

Fredonia

John E. McCarthy
Washington University, St. Louis, MO 63130
mccarthy@wustl.edu

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1 Fredonia

The planet of Fredonia has seven moons, and the number 7 has long had mystical significance. It has also served as the basis for their number system. In ancient times, merchants were able to manipulate numbers as large as 7^7 (which was called F). Then the brilliant mathematician Farchimedes attempted to answer the question of how many specks of dust would fill the universe. To do this, she first needed a system to express very large numbers. Roughly speaking, she defined numbers in the following order:

$$1, 2, \dots, F, F + 1, \dots, 2F, 2F + 1, \dots, 3F, \dots, F^2, \dots, F^3, \dots, 7^F, \dots$$

The largest number expressible using her notation was called F_{red} . The number F_{red} is the largest number in the Fredonian number system, which considers all rational numbers with both numerator and denominator less than F_{red} .

In medieval times, it was a religious dogma that no number larger than F_{red} could exist; those who questioned this were executed for heresy. Today, in the liberal universities, mathematicians chuckle when a student asks timidly “what about F_{red} plus one?” They point out that according to the currently accepted theory of physics, the smallest unit of time is a blip — approximately the time it would take for light to travel the diameter of a proton. The age of the universe is approximately 7^{7^2} blips, so there is no conceivable calculation that could come close to F_{red} . As Fittgenstein put

it, “Whereof one cannot speak, thereof one must be silent. Of no number as large as F_{red} can one speak”.

Fredonians have a pragmatic approach to mathematics. The most celebrated fields of mathematical study are approximation, statistics, combinatorics, and mechanics. The Fythagorean approximation theorem says that if $a^2 + b^2$ is close to c^2 , then the triangle with sides a, b, c is approximately right-angled. It has been known since ancient times that only certain choices of a and b can be used in constructing triangles that are exactly right-angled; for example, 3 and 4 can be used, but not 1 and 2. The theory of which triangles are constructible has been extensively studied; the right-angled case was solved by Feucclid.

In mechanics, the equations of motion for constant acceleration have been known for centuries. Variable acceleration was treated by breaking the interval into subintervals. This was a tedious process to do manually, but the invention of a remarkable mechanical calculator by Fabbage allowed this to be done over very small subintervals. This led to breakthroughs in simulation, modeling and prediction. Aeroplanes could be designed without resort to physical wind-tunnels, and space-craft trajectories could be accurately calculated. Weather prediction has proved more difficult, but it has been slowly improving, aided by advances in calculator architecture and the incorporation of statistical ideas.

In physics, the discreteness of time and space has never been questioned, but the size of the smallest unit of time and space have been revised downwards on several occasions, ever since Feno demonstrated that a smallest unit must exist.

A popular recent idea in theoretical physics is that elementary particles are actually two-dimensional tori of genus one, where this surface is special since it is the only one for which the seven color theorem holds. Experimentalists are not impressed, as the theory has yet to make any verifiable predictions.

2 Questions

It is amusing to imagine different systems, and of course none can be absolutely ruled out as theoretically impossible, since we aren’t sure what the rules are. Nonetheless, it is interesting to argue that some systems are implausible. Once one names F_{red} , does that make it implausible not to be able

to name $F_{red} + 1$? Does the answer to this question differ if F_{red} is like a “limit ordinal”

$$F_{red} = 7^{7^{7^{7^{7^7}}}},$$

as opposed to a number that can’t be written out compactly in the accepted notation

$$F_{red} = 7^{7^{7^{7^7}}} \cdot 7^{7^{7^{7^7}}} \cdot 7^{7^{7^{7^7}}} \dots?$$

Is constructing a right-angle so fundamental to plane geometry that $\sqrt{2}$ must be accepted as a constructible length if geometry is ever developed? If so, can one stop at constructible numbers, or is one led inexorably to real numbers, at least in an intuitive way, if not an axiomatic one?

Is our concept of what makes a satisfactory proof likely to be widespread? Are some proof concepts, such as the proof that

$$\sum_{k=1}^N k = \frac{N(N+1)}{2}$$

by pairing k with $N - k$, more universal than others, like Euclidean proofs that require constructing extra lines that don’t appear in the statement of the theorem? What about proof by contradiction?