequence in $[1, \infty)$, with $y_0 = 1$, and tending to infinity so fast that

(1) $x_n := 1 - \frac{1}{y_n}$ is an interpolating sequence for H^∞,
(2) $\sum_n \frac{y_{n-1}}{y_n} < \infty$, and
(3) $\lim_{n\to\infty} \frac{n}{\log(y_n)} = 0$.

For instance, $y_n = 2^{2^n}$ or $y_n = (n!)^2$ will do. Let $V = \{x_n : n \in \mathbb{N}\} \subset \mathbb{D}$ and let $\mathcal{M} = H^2\big|_V$. We will show that

$$\lim_j \frac{\|\delta_{x_j}\|_{\mathrm{Han}}}{\log(k(x_j, x_j))} = 0.$$

Again by Theorem 3.6 and Lemma 3.10 (c), it suffices to show that

$$\lim_j \frac{1}{\log(k(x_j, x_j))} \sum_n \frac{|k(x_n, x_j)|}{k(x_n, x_n)} = 0.$$

To this end, note that

$$\sum_n \frac{|k(x_n, x_j)|}{k(x_n, x_n)} = \sum_n \frac{1 - x_n^2}{1 - x_n x_j} \lesssim \sum_{n=0}^{j} \frac{1 - x_n^2}{1 - x_n x_j} + \sum_{n=j+1}^{\infty} \frac{1 - x_n}{1 - x_n x_j}.$$

Each summand in the first sum in bounded by 1. In the second sum, note that $x_j \le x_{n-1}$, hence

$$\sum_n \frac{|k(x_n, x_j)|}{k(x_n, x_n)} \lesssim (j+1) + \sum_{n=j+1}^{\infty} \frac{1 - x_n}{1 - x_{n-1}} \le (j+1) + \sum_n \frac{y_{n-1}}{y_n},$$

and the last sum converges by Property (2). Thus,

$$\frac{1}{\log(k(x_j, x_j))} \sum_n \frac{|k(x_n, x_j)|}{k(x_n, x_n)} \lesssim \frac{j+1}{\log(y_j)} \xrightarrow{j \to \infty} 0$$

by Property (3).

Remark 3.18 Suppose that $\mathcal{M}$ is a normalized complete Pick space with kernel $k(x,y) = \frac{1}{1 - \langle b(x), b(y) \rangle}$, where $b : X \to \mathbb{B}_d$. Then the map $f \mapsto f \circ b$ takes H^2_d to $\mathcal{H}$. By Lemma 3.2, it also takes $\mathcal{H}^p(S)$ to $\mathcal{H}^p(k)$ for $1 \le p \le 2$, and it is easy to see that it maps $\mathrm{Mult}(H^2_d)$ to $\mathrm{Mult}(\mathcal{M})$. The preceding two examples show that in general, it does not map $\mathrm{Han}(S)$ to $\mathrm{Han}(k)$, because by part (d) of Theorem 3.6, we have $\|\delta_x\|_{\mathrm{Han}(S)} \approx 1 + \log(S(x,x))$. Similarly, Proposition 3.5 does not hold with Han in place of $\mathcal{H}^p$.

4. Questions

Question 4.1. There are many interesting Hilbert function spaces for which assumption (A) fails, such as ℓ^2, the Hardy space of the upper half-plane, and the Fock space. One can still define an $\mathcal{H}^p$ scale for these spaces for $p \in [1,2]$ by interpolating between $\mathcal{M} \odot \mathcal{M}$ and $\mathcal{M}$. Is there a general method to identify the anti-duals of these spaces with Banach function spaces on X?

Question 4.2. How does one define the $\mathcal{H}^p$ scale for $0 < p < 1$?

Question 4.3. When can we recover the $\mathcal{H}^p$ spaces isomorphically by interpolating between $\mathcal{M} \odot \mathcal{M}$ and $\mathrm{Mult}(\mathcal{M})$? This is true for the Hardy space [14].

Question 4.4. What are the multipliers of $\mathcal{H}^p$? When are they the same as $\mathrm{Mult}(\mathcal{M})$? For complete Pick spaces, Clouâtre and the second named author [10, 13] show that $\mathrm{Mult}(\mathcal{M}) = \mathrm{Mult}(\mathcal{M} \odot \mathcal{M})$. Is this enough to get $\mathrm{Mult}(\mathcal{H}^p) = \mathrm{Mult}(\mathcal{M})$ for $1 \leq p \leq 2$?

Question 4.5. If $k(x, y) = \frac{1}{1 - \langle b(x), b(y) \rangle}$, does the map $f \mapsto f \circ b$ take $\mathcal{H}^p(S)$ to $\mathcal{H}^p(k)$ for $p > 2$?

A positive answer to Question 4.3 would imply a positive answer to Question 4.5, as the map $f \mapsto f \circ b$ takes multipliers to multipliers.

Question 4.6. Are the interpolating sequences for $\mathcal{H}^p$ the same as for $\mathcal{H}^2$ for $1 < p < 2$? What about $2 < p < \infty$?

Question 4.7. What is a function theoretic description of $\mathcal{H}^p$ for the Dirichlet space or for weighted Besov–Sobolev spaces?

References

[1] Jim Agler, *Nevanlinna-Pick interpolation on Sobolev space*, Proc. Amer. Math. Soc. **108** (1990), 341–351. ↑6

[2] Jim Agler and John E. McCarthy, *Complete Nevanlinna-Pick kernels*, J. Funct. Anal. **175** (2000), no. 1, 111–124. ↑7

[3] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002. ↑6, 14

[4] Alexandru Aleman, Michael Hartz, John E. McCarthy, and Stefan Richter, *Radially weighted Besov spaces and the Pick property*, Analysis of operators on function spaces, 2019, pp. 29–61. ↑6

[5] Alexandru Aleman, Michael Hartz, John E. McCarthy, and Stefan Richter, *Interpolating sequences in spaces with the complete Pick property*, Int. Math. Res. Not. IMRN **12** (2019), 3832–3854. ↑14, 15

[6] ———, *Weak products of complete Pick spaces*, 2019. Indiana Math. J., to appear. ↑2, 3, 4, 7

[7] N. Arcozzi, R. Rochberg, E. Sawyer, and B. D. Wick, *Function spaces related to the Dirichlet space*, J. Lond. Math. Soc. (2) **83** (2011), no. 1, 1–18. ↑10, 12

[8] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, Berlin, 1976. ↑1, 3, 4, 5

[9] A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190. ↑4

[10] Raphaël Clouâtre and Michael Hartz, *Multipliers and operator space structure of weak product spaces*, Anal. PDE (to appear). arXiv:1909.12883. ↑17

[11] Fernando Cobos and Tomas Schonbek, *On a theorem by Lions and Peetre about interpolation between a Banach space and its dual*, Houston J. Math. **24** (1998), no. 2, 325–344. ↑4, 5

[12] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193. ↑2

[13] Michael Hartz, *Every complete Pick space satisfies the column-row property*, arXiv:2005.09614. ↑7, 17

[14] Peter W. Jones, *L^∞ estimates for the $\bar{\partial}$ problem in a half-plane*, Acta Math. **150** (1983), no. 1-2, 137–152. ↑17

[15] D. Marshall and C. Sundberg, *Interpolating sequences for the multipliers of the Dirichlet space*, 1994. Preprint; see http://www.math.washington.edu/~marshall/preprints/preprints.html. ↑6, 12

[16] Georg Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann. **77** (1916), 7–23. ↑6

[17] Gilles Pisier, *The operator Hilbert space* OH*, complex interpolation and tensor norms*, Mem. Amer. Math. Soc. **122** (1996), no. 585, viii+103. ↑4

[18] ______, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series, vol. 294, Cambridge University Press, Cambridge, 2003. ↑4

[19] H.S. Shapiro and A.L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513–532. ↑12

[20] Frédérique Watbled, *Complex interpolation of a Banach space with its dual*, Math. Scand. **87** (2000), no. 2, 200–210. ↑4

[21] K. Zhu, *Operator theory in function spaces*, Monographs and textbooks in pure and applied mathematics, vol. 139, Marcel Dekker, Inc., New York, 1990. ↑10

Lund University, Mathematics, Faculty of Science, P.O. Box 118, S-221 00 Lund, Sweden
Email address: alexandru.aleman@math.lu.se

Fachrichtung Mathematik, Universität des Saarlandes, 66123 Saarbrücken, Germany
Email address: hartz@math.uni-sb.de

Department of Mathematics, Washington University in St. Louis, One Brookings Drive, St. Louis, MO 63130, USA
Email address: mccarthy@wustl.edu

Department of Mathematics, University of Tennessee, 1403 Circle Drive, Knoxville, TN 37996-1320, USA
Email address: srichter@utk.edu