

Abelian Gauge Theory, Knots and Odd Khovanov Homology

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Abstract

A homological invariant of 3-manifolds is defined using abelian Yang-Mills gauge theory. It is shown that the construction, in an appropriate sense, is functorial with respect to the families of 4-dimensional cobordisms. This construction and its functoriality are used to define several link invariants. The strongest version of these invariants has the form of a filtered chain complex that can recover Khovanov homology of the mirror image as a bi-graded group.

Gauge theoretical methods are proven to be useful tools to study low dimensional manifolds. As one of the first instances of this approach, Donaldson constructed a family of invariants, usually known as *Donaldson invariants*, for smooth closed 4-manifolds that satisfy a mild topological assumption [4]. To construct these invariants, one fixes a Hermitian vector bundle of rank 2 on a Riemannian 4-manifold and considers the space of connections that satisfy a geometrical PDE (aka *anti-self-duality* equation) modulo the action of bundle automorphisms. Although the anti-self-duality equation depends on the Riemannian metric, a careful study of the moduli spaces of anti-self-dual connections leads to numerical invariants that depend only on the diffeomorphism type of the 4-manifold. More recently, Kronheimer applied a modification of the original construction to Hermitian vector bundles of rank n , with $n \geq 3$, and defined similar 4-manifold invariants. In the case $n = 1$, the relevant moduli spaces can be characterized by the classical topological invariants (i.e. cohomology groups of the underlying 4-manifolds) and hence they do not give rise to any interesting numerical invariant that can be used to study the topology of 4-manifolds.

Let Y be a 3-manifold which has the same integral homology groups as the 3-dimensional sphere. In [7], Floer used the moduli space of anti-self-dual connections on $\mathbb{R} \times Y$ to produce topological invariants of Y . In parallel to the case of 4-manifolds, one expects that anti-self-dual connections on vector bundles of higher rank can be also utilized to define 3-manifold invariants. Kronheimer and Mrowka used this approach in a greater generality, and among other topological invariants, they constructed invariants of 3-manifolds using higher rank bundles.

The purpose of this article is to show that the moduli space of anti-self-dual connections on a Hermitian line bundle can be used to extract non-trivial data in 3-manifold topology. We will use a slight variation of these spaces to define an invariant that is called *plane Floer homology*. Plane Floer homology of a 3-manifold Y is a graded vector space over a field Λ and is denoted by $\text{PFH}(Y)$. As any other Floer homology theory, this vector space is the homology of a chain complex which in this case is denoted by $\text{PFC}(Y)$. An important feature of plane Floer homology is functoriality

with respect to cobordisms. That is to say, if $W : Y_0 \rightarrow Y_1$ is a 4-dimensional cobordism from Y_0 to Y_1 , then there is a chain map $\text{PFC}(W) : \text{PFC}(Y_0) \rightarrow \text{PFC}(Y_1)$. This cobordism map is defined with the aid of an appropriate metric on W . More generally, for a family of metrics on the cobordism W , we can define cobordism maps from $\text{PFC}(Y_0)$ to $\text{PFC}(Y_1)$. The following theorem is the heart of this article:

Theorem 1. *If L is a framed link with n connected components in a 3-manifold Y , then there is a chain complex $\text{PFC}_L(Y)$ that has the same chain homotopy type as $\text{PFC}(Y)$. The underlying chain group of $\text{PFC}_L(Y)$ is equal to:*

$$\bigoplus_{\mathbf{m} \in \{0,1\}^n} \text{PFC}(Y_{\mathbf{m}})$$

For $\mathbf{m} = (m_1, \dots, m_n) \in \{0,1\}^n$, the 3-manifold $Y_{\mathbf{m}}$ is given by performing the m_i -surgery along the i^{th} connected component of L for each i . The differential of this complex is also defined using cobordism maps associated with certain families of metrics.

For a detailed version of this theorem see Theorem 2.9 and Corollary 2.10. To have a closer look at Theorem 1, let L have one connected component and $Y_{(1)}$ (respectively, $Y_{(0)}$) be the result of 1-surgery (respectively, 0-surgery) along L . Let also $W : Y_{(1)} \rightarrow Y_{(0)}$ be the standard cobordism that is given by attaching a 2-handle to $[0,1] \times Y_{(1)}$ along L . Therefore, there is a chain map $\text{PFC}(W) : \text{PFC}(Y_{(1)}) \rightarrow \text{PFC}(Y_{(0)})$. In this special case, $\text{PFC}_L(Y)$ is the mapping cone of the chain map $\text{PFC}(W)$. As a consequence, there is an exact triangle of the following form:

$$\begin{array}{ccc} & \text{PFH}(Y) & \\ & \nearrow & \searrow \\ \text{PFH}(Y_{(0)}) & \longleftarrow & \text{PFH}(Y_{(1)}) \end{array} \quad (1)$$

In the more general case that L has more than one connected component, there is again a filtration on the chain complex $\text{PFC}_L(Y)$. This filtration determines a spectral sequence that abuts to $\text{PFH}(Y)$. This spectral sequence should be thought as the counterpart of the *surgery spectral sequence* of [16] for plane Floer homology. Similar spectral sequences for various other Floer homologies are established in [11, 3, 17]. Note that in all these cases, the pages of the spectral sequence depend on the framed link L and are not 3-manifold invariants.

Let K be a link in S^3 and $\Sigma(K)$ be the branched double cover of S^3 branched along K . The chain complexes given by Theorem 1 can be used to construct additional structures on $\text{PFH}(\Sigma(K))$ which are not otherwise apparent in the original chain complex. Let D be a link diagram for K . One can use this diagram to produce a framed link in $\Sigma(K)$ (cf. section 3). Therefore, it can be used as an input for Theorem 1 to construct a chain complex that is denoted by $\text{PKC}(D)$. We shall define two gradings on $\text{PKC}(D)$ which are called *homological* and δ -gradings. The differential of the chain complex $\text{PKC}(D)$ does not decrease the homological grading and increases the δ -grading by 1. We call a chain complex with a bi-grading that satisfies the above properties a *filtered \mathbb{Z} -graded chain complex*:

Theorem 2. *The homotopy type of $\text{PKC}(D)$, as a filtered \mathbb{Z} -graded chain complex, depends only on K and hence is a link invariant.*

For the definition of the homotopy type of a filtered \mathbb{Z} -graded chain complex see subsection 3.2. The homological filtration and the δ -grading of $\text{PKC}(D)$ induces a spectral sequence $\{\text{PKE}_r(K)\}$ where $\text{PKE}_r(K)$ is a bi-graded vector space. As an immediate corollary of Theorem 2, we have:

Corollary 1. *For each $r \geq 2$, there is a bi-graded vector space $\text{PKE}_r(K)$ that is an invariant of K .*

Remark 1. *The spectral sequence in Corollary 1 can be utilized to define a series of concordance homomorphisms. In fact, an equivariant version of $\text{PKC}(D)$ gives rise to other spectral sequences similar to that of Corollary 1. These spectral sequences are the starting point to define other concordance invariants. This circle of ideas will be explored in a forthcoming work.*

Corollary 1 provides us with a series of homological link invariants which begs for further study. In the last part of this article, we take up the task of understanding the first invariant in this list and show that it is related to *Khovanov homology*. Khovanov homology is a categorification of Jones polynomial defined in [8]. In [15], an elaborate modification of the definition of Khovanov homology was used to define another categorification of Jones polynomial that is known as odd Khovanov homology. Subsequently, Khovanov's original link invariant is also known as even Khovanov homology. Even and odd Khovanov homology agree when one uses a characteristic two ring to define these homological link invariants. However, if a characteristic zero coefficient ring is used, then these invariants diverge significantly from each other.

Theorem 3. *The invariant $\text{PKE}_2(K)$ is isomorphic to the Khovanov homology of the mirror image of K with coefficients in the field Λ .*

The field Λ has characteristic two and is the quotient of a ring $\tilde{\Lambda}$ with characteristic zero (cf. subsection 1.4). The vector space $\text{PFH}(Y)$ can be lifted to an invariant $\widetilde{\text{PFH}}(Y)$ which is a $\tilde{\Lambda}$ -module and is called the *oriented plane Floer homology* of Y . There is an analogue of the spectral sequence $\{\text{PKE}_r(K)\}$ in the context of oriented plane Floer homology:

Theorem 4. *There is a spectral sequence $\{\widetilde{\text{PKE}}_r(K)\}$ with coefficients in $\tilde{\Lambda}$ that abuts to $\widetilde{\text{PFH}}(\Sigma(K))$. The second page of this spectral sequence, $\widetilde{\text{PKE}}_2(K)$, is isomorphic to odd Khovanov homology of the mirror image of K with coefficients in the ring Λ .*

This article is organized as follows. In section 1, we focus on the definition of plane Floer homology of 3-manifolds and the cobordism maps. Section 2 is devoted to providing the proof of Theorem 1. In section 3, we use the results of section 2 in the special case of the branched double covers and upgrade plane Floer homology of branched double covers to a stronger invariant. In particular, Theorem 2 is proved in this section. In Section 4, we present a slight reformulation of odd Khovanov homology and prove Theorems 3 and 4.

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1 Abelian Anti-Self-Dual Connections

Plane Floer homology is defined with the aid of the moduli spaces of spin^c connections that satisfy a version of anti-self-duality equation. The local behavior of these spaces is governed by the anti-self-duality operator. In subsection 1.1, we recall the definition of appropriate versions of this operator that serve better for our purposes. Then we review the basic properties of this operator. These are fairly standard facts whose proofs are scattered in the literature. The moduli space of anti-self-dual spin^c connections are constructed in subsection 1.2. We use these spaces to define a primary version of plane Floer homology and cobordism maps in subsection 1.3. In the last subsection of this section, we discuss orientations of the moduli spaces and certain local coefficient systems on these spaces. These structures are the main ingredients in the definition of cobordism maps for oriented plane Floer homology in subsection 1.4.

1.1 Abelian ASD Operator

Suppose $W^\circ : Y_0 \rightarrow Y_1$ is a compact, connected cobordism between Riemannian 3-manifolds Y_0 and Y_1 . Although Y_1 can have several (possibly zero) connected components, we require that Y_0 be connected. We equip W° with a Riemannian metric such that the metric in a collar neighborhood of Y_0 and Y_1 is isometric to the product metric corresponding to the metrics on Y_0 and Y_1 . One can glue cylindrical ends $\mathbb{R}^{\leq 0} \times Y_0$ and $\mathbb{R}^{\geq 0} \times Y_1$ with the product metric to W° in order to produce a non-compact and complete Riemannian manifold denoted by W . For a real number δ , let ψ_δ be a smooth positive function on W such that its value for $(t, y) \in \mathbb{R}^{\leq 0} \times Y_0 \amalg \mathbb{R}^{\geq 0} \times Y_1$ is equal to $e^{\delta|t|}$. The weighted Sobolev space $L_{k,\delta}^2(W)$ is defined to be the space of functions f such that $\psi_\delta f \in L_k^2(W)$. Roughly speaking, if δ is positive, then the elements of this space have exponential decay over the ends of W . Otherwise, they are allowed to have controlled exponential growth. This Banach space is independent of the choice of ψ_δ . If different weights δ_0 and δ_1 are used on the ends $\mathbb{R}^{\leq 0} \times Y_0$ and $\mathbb{R}^{\geq 0} \times Y_1$, then the resulting Banach space is denoted by $L_{k,\delta_0,\delta_1}^2(W)$. Given a vector bundle E with an inner product, the definition of $L_{k,\delta}^2(W)$ can be extended to $L_{k,\delta}^2$ sections of E . The dual of the Banach space of $L_{k,\delta}^2$ sections of E is the space of $L_{k,-\delta}^2$ sections of the same vector bundle. The L_k^2 -inner product defines the pairing between these Banach spaces.

The exterior derivative gives rise to a map $d : L_{k,\delta}^2(W, \Lambda^l) \rightarrow L_{k-1,\delta}^2(W, \Lambda^{l+1})$. As usual the formal adjoint of d is denoted by $d^* : L_{k,\delta}^2(W, \Lambda^{l+1}) \rightarrow L_{k-1,\delta}^2(W, \Lambda^l)$. Since W is a Riemannian 4-manifold, a 2-form on W decomposes as a summation of a self-dual form and an anti-self-dual form. Let $d^+ : L_{k,\delta}^2(W, \Lambda^1) \rightarrow L_{k-1,\delta}^2(W, \Lambda^+)$ be the composition of d and the projection on the space of self-dual 2-forms. Define the *anti-self-duality operator (ASD operator)* to be the differential operator $-d^* \oplus d^+ : L_{k,\delta}^2(W, \Lambda^1) \rightarrow L_{k-1,\delta}^2(W, \Lambda^0 \oplus \Lambda^+)$.

In order to understand the ASD operator on a cobordism with cylindrical ends, we firstly consider the ASD operator on a cylinder $(a, b) \times Y$ with the product metric. A 1-form A and an anti-self-dual form B on this cylinder can be decomposed in the following way:

$$A = \alpha(t)dt + \beta(t) \quad B = \frac{1}{2}(dt \wedge \gamma(t) + *_3\gamma(t)) \quad (2)$$

where for each $t \in (a, b)$, $\alpha(t)$ is a 0-form, and $\beta(t), \gamma(t)$ are 1-forms on Y . Therefore, a 1-form on the cylinder is given by a 1-parameter family of 0-forms and 1-forms, and an anti-self-dual 2-form is determined by a 1-parameter family of 1-forms. Given this, we can rewrite the ASD operator as $\frac{d}{dt} - Q$ where Q is equal to the following differential operator acting on $\Lambda^0(Y) \oplus \Lambda^1(Y)$:

$$Q := \begin{pmatrix} 0 & d_3^* \\ d_3 & -*_3 d_3 \end{pmatrix} \quad (3)$$

Here d_3, d_3^* , and $*_3$ are the exterior derivative, its adjoint, and the Hodge operator on Y . The operator Q is an unbounded and self-adjoint operator acting on $L^2(Y, \Lambda^0 \oplus \Lambda^1)$. There exists an orthonormal basis $\{\phi_i\}_{i=1}^\infty$ for $L^2(Y, \Lambda^0 \oplus \Lambda^1)$ that consists of the eigenvectors of Q . If λ_i is the eigenvalue corresponding to ϕ_i , then $\{\lambda_i\}_{i=1}^\infty$ is a discrete subset of \mathbb{R} . Therefore the set of non-zero eigenvalues has an element with the smallest magnitude. Fix δ to be a positive number smaller than the magnitude of this element. The kernel of Q can be also identified as:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \ker(Q) \iff d\alpha = 0, d\beta = 0, d^*\beta = 0 \quad (4)$$

That is to say α is a constant function and β is a harmonic 1-form. Therefore, $\ker(Q)$ has dimension $b_1(Y) + 1$.

In general, a 1-form $a = \alpha(t)dt + \beta(t)$ on a cylinder can be decomposed with respect to the spectrum of Q in the following way:

$$\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \sum_i f_i(t)\phi_i \quad (5)$$

Now assume that a is defined on $\mathbb{R}^{\geq 0} \times Y$ (respectively, on $\mathbb{R}^{\leq 0} \times Y$) and $d^*(a) = 0, d^+(a) = 0$. Vanishing of a by the ASD operator implies that $f_i(t) = c_i e^{\lambda_i t}$. The 1-form a has a finite $L_{k,\delta}^2$ norm if and only if a has a finite L_k^2 -norm because if $\|a\|_{L_k^2}$ is finite, then $c_i = 0$ unless λ_i is negative (respectively, positive). This in turn suffices to ensure that $\|a\|_{L_{k,\delta}^2}$ is finite. For our purposes, we need to work with slightly larger Banach spaces. The *extended weighted Sobolev space* $L_{k,\text{ext}}^2(Y \times \mathbb{R}^{\geq 0}, \Lambda^1)$ consists of the 1-forms a that can be written as a sum $b + h$ where $b \in L_{k,\delta}^2(Y \times \mathbb{R}^{\geq 0}, \Lambda^1)$ and h is the pull-back of a harmonic 1-form on Y . Note that this decomposition is unique and hence $L_{k,\text{ext}}^2$ is a Banach bundle over the set of harmonic 1-forms with fiber $L_{k,\delta}^2$.

Going back to the case of a cobordism $W : Y_0 \rightarrow Y_1$, let δ be smaller than the smallest of the eigenvalues of the operator Q associated with the 3-manifold $\overline{Y_0} \amalg Y_1$. The Banach space $L_{k,\text{ext}}^2(W, \Lambda^1)$ for a cobordism W consists of the 1-forms that have finite L_k^2 norm on the compact subsets of W and their restrictions to the ends lie in $L_{k,\text{ext}}^2(Y_0 \times \mathbb{R}^{\leq 0}, \Lambda^1)$ and $L_{k,\text{ext}}^2(Y_1 \times \mathbb{R}^{\geq 0}, \Lambda^1)$.

Let C_0 and C_1 be the spaces of harmonic 1-forms on Y_0 and Y_1 . Also, suppose $\varphi : C_0 \oplus C_1 \rightarrow L_{k,\text{ext}}^2(W, \Lambda^1)$ is a linear map that sends an element $h \in C_0 \oplus C_1$ to a smooth 1-form that is equal to the pull back of h on the ends. This map can be constructed for example by the aid of appropriate cut-off functions on the ends. Each element of $L_{k,\text{ext}}^2(W, \Lambda^1)$ can be uniquely written as the sum $b + \varphi(h)$ where $b \in L_{k,\delta}^2(W, \Lambda^1)$ and $h \in C_0 \oplus C_1$. Therefore, $L_{k,\text{ext}}^2(W, \Lambda^1)$ is isomorphic to $L_{k,\delta}^2(W, \Lambda^1) \oplus C_0 \oplus C_1$. This decomposition can be used to equip $L_{k,\text{ext}}^2(W, \Lambda^1)$ with an inner product. The dual of this Banach space is $L_{k,-\delta}^2(W, \Lambda^1) \oplus C_0 \oplus C_1$. In the following, we will write ι_i for the projection of $L_{k,\text{ext}}^2(W, \Lambda^1)$ into C_i .

It is a standard fact that the ASD operator, acting on $L_{k,\delta}^2(W, \Lambda^1)$, (and hence on $L_{k,\text{ext}}^2(W, \Lambda^1)$) is elliptic and Fredholm. In the following lemma we characterize the kernel and the cokernel of the ASD operator. Firstly we need to introduce the following notation: let $H_{\text{comp}}^i(W)$ be the i^{th} de Rham cohomology group with compact support. The intersection pairing $Q : H_{\text{comp}}^2(W) \times H^2(W) \rightarrow \mathbb{Z}$ on W induces a non-degenerate bi-linear form on $I(W) := \text{image}(i : H_{\text{comp}}^2(W) \rightarrow H^2(W))$. This space can be decomposed as $I^+(W) \oplus I^-(W)$ where $I^+(W)$ and $I^-(W)$ are choices of a maximal positive definite and a maximal negative definite subspaces of $I(W)$. The dimension of these spaces are denoted by $b^+(W)$ and $b^-(W)$.

Lemma 1.1. *The kernel and the cokernel of $-d^* \oplus d^+ : L_{k,\text{ext}}^2(W, \Lambda^1) \rightarrow L_{k-1,\delta}^2(W, \Lambda^0 \oplus \Lambda^+)$ can be identified with $H^1(W; \mathbb{R})$ and $I^+(W) \oplus \mathbb{R}$, respectively.*

Proof. For $\alpha \in L_{k,\text{ext}}^2(W, \Lambda^1)$ in the kernel of $-d^* \oplus d^+$, we have:

$$0 = \int_W d\alpha \wedge d\alpha = \int_W |d^+\alpha|^2 - |d^-\alpha|^2 = - \int_W |d^-\alpha|^2 \implies d\alpha = 0 \quad (6)$$

where the first equality holds by Stokes' theorem and the assumption $k \geq 1$. Therefore, $\ker(-d^* \oplus d^+)$ consists of $L_{k,\text{ext}}^2$ 1-forms that are annihilated by both d and d^* . There is a natural map from this space, denoted by $\mathcal{H}^1(W)$, to $H^1(W)$. We shall show that this map is an isomorphism (cf. also [2]).

If $\alpha \in \mathcal{H}^1(W)$ represents a zero cohomology class, then the restriction of α to $\{-t\} \times Y_0 \amalg \{t\} \times Y_1$ for any value of t is an exact 1-form. But this 1-form is exponentially asymptotic to a harmonic 1-form on $\overline{Y_0} \amalg Y_1$. Therefore, the harmonic 1-form that α is asymptotic to on the ends of W vanishes and $\alpha \in L_{k,\delta}^2(W, \Lambda^1)$. In particular, α is an L^2 harmonic 1-form that represents the zero cohomology class. In the proof of Proposition 4.9 of [2] it is shown that any such α has to be zero. To address surjectivity, fix a cohomology class $\zeta \in H^1(W) = H^1(W^\circ)$ and let β be a closed 1-form on W° representing ζ . By subtracting an exact 1-form, we can assume that β in a collar neighborhood of W° is the pull-back of a harmonic 1-form on $\overline{Y_0} \amalg Y_1$. Note that $W^\circ \setminus \partial W^\circ$ is diffeomorphic to W . We can use such a diffeomorphism to pull back the 1-form β to W to produce the closed 1-form γ that its restriction to the ends is pull back of harmonic 1-forms and it represents ζ . Now the Fredholm alternative for the Fredholm operator $d^*d : L_{k+1,\delta}^2(W) \rightarrow L_{k-1,\delta}^2(W)$ implies that there exists $f \in L_{k+1,\delta}^2(W)$ such that $d^*df = d^*\gamma$. Thus $df \in L_{k,\delta}^2(W, \Lambda^1)$ and the harmonic 1-form $\gamma - df \in L_{k,\text{ext}}^2(W, \Lambda^1)$ represents the cohomology class ζ .

Suppose $(f, \omega) \in L_{k,-\delta}^2(W, \Lambda^0 \oplus \Lambda^+)$ is in the co-kernel of $-d^* \oplus d^+$. If α is a smooth 1-form

with compact support, then:

$$0 = \int_W \langle (d^* \alpha, d^+ \alpha), (f, \omega) \rangle = \int_W \langle d^* \alpha, f \rangle + \int_W \langle d\alpha, \omega \rangle = \int_W \langle \alpha, df + d^* \omega \rangle$$

Thus $df + d^* \omega = 0$. In particular, (f, ω) on the cylindrical ends is asymptotic to an element in $\ker(Q)$. This in particular justifies the following identities:

$$\|df\|_{L^2}^2 = \|d^* \omega\|_{L^2}^2 = -\langle df, d^* \omega \rangle = 0 \implies df = 0, d^* \omega = 0, d\omega = 0$$

Furthermore, if ω is asymptotic to $h \in C_0 \oplus C_1$, then for an element $h' \in C_0 \oplus C_1$, we have:

$$\int_W \langle (d^* \varphi(h'), d^+ \varphi(h')), (f, \omega) \rangle = \int_W \langle \varphi(h'), df + d^* \omega \rangle + \int_{Y_0 \amalg Y_1} \langle h', h \rangle = \int_{Y_0 \amalg Y_1} \langle h', h \rangle \implies h = 0$$

Consequently, $\omega \in L_{k,\delta}^2$. Therefore, cokernel of $-d^* \oplus d^+$ consists of the pairs (f, ω) where f is a constant function and ω is an $L_{k,\delta}^2$ self-dual and harmonic 2-form. By [2] this space represents $I^+(W) \oplus \mathbb{R}$. \square

Remark 1.2. Lemma 1.1 implies that the index of $-d^* \oplus d^+ : L_{k,\text{ext}}^2(W, \Lambda^1) \rightarrow L_{k-1,\delta}^2(W, \Lambda^0 \oplus \Lambda^+)$ is equal to $b^1(W) - b^+(W) - 1$. An examination of the long exact sequence of cohomology groups for the pair $(W, Y_0 \amalg Y_1)$ shows that this number is equal to:

$$-\frac{\chi(W) + \sigma(W)}{2} + \frac{b_1(Y_0) + b_1(Y_1)}{2} - \frac{b_0(Y_0) + b_0(Y_1)}{2}.$$

There are two other operators that are of interest to us:

$$\mathcal{D}_W : L_{k,\text{ext}}^2(W, \Lambda^1) \rightarrow L_{k-1,\delta}^2(W, \Lambda^0)_0 \oplus L_{k-1,\delta}^2(W, \Lambda^+) \oplus C_1 \quad \mathcal{D}'_W : L_{k,-\delta,\delta}^2(W, \Lambda^1) \rightarrow L_{k,-\delta,\delta}^2(W, \Lambda^0 \oplus \Lambda^+)$$

$$\mathcal{D}_W(\alpha) := (-d^*(\alpha), d^+(\alpha), \iota_1(\alpha))$$

$$\mathcal{D}'_W(\alpha) := (-d^*(\alpha), d^+(\alpha))$$

where $L_{k-1,\delta}^2(W, \Lambda^0)_0$ is the subspace of $L_{k-1,\delta}^2(W, \Lambda^0)$ that consists of the functions that their integrals over W vanish. An argument similar to the proof of Lemma 1.1 shows that:

$$\ker(\mathcal{D}_W) = \ker(\mathcal{D}'_W) = \{\alpha \in L_{k,\text{exp}}^2(W, \Lambda^1) \mid d\alpha = 0, d^* \alpha = 0, \iota_1(\alpha) = 0\} \quad (7)$$

$$\text{coker}(\mathcal{D}_W) = \text{coker}(\mathcal{D}'_W) = \{\omega \in L_{k,-\delta,\delta}^2(W, \Lambda^+) \mid d\omega = 0\} \quad (8)$$

A step in the proof of (7) involves showing that if $\alpha \in L_{k,-\delta,\delta}^2(W, \Lambda^1)$ is a closed and co-closed 1-form then $\alpha \in L_{k,\text{ext}}^2(W, \Lambda^1)$ with $\iota_1(\alpha) = 0$. Because $\alpha \in L_{k,-\delta,\delta}^2$, the 1-form α is asymptotic to zero on the end Y_1 . On the other hand, the relations $d\alpha = 0$ and $d^* \alpha = 0$ assert that α is asymptotic to an element of $\ker(Q)$ on the end Y_0 . Since Y_0 is connected, an application of Stokes theorem for the closed 3-form $*\alpha$ shows that this element in $\ker(Q)$ cannot have a component in $\Omega^0(Y)$ and hence $\alpha \in L_{k,\text{ext}}^2(W, \Lambda^1)$. Lemma 1.1 shows that the index of the operator \mathcal{D}_W (and hence \mathcal{D}'_W) is equal to:

$$-\frac{\chi(W) + \sigma(W)}{2} + \frac{b_1(Y_0) - b_1(Y_1)}{2} - \frac{b_0(Y_0) + b_0(Y_1)}{2} + 1 \quad (9)$$

A *family of metrics* on a cobordism W° parametrized by a manifold G is a fiber bundle \mathfrak{W}° with the base G and the fiber W° that is equipped with a *partial metric*. A partial metric on \mathfrak{W}° is a 2-tensor $g \in \Gamma(T\mathfrak{W}^\circ \otimes T\mathfrak{W}^\circ)$ such that its restriction to each fiber is a metric. Furthermore, we assume that in a collar neighborhood of the boundary the metric is the product metric corresponding to the fixed metrics on Y_0 and Y_1 . That is to say, there exists a sub-bundle $([0, 1] \times Y_0 \amalg [-1, 0] \times Y_1) \times G$ of \mathfrak{W}° such that the restriction of the partial metric to this sub-bundle is the product metric for the fixed metric metric on ∂W . Adding the cylindrical ends to the fibers of \mathfrak{W}° results in a fiber bundle over G such that each fiber is diffeomorphic to W . We will write \mathfrak{W} for this family of metrics with cylindrical ends.

The family of metrics \mathfrak{W} define a family of Fredholm operators parametrized by G . For each element $g \in G$, the corresponding operator is \mathcal{D}_{W^g} where W^g is the fiber of \mathfrak{W} over $g \in G$. The index of this family of Fredholm operators is an element of the real K -group of the base G (cf. [1]), and is denoted by $\text{ind}(\mathfrak{W}; G)$. We can equivalently use the operator \mathcal{D}'_W to define $\text{ind}(\mathfrak{W}; G)$. However, the operator \mathcal{D}_W is more suitable for the geometrical set up of this paper. There is only one point (Lemma 1.3) that it is more convenient for us to work with \mathcal{D}'_W .

Above discussion can be further generalized by working with *broken Riemannian metrics* [11]. A broken Riemannian metric, strictly speaking, is not a metric on W . However, it can be considered as the limit of a sequence of metrics on W . We firstly discuss model cases for such family of metrics. Let T be a compact orientable codimension 1 sub-manifold of W with i connected components T_1, \dots, T_i . Also, define $T_0 := Y_0, T_{i+1} = Y_1$. We call T a *cut*, if removing T decomposes W into a union of $i + 1$ cobordisms W_0, \dots, W_i where W_k is a cobordism with exactly one incoming end, which is T_k , and several (possibly zero) outgoing ends. For each $1 \leq k \leq i + 1$, the connected 3-manifold T_k appears as one of the outgoing ends of a cobordism that is denoted by $W_{o(k)}$. The simplest arrangement is when $W_k : T_k \rightarrow T_{k+1}$ and $W = W_0 \circ \dots \circ W_i$. In this case $o(k) = k - 1$. For the most part, we are interested in such cuts. However, we need the more general cuts in the proof of exact triangles. It is worthwhile to point out that the formula (9) for such decompositions of W is additive, i.e., $\text{ind}(\mathcal{D}_W) = \text{ind}(\mathcal{D}_{W_0}) + \dots + \text{ind}(\mathcal{D}_{W_i})$.

A broken metric on W with a cut along T is a metric g with cylindrical ends on the union of the cobordisms $W_0 \amalg \dots \amalg W_i$ such that the product metrics on $\mathbb{R}^{\geq 0} \times T_k \subset W_{o(k)}$ and $\mathbb{R}^{\leq 0} \times T_k \subset W_k$ are modeled on the same metric of T_k . Given g , we can construct a family of (possibly broken) metrics parametrized by $[0, \infty]^i$ on W in the following way: for $(t_1, \dots, t_i) \in [0, \infty]^i$, if $t_k \neq \infty$, we remove the cylindrical ends $\mathbb{R}^{\geq 0} \times T_k \subset W_{o(k)}$, $\mathbb{R}^{\leq 0} \times T_k \subset W_k$, and glue $[-t_k, t_k] \times T_k$ by identifying $\{-t_k\} \times T_k$ with $\{0\} \times T_k \subset W_{o(k)}$ and $\{t_k\} \times T_k$ with $\{0\} \times T_k \subset W_k$. On the remaining points of W , we use the same metric as g . More generally, let \mathfrak{W}_k be a family of (non-broken) metrics on W_k parametrized with G_k such that for each k , the metrics on the ends $\mathbb{R}^{\leq 0} \times T_k$ and $\mathbb{R}^{\geq 0} \times T_k$, induced by \mathfrak{W}_k and $\mathfrak{W}_{o(k)}$, are modeled on the same metric of T_k . Then we can construct a family of (possibly broken) metrics on W parametrized by $[0, \infty]^i \times G_0 \times \dots \times G_i$. From this point on, we work with the following more general definition of family of metrics: a family of metrics \mathfrak{W} on the cobordisms W parametrized by the cornered manifold G is a bundle over G with a partial metric such that for any codimension i face of G , the family over the interior of the face has the form $\mathfrak{W}_0 \times \dots \times \mathfrak{W}_i$. Furthermore, the family over a neighborhood of the interior of this face has the form of the above family parametrized by $[0, \infty]^i \times G_0 \times \dots \times G_i$. For an organized review of manifolds with corners we refer the reader to [13]. Our treatment of cornered manifolds in this

paper, for the sake of simplicity of exposition, is rather informal.

Lemma 1.3. *Suppose \mathfrak{W} is a family of metrics on W parametrized by G . Then there exists an element of $KO(G)$, called the index bundle of the family \mathfrak{W} and denoted by $\text{ind}(\mathfrak{W}; G)$, such that the following holds. Let G' be a face of G corresponding to a cut $T \subset W$. Assume that $W \setminus T = W_0 \coprod \dots \coprod W_i$, $G' = G_0 \times \dots \times G_i$ and the family of metrics over G' has the form $\mathfrak{W}_0 \times \dots \times \mathfrak{W}_i$ where \mathfrak{W}_k is a family of (non-broken) metrics on W_k parametrized by G_k . Then the restriction of the index bundle to G' is isomorphic to the following element of $KO(G')$:*

$$\text{ind}(\mathfrak{W}_0, G_0) \oplus \dots \oplus \text{ind}(\mathfrak{W}_i, G_i) \quad (10)$$

Note that \mathfrak{W}_k consists of non-broken metrics and hence $\text{ind}(\mathfrak{W}_k, G_k)$ in (10) is already defined and we do not need to appeal to the lemma to define it. With a slight abuse of notation, $\text{ind}(\mathfrak{W}_j, G_j)$ in (10) denotes an element of $KO(G')$ which is given by the pull-back of $\text{ind}(\mathfrak{W}_j, G_j) \in KO(G_i)$ via the projection map.

Proof. This lemma is a global manifestation of the additivity of the numerical index of the operator \mathcal{D}_W . Firstly let $\mathfrak{W}_0 \times \dots \times \mathfrak{W}_i$ be a family of broken metrics parametrized by $G_1 \times \dots \times G_i$ on W corresponding to a decomposition of W to cobordisms W_0, \dots, W_i . Here we assume that G_j is a compact space (not necessarily a manifold) that parametrizes the family of non-broken metrics \mathfrak{W}_j . These families determine a family of metrics parametrized by $[0, \infty]^i \times G_0 \times \dots \times G_i$ on W . If T_0 is large enough and one uses the operator \mathcal{D}'_W in the definition of the index bundles, then the arguments in [5, section 3.3] shows that the pull-back of the index bundle:

$$\text{ind}(\mathfrak{W}_0, G_0) \oplus \dots \oplus \text{ind}(\mathfrak{W}_i, G_i)$$

to $(T_0, \infty]^i \times G_0 \times \dots \times G_i$ gives an element of $KO((T_0, \infty]^i \times G_0 \times \dots \times G_i)$ whose restriction to each face is isomorphic to the index bundle of the corresponding family of metrics. In the general case, the parametrizing set G is a union of the sets of the above form. As it can be seen from the arguments of [5, section 3.3], the isomorphisms in the intersection of these sets can be made compatible. Therefore, one can construct the desired determinant bundle over G . \square

The orientation bundle of $\text{ind}(\mathfrak{W}; G)$ forms a $\mathbb{Z}/2\mathbb{Z}$ -bundle over G which is called the *determinant bundle* of the family of metrics \mathfrak{W} and is denoted by $o(\mathfrak{W})$. For each $g \in G$ the fiber $o(\mathfrak{W})|_g$ consists of the set of orientations of the line $\Lambda^{\max}(\ker(\mathcal{D}_g)) \otimes (\Lambda^{\max}(\text{coker}(\mathcal{D}_g)))^*$.

As it is mentioned in the proof of Lemma 1.3, in the construction of the index bundles, we need to identify the index bundles for broken metrics with the index of non-broken metrics that are converging to the broken ones. These identifications (which are generalization of those of [5]) involve choosing cut-off functions and hence are far from being unique. However, the set of choices is contractible. Therefore, any two different choices in the construction of the index bundles give rise to isomorphic determinant bundles and the choice of the isomorphism is unique.

For a cobordism W and an arbitrary metric g on W , the determinant bundle for the one point set $\{g\}$ consists of two points. A homology orientation for W is a choice of one of these two points. Any other metric g' can be connected to g by a path of metrics. This path produces an isomorphism

of the determinant bundles for g and g' . Furthermore, this isomorphism is independent of the choice of the path. As a result, choice of the homology orientation for one metric determines a canonical choice of the homology orientation for any other metric. Therefore, it is legitimate to talk about homology orientations of W without any reference to a metric on W . We will write $o(W)$ for the set of homology orientations of W .

Suppose the cobordism W is the composition of cobordisms $W_0 : Y_0 \rightarrow Y_1$ and $W_1 : Y_1 \rightarrow Y_2$. Let also g_0 and g_1 be metrics on these two cobordisms. There is a 1-parameter family of metrics parametrized by $(0, \infty]$ such that the metric over ∞ is the broken one on $W = W_0 \circ W_1$ determined by g_0 and g_1 , and the metrics over the other points of $(0, \infty]$ are non-broken. The index bundle for this family over the point ∞ has the form $\text{ind}(W_0^{g_0}) \oplus \text{ind}(W_1^{g_1})$. Consequently, this family of metrics produces an isomorphism of $o(W_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} o(W_0)$ and $o(W)$. In particular, if W_0 and W_1 are given homology orientations, then W inherits a homology orientation that is called the *composition* of the homology orientations of W_0 and W_1 . As in the previous paragraph it is easy to see that the composition of homology orientations is independent of the choice of g_0 and g_1 .

Suppose W_0, \dots, W_i is a list of cobordisms with homology orientations. These homology orientations can be used to define a homology orientation for $W_0 \coprod \dots \coprod W_i$ which is denoted by $o(W_0, \dots, W_i)$. This homology orientation depends on the order of W_0, \dots, W_i . For example:

$$o(W_0, \dots, W_{j+1}, W_j, \dots, W_i) = (-1)^{\text{ind}(\mathcal{D}W_j) \cdot \text{ind}(\mathcal{D}W_{j+1})} o(W_0, \dots, W_j, W_{j+1}, \dots, W_i)$$

Next, suppose that there is a cut T in a cobordism W such that $W \setminus T = W_0 \coprod \dots \coprod W_i$. Lemma 1.3 asserts that the homology orientations of W_0, \dots, W_i define a homology orientation for W . If it does not make any confusion, we denote this homology orientation of W with $o(W_0, \dots, W_i)$, too. This is a generalization of the composition of homology orientations in the previous paragraph.

1.2 Abelian ASD equation

Suppose Y is a (possibly disconnected) closed Riemannian 3-manifold. A spin^c structure \mathfrak{t} on Y is a principal $\text{Spin}(3)$ -bundle (or equivalently a principal $U(2)$ -bundle) P such that the induced $SO(3)$ -bundle, determined by the adjoint action $\text{ad} : \text{Spin}^c(3) \cong U(2) \rightarrow SO(3)$, is identified with the framed bundle of TY . We can also use the determinant map $\det : U(2) \rightarrow U(1)$ to construct a complex line bundle that is called the *determinant bundle of \mathfrak{t}* and is denoted by $L_{\mathfrak{t}}$. A smooth connection \underline{B} on \mathfrak{t} is spin^c if the induced connection on $\text{ad}(\mathfrak{t})$ is the Levi-Civita connection. The spin^c connection \underline{B} also determines a connection on $L_{\mathfrak{t}}$ that is called the *central part* of \underline{B} and is denoted by B . Since the data of connections on \mathfrak{t} is equivalent to that of connections on TY and $L_{\mathfrak{t}}$, a spin^c connection \underline{B} is uniquely determined by its central part. We will write $A_f(Y, \mathfrak{t})$ for the space of all spin^c connections on \mathfrak{t} with flat central parts. Note that $L_{\mathfrak{t}}$ admits a flat connection if and only if $c_1(L_{\mathfrak{t}})$ is a torsion element of $H^2(Y; \mathbb{Z})$, and then any two flat connections on this bundle differs by a closed 1-form on Y . Therefore, if $c_1(L_{\mathfrak{t}})$ is torsion, then $A_f(Y, \mathfrak{t})$ is an affine space modeled on the space of closed 1-forms.

An automorphism of the spin^c structure \mathfrak{t} is a smooth automorphism of \mathfrak{t} as a principal $U(2)$ -bundle that acts trivially on the tangent bundle of Y . The *gauge group of Y* , denoted by $\mathcal{G}(Y)$, is the group of all such elements, and can be identified with the space of smooth maps from Y to

S^1 . The space $\mathcal{G}(Y)$ acts on the space of spin^c connections by pulling back the connections. Given $u : Y \rightarrow S^1$ and a spin^c connection \underline{B} on \mathfrak{t} , this action sends \underline{B} to a connection with the central part $B - 2u^{-1}du$. Therefore, this action changes the central part B by the twice of an integral closed 1-form. The quotient of $A_f(Y, \mathfrak{t})$ with respect to the action of $\mathcal{G}(Y)$ is denoted by $\mathcal{R}(Y, \mathfrak{t})$. Let \mathfrak{t} be a torsion spin^c structure, and fix a base point in $A_f(Y, \mathfrak{t})$. Then $\mathcal{R}(Y, \mathfrak{t})$ can be identified with $J(Y) := H^1(Y; \mathbb{R})/2H^1(Y; \mathbb{Z})$ which is a rescaling of the Jacobian torus of Y . We will write $\mathcal{R}(Y)$ for the union $\cup_{\mathfrak{t}} \mathcal{R}(Y, \mathfrak{t})$. Therefore, this space has a copy of $J(Y)$ for each torsion spin^c structure \mathfrak{t} .

Next consider a cobordism $W^\circ : Y_0 \rightarrow Y_1$ as in the previous section. In particular, we assume that Y_0 is a connected 3-manifold. As in the case of 3-manifolds, a spin^c structure \mathfrak{s} on W° (or equivalently W) is a principal $\text{Spin}^c(4)$ -bundle such that the induced $SO(4)$ -bundle, induced by the adjoint action $\text{ad} : \text{Spin}^c(4) \rightarrow SO(4)$, is identified with the frame bundle of TW° . This principal bundle gives rise to a determinant bundle that is denoted by $L_{\mathfrak{s}}$. A spin^c connection \underline{A} on \mathfrak{s} is also uniquely determined by the induced connection on $L_{\mathfrak{s}}$. Again, this induced connection is called the central part of \underline{A} and is denoted by A . Here we are interested in spin^c connections defined on W (rather than W°) and for analytical purposes we have to work with connections which are in an appropriate Sobolev space and behaves in a controlled way on the ends of W . To give the precise definition of this space of connections, fix a connection \underline{A}_0 on \mathfrak{s} such that the restriction of \underline{A}_0 to the ends $\mathbb{R}^{\leq 0} \times Y_0 \amalg \mathbb{R}^{\geq 0} \times Y_1$ is the pull-back of flat connections on Y_0 and Y_1 . The *space of weighted Sobolev spin^c connections* for the spin^c structure \mathfrak{s} is defined as:

$$\mathcal{A}(W, \mathfrak{s}) := \{\underline{A} \mid A = A_0 + ia, a \in L_{k, \text{ext}}^2(W, \Lambda^1)\}$$

where k is an integer number greater than 1.

The restriction of \mathfrak{s} to $\{0\} \times Y_0$ and $\{0\} \times Y_1$ gives rise to the spin^c structures \mathfrak{t}_0 and \mathfrak{t}_1 on Y_0 and Y_1 , respectively. Furthermore, the limit of $\underline{A}|_{\{-t\} \times Y_0}$ and $\underline{A}|_{\{t\} \times Y_1}$ as t goes to ∞ induces spin^c connections on \mathfrak{t}_0 and \mathfrak{t}_1 with flat central parts. Therefore, there is a map $R : \mathcal{A}(W, \mathfrak{s}) \rightarrow A_f(Y_0, \mathfrak{t}_0) \times A_f(Y_1, \mathfrak{t}_1)$. Note that if \mathfrak{t}_i is non-torsion, then $\mathcal{A}(W, \mathfrak{s})$ is empty, and from now on we assume that $\mathfrak{s}|_{Y_i}$ is torsion.

A smooth map $w : W \rightarrow S^1$ is *harmonic on the ends* if the restriction of w to $\mathbb{R}^{\leq 0} \times Y_0 \amalg \mathbb{R}^{\geq 0} \times Y_1$ is pull-back of a harmonic circle-valued map on Y_0 and Y_1 . The *weighted gauge group* $\mathcal{G}(W)$ is defined as follows:

$\mathcal{G}(W) := \{u : W \rightarrow S^1 \mid \exists v, w : u = vw, v \in L_{k+1, \delta}^2(W, S^1), w : W \rightarrow S^1 \text{ is harmonic on the ends}\}$
 Here $L_{k+1, \delta}^2(W, S^1)$ is the set of maps $v : Y \rightarrow S^1$ such that $v^{-1}dv \in L_{k, \delta}^2(W, \mathbb{C})$. Roughly speaking, an element of $\mathcal{G}(W)$ is exponentially asymptotic to harmonic circle-valued maps on Y_0 and Y_1 . It is straightforward to see $\mathcal{G}(W)$ is an abelian Banach Lie group with the Lie algebra $L_{k+1, \delta}^2(W, i\mathbb{R})$. The set of connected components of $\mathcal{G}(W)$ can be identified with $H^1(W, \mathbb{Z}) \cong [W, S^1]$. The gauge group $\mathcal{G}(W)$ acts on $\mathcal{A}(W, \mathfrak{s})$ by pulling-back the connections. If $u : W \rightarrow S^1$ is an element of $\mathcal{G}(W)$ and \underline{A} is a spin^c connection, then the action of u maps \underline{A} to a spin^c connection with the central part $A - 2u^{-1}du$. The stabilizer of any point in $\mathcal{A}(W, \mathfrak{s})$ consists of the constant maps in $\mathcal{G}(W)$.

The quotient space $\mathcal{B}(W, \mathfrak{s}) := \mathcal{A}(W, \mathfrak{s})/\mathcal{G}(W)$ is a smooth Banach manifold and is called the *configuration space of weighted Sobolev spin^c connections* on \mathfrak{s} . The tangent space at $[\underline{A}] \in \mathcal{B}(W, \mathfrak{s})$

is isomorphic to the kernel of $d^* : L_{k,\text{ext}}^2(W, i\Lambda^1) \rightarrow L_{k-1,\delta}^2(W, i\mathbb{R})_0$. Moreover, the action of the gauge group is compatible with $R : \mathcal{A}(W, \mathfrak{s}) \rightarrow A_f(Y_0, \mathfrak{t}_0) \times A_f(Y_1, \mathfrak{t}_1)$. Thus R produces the restriction maps $r_0 : \mathcal{B}(W, \mathfrak{s}) \rightarrow \mathcal{R}(Y_0, \mathfrak{t}_0)$ and $r_1 : \mathcal{B}(W, \mathfrak{s}) \rightarrow \mathcal{R}(Y_1, \mathfrak{t}_1)$. We will write $\mathcal{B}(W)$ for the union $\bigcup_{\mathfrak{s}} \mathcal{B}(W, \mathfrak{s})$. There are resstriction maps from $\mathcal{B}(W)$ to $\mathcal{R}(Y_0)$ and $\mathcal{R}(Y_1)$ which are also denoted by r_0 and r_1 .

Remark 1.4. In fact, $\mathcal{B}(W, \mathfrak{s})$ can be globally identified with the quotient of $\ker(d^*)$ with respect to the action of the discrete group $H^1(W, \mathbb{Z})$.

By definition, the central part of a connection $\underline{A} \in \mathcal{A}(W, \mathfrak{s})$ can be written as a sum of the smooth connection A_0 that is flat on the ends and an imaginary 1-form $ia \in L_{k,\text{ext}}^2(W, i\Lambda^1)$. Thus $F_c(\underline{A})$, the curvature of A , is an element of $L_{k-1,\delta}^2(W, i\Lambda^2)$. Consequently, $F_c^+(\underline{A})$, the self-dual part of the curvature of A , is in $L_{k-1,\delta}^2(W, i\Lambda^+)$. The action of the gauge group does not change the curvature and we have a well-defined map $F_c^+ : \mathcal{B}(W, \mathfrak{s}) \rightarrow L_{k-1,\delta}^2(W, i\Lambda^+)$. In local coordinates around $[\underline{A}]$, this map is equal to $d^+ + F_c^+(\underline{A}) : \ker(d^*) \subset L_{k,\text{ext}}^2(W, i\Lambda^1) \rightarrow L_{k-1,\delta}^2(W, i\Lambda^+)$ where d^+ is the self-dual part of the exterior derivative. For an arbitrary $\nu \in L_{k-1,\delta}^2(W, \Lambda^2)$, define the following moduli spaces:

$$M_\nu(W, \mathfrak{s}) := \{[\underline{A}] \in \mathcal{B}(W, \mathfrak{s}) \mid F_c^+(\underline{A}) = i\nu^+\}$$

$$M_\nu(W) := \bigcup_{\mathfrak{s}} M_\nu(W, \mathfrak{s})$$

If there is a chance of confusion about the metric g on W which is used to define $M_\nu(W)$, we denote this moduli space with $M_\nu(W^g)$. With a slight abuse of notation, we will also write $r_0 : M_\nu(W) \rightarrow \mathcal{R}(Y_0)$ and $r_1 : M_\nu(W) \rightarrow \mathcal{R}(Y_1)$ for the restriction maps restricted to the moduli space $M_\nu(W)$.

Lemma 1.5. *The map $F_c^+ : \mathcal{B}(W) \rightarrow L_{k-1,\delta}^2(W, i\Lambda^2)$ is smooth and Fredholm with $\dim(\ker) = b^1(W)$ and $\dim(\text{coker}) = b^+(W)$.*

Proof. Locally, the map F_c^+ is affine and hence is smooth. The rest of the lemma follows easily from Lemma 1.1. \square

This lemma implies that the index of F_c^+ is equal to:

$$b^1(W) - b^+(W) = -\frac{\chi(W) + \sigma(W)}{2} + \frac{b_1(Y_0) + b_1(Y_1)}{2}.$$

A global description of the moduli space $M_\nu(W)$ is given in the following lemma:

Lemma 1.6. *The moduli space $M_\nu(W, \mathfrak{s})$ is either empty or can be identified with $J(W) := H^1(W; \mathbb{R})/2H^1(W; \mathbb{Z})$, and this identification is canonical up to a translation on $J(W)$. If $b^+(W) > 0$, then ν can be chosen in such a way that $M_\nu(W, \mathfrak{s})$ is empty, and in the case $b^+(W) = 0$, the moduli space, for any choice of ν , is isomorphic to $J(W)$.*

Proof. Suppose $M_\nu(W, \mathfrak{s})$ is not empty and $[\underline{A}_0] \in M_\nu(W, \mathfrak{s})$. For $[\underline{A}] \in \mathcal{B}(W, \mathfrak{s})$:

$$F_c^+(\underline{A}) = F_c^+(\underline{A}_0 + (\underline{A} - \underline{A}_0)) = F_c^+(\underline{A}_0) + d^+(A - A_0) = \nu^+ + d^+(A - A_0)$$

Thus $[\underline{A}] \in M_\nu(W, \mathfrak{s})$ if and only if $d^+(A - A_0) = 0$, which in turn is equivalent to $d(A - A_0) = 0$ by (6). Two connections \underline{A} and \underline{A}' give rise to the same element of $M_\nu(W, \mathfrak{s})$ if $A' = A - 2u^{-1}du$ for $u \in \mathcal{G}(W)$, i.e., A' and A differ by twice of an integral closed 1-form. In summary, after fixing \underline{A}_0 , $M_\nu(W, \mathfrak{s})$ can be identified with $J(W) = H^1(W; \mathbb{R})/2H^1(W; \mathbb{Z})$.

In order to prove the second part, start with an arbitrary spin^c connection \underline{A}_0 and try to modify this element by $a \in L_{k,\delta}^2(W, \Lambda^1)$:

$$F^+(A_0 + ia) = i\nu^+ \iff d^+ia = i\nu^+ - F^+(A_0) \quad (11)$$

Lemma 1.1 states that d^+ is surjective in the case $b^+(W) = 0$, and hence Equation (11) has solution for any choice of ν . If $b^+(W) > 0$, then d^+ is not surjective and ν can be picked such that Equation (11) does not have any solution. \square

Remark 1.7. By our previous discussions, $J(Y)$ and $J(W)$ act on $\mathcal{R}(Y, \mathfrak{t})$ and $M_\nu(W, \mathfrak{s})$, respectively. These actions give isomorphisms of $M_\nu(W, \mathfrak{s})$, $\mathcal{R}(Y, \mathfrak{t})$ with $J(W)$, $J(Y)$, in the case that $M_\nu(W, \mathfrak{s})$, $\mathcal{R}(Y, \mathfrak{t})$ are non-empty. If these isomorphisms are chosen appropriately, then the restriction map $r = (r_0, r_1) : M_\nu^g(W, \mathfrak{s}) \rightarrow \mathcal{R}(Y_0, \mathfrak{t}_0) \times \mathcal{R}(Y_1, \mathfrak{t}_1)$ can be identified with the restriction map $i = (i_0, i_1) : J(W) \rightarrow J(Y_0) \times J(Y_1)$, induced by the inclusion of Y_0 and Y_1 in W .

We can also define similar moduli spaces in the case that W is equipped with a broken metric g . Suppose g is a broken metric in correspondence with the cut T , and removing T decomposes W into the union of cobordisms W_0, \dots, W_i . Let T_1, \dots, T_i be the connected components of T such that T_j is the incoming end of the cobordism W_j and one of the outgoing ends of $W_{o(j)}$. Let also $T_0 = Y_0$ and $T_{i+1} = Y_1$. The metric g induces a metric g_j on the cobordism W_j and hence we can define the moduli spaces $M_{\nu_j}(W_j^{g_j})$ for a perturbation term ν_j . Since T_j is the incoming end of W_j , there is a restriction map $r_j^0 : M_{\nu_j}(W_j^{g_j}) \rightarrow \mathcal{R}(T_j)$. Furthermore, there is a restriction map from $M_{\nu_{o(j)}}(W_{o(j)}^{g_{o(j)}})$ to $\mathcal{R}(T_j)$ which will be denoted by r_j^1 . Define:

$$M_\nu(W^g) := \{([\underline{A}_0], \dots, [\underline{A}_i]) \in M_{\nu_0}(W_0^{g_0}) \times \dots \times M_{\nu_i}(W_i^{g_i}) \mid r_0^j([\underline{A}_j]) = r_1^j([\underline{A}_{o(j)}]) \quad 1 \leq j \leq i\}$$

One can still define the restriction maps $r_0 : M_\nu(W^g) \rightarrow \mathcal{R}(Y_0)$ and $r_1 : M_\nu(W^g) \rightarrow \mathcal{R}(Y_1)$ with the aid of the restriction maps on the moduli spaces $M_{\nu_0}(W_0^{g_0})$ and $M_{\nu_{o(i+1)}}(W_{o(i+1)}^{g_{o(i+1)}})$.

The broken metric g determines a family of metrics parametrized with $[0, \infty]^i$ on W . For each element $g' = (t_1, \dots, t_i) \in [1, \infty]^i$ of this family and $([\underline{A}_0], \dots, [\underline{A}_i]) \in M_\nu(W^g)$, we can construct a spin^c connection on W . To that end, fix a function $\chi : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that $f(t) = 1$ for $t \leq \frac{1}{2}$ and $f(t) = 0$ for $t \geq 1$. This function can be used to define a function $\chi_{t_j} : W_j \rightarrow \mathbb{R}^{\geq 0}$ for $0 \leq j \leq i$. The function χ_{t_j} is defined to be equal to 1 on the compact part W_j° . For a point $(t, y) \in \mathbb{R}^{\leq 0} \times Y_j$ in the incoming end, define $\chi_{t_j}(t, y) := \chi(-\frac{t}{t_j})$. Use a similar definition to extend χ_{t_j} to the outgoing ends. Now suppose \underline{A}'_j is the spin^c connection on W_j which is equal to \underline{A}_j on W_j° and is equal to $\chi_{t_j}\underline{A}_j + (1 - \chi_{t_j})\pi_2^*(R_k(\underline{A}_j))$ for each end of the form $\mathbb{R}^{\leq 0} \times T_k$ or

$\mathbb{R}^{\geq 0} \times T_k$ where $1 \leq k \leq i$. Here $\pi_2 : \mathbb{R}^{\leq 0} \times T_k \amalg \mathbb{R}^{\geq 0} \times T_k \rightarrow T_k$ is the projection to the second factor, and $R_k : \mathcal{A}(W) \rightarrow A_f(T_k)$ is the analogue of the map R , defined earlier. Therefore, \underline{A}'_j is a spin^c connection whose restriction to the subset $(-\infty, -t_j] \times T_j$ is pull back of a connection on T_j . A similar property holds for the outgoing ends. Because each cobordism W_j has one incoming end, there is a unique way to glue the connections \underline{A}'_j to define an element $\phi_{g'}(\underline{A}_0, \dots, \underline{A}_i)$ of the configuration space of spin^c connections on W with the (possibly broken) metric g' . Note that the construction of this element depends on the choice of the lift of $[\underline{A}_j] \in \mathcal{B}(W_j)$ to $\underline{A}_j \in \mathcal{A}(W_j)$. However, for a non-broken metric $g' \in [1, \infty)^i$, $\phi_{g'}(\underline{A}_0, \dots, \underline{A}_i)$ determines a spin^c connection on W that is independent of the choice of g' . We assign this spin^c connection to $([\underline{A}_0], \dots, [\underline{A}_i])$. In particular, the partition of the moduli space $M_\nu(W^g)$ with respect to the spin^c structures is still well-defined in the case of broken metrics.

Let \mathfrak{W} be a family of metrics on W parametrized by a cornered manifold G . For each choice of integers l and k , we shall construct a space $\mathcal{V}_k^l(\mathfrak{W})$ with a projection map $\pi : \mathcal{V}_k^l(\mathfrak{W}) \rightarrow G$. The fiber of π over a non-broken metric $g \in G$ is equal to $L_{k,\delta}^2(W^g, \Lambda^l)$. These fibers together form a Banach bundle on the interior of G . Let $g \in G$ be a broken metric determined by a Riemannian metric on the decomposition $W_0 \amalg \dots \amalg W_i$ of the complement of a cut T in W . We define the fiber of $\mathcal{V}_k^l(\mathfrak{W})$ over g to be equal to $L_{k,\delta,c}^2(W_0, \Lambda^l) \oplus L_{k,c,c}^2(W_1, \Lambda^l) \oplus \dots \oplus L_{k,c,c}^2(W_{i-1}, \Lambda^l) \oplus L_{k,c,\delta}^2(W_i, \Lambda^l)$. The space $L_{k,\delta,c}^2(W_0, \Lambda^l)$ is the subspace of elements of $L_{k,\delta}^2(W_0, \Lambda^l)$ that are supported in $\mathbb{R}^{\leq 0} \times Z_0 \cup W_0^\circ \cup [0, 1] \times (\partial^{\text{out}} W_0)$ where $\partial^{\text{out}} W_0$ is the outgoing boundary of W_0 . The index c in the other Banach spaces should be interpreted similarly. If we instead use the Banach space $L_{k,c,c}^2(W_0, \Lambda^l) \oplus L_{k,c,c}^2(W_1, \Lambda^l) \oplus \dots \oplus L_{k,c,c}^2(W_{i-1}, \Lambda^l) \oplus L_{k,c,c}^2(W_i, \Lambda^l)$, the resulting space will be denoted by $\mathcal{V}_k^l(\mathfrak{W}^\circ)$. Starting with such a metric, there is a family of metrics on W parametrized by $[0, \infty]^i$ that forms part of G . The point of considering forms that vanish on the intermediate ends is that any element of $L_{k,\delta,c}^2(W_0, \Lambda^l) \oplus \dots \oplus L_{k,c,\delta}^2(W_i, \Lambda^l)$ determines an element of $L_{k,\delta}^2(W^{g'}, \Lambda^l)$ where $g' \in [0, \infty]^i$. We can also define $\mathcal{V}_k^+(\mathfrak{W})$ as a subspace of $\mathcal{V}_k^2(\mathfrak{W})$ that consists of the self-dual 2-forms.

The definition of the moduli spaces can be extended to the family of metrics \mathfrak{W} . A perturbation term for this family is a smooth section $\eta : G \rightarrow \mathcal{V}_{k-1}^2(\mathfrak{W}^\circ)$. Smoothness of η at the boundary points of G should be interpreted in the following way. Fix a face $G' = G_0 \times \dots \times G_i$ of codimension i in G . This face parametrizes a family of metrics broken along a cut T . Removing T decomposes W into the union of cobordisms $W_0 \amalg \dots \amalg W_i$, and G_k parametrizes a family of metrics \mathfrak{W}_k on W_k such that $\mathfrak{W}|_{G'} = \mathfrak{W}_0 \times \dots \times \mathfrak{W}_i$. Smoothness of η over the face G' implies that there are $\eta_k \in \mathcal{V}_{k-1}^2(\mathfrak{W}_k^\circ)$ and a neighborhood of G' in G of the form $[0, \infty]^i \times G_0 \times \dots \times G_i$ such that:

$$\eta(t_1, \dots, t_i, g_0, \dots, g_i)|_{W_k} = \eta_k(g_k)$$

The moduli space $M_\eta(\mathfrak{W})$ is defined to be:

$$\bigcup_{g \in G} M_{\eta(g)}(W^g)$$

Again, use π to denote the projection map from $M_\eta(\mathfrak{W})$ to G . The definition of topology on this moduli space is standard and we refer the reader to [9]. Given a face G' , we will write $M_\eta(\mathfrak{W}|_{G'})$ for the part of the moduli space that is mapped to G' by π .

The decomposition of the moduli spaces $M_\eta(\mathfrak{W})$ with respect to spin^c structures on W can be recovered partly. Firstly let $[\underline{A}] \in \mathcal{B}(W, \mathfrak{s})$. Then the restriction of \mathfrak{s} to ∂W is a torsion spin^c structure and hence self-intersection of $c_1(\mathfrak{s})$, as a rational cohomology class, is well-defined. Therefore:

$$E([\underline{A}]) := -c_1(\mathfrak{s}) \cdot c_1(\mathfrak{s}) = -c_1(L_{\mathfrak{s}}) \cdot c_1(L_{\mathfrak{s}}) = \frac{1}{4\pi^2} \int_W F(A) \wedge F(A)$$

assigns a rational number to $[\underline{A}]$ that is called the *energy* of $[\underline{A}]$. In fact, if m is the number of torsion elements in $H^2(\partial W, \mathbb{Z})$, then the restriction of $mc_1(\mathfrak{s})$ to ∂W , as an integer class, vanishes. Consequently, $m^2 e([\underline{A}])$ is an integer number. In particular, the set of possible values for $E([\underline{A}])$ is a discrete subset of the rational numbers. If g is a broken metric, inducing the decomposition $W_1 \coprod \dots \coprod W_i$ of W , and $[\underline{A}] \in M_\nu(W^g)$, then:

$$E([\underline{A}]) := \sum_{j=1}^i E([\underline{A}_j])$$

where $[\underline{A}_j]$ is the induced connection on W_j . If \mathfrak{s} is the spin^c structure associated with the broken spin^c connection $[\underline{A}]$, then it is still true that $E([\underline{A}]) = -c_1(\mathfrak{s}) \cdot c_1(\mathfrak{s})$. Therefore, the energy of an element of the moduli space $M_\nu(W^g)$ again has the form $\frac{k}{m^2}$ for an integer number k .

For a metric g (broken or non-broken), let $M_\nu(W^g, e)$ be the set of elements of $M_\nu(W^g)$ with energy e . More generally, if \mathfrak{W} is a family of metrics, then define:

$$M_\eta(\mathfrak{W}; e) := \{[\underline{A}] \in M_\eta(\mathfrak{W}) \mid E([\underline{A}]) = e\}$$

Since the set of possible values for e is discrete, $M_\eta(\mathfrak{W}; e)$ is a union of the connected components of $M_\eta(\mathfrak{W})$.

Lemma 1.8. *Suppose \mathfrak{W} is a family of metrics parametrized by a compact cornered manifold G . For any choice of a smooth perturbation η , the moduli space $M_\eta(\mathfrak{W}; e)$ is compact. Moreover, there is a positive number ϵ such that if $\|\eta(g)\|_2 < \epsilon$ for any $g \in G$, then $M_\eta(\mathfrak{W}; e)$ is empty for negative values of e .*

Proof. Suppose $[\underline{A}] \in M_\eta(\mathfrak{W}; e)$ and $g \in G$ is the corresponding metric. Then:

$$E([\underline{A}]) = -\frac{1}{4\pi^2} (\|F^+(A)\|_2^2 - \|F^-(A)\|_2^2) \geq -\frac{1}{4\pi^2} \|\eta(g)\|_2^2 \quad (12)$$

This inequality verifies the second claim in the lemma, because the set of possible values for E is discrete.

Compactness of $M_\eta(\mathfrak{W}; e)$ can be proved by a standard argument. For the convenience of the reader, we sketch the main steps of the proof. Let $\{z_i\}_{i \in \mathbb{N}} \subset M_\eta(\mathfrak{W}; e)$ and $\pi([\underline{z}_i]) = g_i$. Compactness of G implies that, after passing to a subsequence, there is $g_0 \in G$ such that the sequence converges to g_0 . By abuse of notation, we use the same notation $\{z_i\}_{i \in \mathbb{N}}$ each time that we pass to a subsequence. Firstly assume that g_0 is in the interior of G and parametrizes a non-broken metric on W . Therefore, we can assume that g_i is a non-broken metric. By trivializing \mathfrak{W}

in a neighborhood of g_0 , each element of the sequence $\{z_i\}$ is represented by a spin^c connection \underline{A}_i on a fixed cobordism W . As a consequence of (12):

$$\|F(A_i)\|_2^2 = \|F^{+i}(A_i)\|_2^2 + \|F^{-i}(A_i)\|_2^2 \leq 4\pi^2 e + 2\|\eta(g_i)\|_2^2 \quad (13)$$

where $F^{+i}(A_i)$ and $F^{-i}(A_i)$ are the self-dual and the anti-self-dual parts of $F(A_i)$ with respect to the metric g_i . The L^2 norm in (13) is also computed with respect to g_i . Because g_i is convergent to g_0 and η is a continuous section over G , the L^2 norm of $F(A_i)$ is uniformly bounded. By Uhlenbeck compactness theorem, there is a sequence $u_i \in \mathcal{G}(W)$ such that after passing to a subsequence, $u_i \cdot \underline{A}_i$ is L_k^2 convergent over a regular neighborhood of W° . Note that we are using the Uhlenbeck compactness theorem for $U(1)$ connections (which is essentially a result of the Hodge theory). In particular, the uniform bound on $\|F(A_i)\|_2$ can be an arbitrary number (as oppose to the non-abelian case that the uniform bound has to be small enough.) Since the perturbations $\eta(g_i)$ is zero on the cylindrical ends, the restriction of the connection A_i to the ends is ASD and the technique of [5, Chapter 4] can be invoked to show that, after passing to a subsequence, the restriction of A_i to the ends is $L_{k,\delta}^2$ -convergent. (In fact k can be any arbitrary positive number.) Consequently, the constructed subsequence is convergent over W by a standard patching argument [6]. It is worth noting that because of the linear nature of our moduli spaces, the energy of a subsequence of the elements of $M_\eta(\mathfrak{W}; e)$ does not slide off the ends.

Next, let g_0 be a broken metric and $W_0 \amalg \dots \amalg W_i$ be the corresponding decomposition of W . A similar argument as before proves the convergence of a subsequence on the compact set $W_0^\circ \amalg \dots \amalg W_i^\circ$, the incoming end of W_0 and the outgoing end of $W_{o(i+1)}$. For the intermediate ends, an analogue of [5, Proposition 4.4] can be applied to finish the proof of compactness. \square

We need to show that the moduli space $M_\eta(\mathfrak{W})$ is a “nice” space for a “generic” choice of η . We start with the case that \mathfrak{W} consists of only non-broken metrics. Define the *family of configuration spaces* $\mathcal{B}(\mathfrak{W})$ to be the Banach bundle over G that its fiber over $g \in G$ is $\mathcal{B}(W^g)$. We denote an element of $\mathcal{B}(\mathfrak{W})$ by $([\underline{A}], g)$ to emphasize that $[\underline{A}]$ is in the fiber over g . The space $\mathcal{V}_{k-1}^+(\mathfrak{W})$ also defines a Banach bundle over G . For a perturbation $\eta : G \rightarrow \mathcal{V}_{k-1}^2$, we can define:

$$F_G^+ : \mathcal{B}(\mathfrak{W}) \rightarrow \mathcal{V}_{k-1}^+(\mathfrak{W})$$

$$F_G^+([\underline{A}], g) = F^{+g}(A) - i\eta(g)^{+g}$$

The moduli space $M_\eta(\mathfrak{W})$ can be realized as the inverse image of the zero section of $\mathcal{V}_{k-1}^+(\mathfrak{W})$ by the map F_G^+ . Lemma 1.5 implies that F_G^+ is Fredholm and its index is equal to $b^1(W) - b^+(W)$. Therefore, if F_G^+ is transverse to the zero section of $\mathcal{V}_{k-1}^+(\mathfrak{W})$, then $M_\eta(\mathfrak{W})$ is a smooth manifold of dimension $b^1(W) - b^+(W) + \dim(G)$. The following lemma states that for an appropriate choice of η a stronger version of this transversality assumption holds:

Lemma 1.9. *Let \mathfrak{W} be a family of non-broken metrics on the cobordism W parametrized by the manifold G . A smooth manifold X and a smooth map $\Phi : X \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1)$ are also given. Then there is a perturbation η such that F_G^+ is transversal to the zero section of $\mathcal{V}_{k-1}^+(\mathfrak{W})$ and $r = (r_0, r_1) : M_\eta(\mathfrak{W}) \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1)$ is transversal to Φ .*

Proof. Firstly it will be shown that there is η such that the first required transversality holds. Choose two open covers $\{U_i\}$ and $\{V_i\}$ of G such that:

$$G = \bigcup_{i=1}^n U_i = \bigcup_{i=1}^n V_i \quad \bar{V}_i \subset U_i$$

and $\mathfrak{W}|_{U_i}$ is a trivial bundle. We construct the perturbation η inductively on $\bigcup_{i=1}^k U_i$ such that the transversality assumption holds on $\bigcup_{i=1}^k V_i$. Suppose η is constructed on $\bigcup_{i=1}^k U_i$ and ϕ_1 and ϕ_2 are two smooth functions on G with the following properties: ϕ_1 is supported in $\bigcup_{i=1}^k U_i$ and is non-zero on $\bigcup_{i=1}^k V_i$. Similarly, ϕ_2 is supported in U_{k+1} and non-zero on V_{k+1} . Furthermore, we assume that $\phi_1 + \phi_2|_{\bigcup_{i=1}^{k+1} V_i} = 1$. Define:

$$\begin{aligned} \Psi : \mathcal{B}(\pi^{-1}(\bigcup_{i=1}^{k+1} U_i)) \times L_{k-1,c}^2(W, \Lambda^2) &\rightarrow \mathcal{V}_{k-1}^+(\mathfrak{W}) \\ ([\underline{A}], g, \nu) &\rightarrow F^{+g}(A) - i\phi_1\eta^{+g}(g) - i\phi_2\nu^{+g} \end{aligned}$$

Note that the fibers of \mathfrak{W} over U_{k+1} can be identified with W . In the definition of Ψ we fix one such identification and hence elements of $L_{k-1,c}^2(W, \Lambda^2)$ can be considered as 2-forms on W^g when $g \in U_{k+1}$. Because of our assumption on the supports of ϕ_1 and ϕ_2 , it is clear that Ψ is well-defined.

For $g \in \bigcup_{i=1}^{k+1} V_i$, if $\phi_2(g) = 0$, then $g \in \bigcup_{i=1}^k V_i$ and $\phi_1(g) = 1$. Our assumption on η implies that Ψ is transverse to the zero section in the fiber over g . If $\phi_2(g) > 0$ and $\Psi([\underline{A}], g, \nu) = 0$, then post-composing the derivative of Ψ at $f = ([\underline{A}], g, \nu)$ with the projection to the fiber of $\mathcal{V}_{k-1}^+(\mathfrak{W})$ gives rise to a linear map:

$$\begin{aligned} D_f \Psi : \ker(d^*) \oplus T_g G \oplus L_{k-1,c}^2(W, \Lambda^2) &\rightarrow L_{k-1,\delta}^2(W, \Lambda^{+g}) \\ (a, 0, \mu) &\rightarrow d^{+g}(a) - \phi_2\mu^{+g} \end{aligned}$$

In order to construct the right inverse for $D_f \Psi$, observe that the set of harmonic g -self-dual forms, denoted by \mathcal{H}^+ , does not have any non-zero element that is orthogonal to $L_{k-1,c}^2(W, \Lambda^2)$. Because otherwise that element would be supported in the cylindrical ends and this is absurd according to the discussion of the solutions of the equation $\frac{d}{dt} - Q$ in the previous subsection. Therefore, there is a linear map $T_1 : \mathcal{H}^+ \rightarrow L_{k-1,c}^2(W, \Lambda^{+g})$ such that

$$\omega - T_1(\omega) \in (\mathcal{H}^+)^\perp \subset L_{k-1,\delta}^2(W, \Lambda^{+g}).$$

where the orthogonal complement $(\mathcal{H}^+)^\perp$ of \mathcal{H}^+ is defined with respect to the L^2 norm defined by g . On the other hand, there is a linear operator $T_2 : (\mathcal{H}^+)^\perp \rightarrow \ker(d^*)$ such that $d^{+g} \circ T_2 = id$. Using these two operators, we can construct a right inverse for $D_f \Psi$:

$$\begin{aligned} L : L_{k-1,\delta}^2(W, \Lambda^{+g}) = \mathcal{H}^+ \oplus (\mathcal{H}^+)^\perp &\rightarrow \ker(d^*) \oplus T_g G \oplus L_{k-1,c}^2(W, \Lambda^2) \\ L(\omega, v) &= (T_2(v + \omega - T_1(\omega)), 0, -T_1(\omega)) \end{aligned}$$

By the implicit function theorem for maps between Banach manifolds, the inverse image of the zero section, $\Psi^{-1}(0) \subset \mathcal{B}(\pi^{-1}(\bigcup_{i=1}^{k+1} U_i)) \times L_{k-1,c}^2(W, \Lambda^2)$, is a regular Banach manifold. The

projection map from the Banach manifold $\Psi^{-1}(0)$ to $L_{k-1,c}^2(W, \Lambda^2)$ is a Fredholm map whose index is equal to $\dim(G) + b^1(W) - b^+(W)$. Thus by the Sard-Smale theorem there is a residual subset of $L_{k-1,c}^2(W, \Lambda^2)$ which are regular values of this projection map.

Next we want to show that η can be modified such that r is transverse to Φ . For the simplicity of the exposition we assume that G consists of one point g . Essentially the same argument treats the general case. The key point is that replacing $\eta(g)$ with $\eta(g) + d^{+g}a$ for an arbitrary $a \in L_{k,\text{ext}}^2(W, \Lambda^1)$ produces a moduli space that can be identified with $M_\eta(W^g)$. In fact, if $F^{+g}(A) = i\eta(g)^{+g}$, then $F^{+g}(A + ia) = i\eta(g)^{+g} + d^{+g}a$. Moreover, regularity of $M_\eta(W^g)$ at $[\underline{A}]$ implies the regularity of $M_{\eta+d^+a}(W^g)$ at $[\underline{A} + ia]$. Consider the map:

$$\begin{aligned} \Psi' : M_\eta(W^g) \times C_0 \oplus C_1 &\rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \\ ([\underline{A}], h) &\rightarrow r([\underline{A} + i\varphi(h)]) \end{aligned}$$

This map is clearly a submersion. Therefore again for a generic choice of $h \in C_0 \oplus C_1$ the restriction map of $M_{\eta+d^+\varphi(h)}(W^g) = M_\eta(W^g) + \varphi(h)$ is transversal to Φ . \square

Remark 1.10. The proof of Lemma 1.9 shows that the following relative version is also valid. Suppose $H \subset G$ is a compact sub-manifold and a perturbation η_0 , defined on an open neighborhood of H , is given such that the transversality assumptions of Lemma 1.9 holds. Then there exists a smooth perturbation η defined on G such that η agrees with η_0 in a probably smaller neighborhood of H .

A perturbation η for a family of non-broken metrics \mathfrak{W} is called *admissible* if F_G^+ is transverse to the zero section of $\mathcal{V}_{k-1}^+(\mathfrak{W})$. In the presence of broken metrics, we need to require more in order to have a well-behaved moduli spaces. In this case, if G' is a face of G , then it parametrizes a family of non-broken metrics on a cobordism $W \setminus T = W_0 \coprod \dots \coprod W_i$ for a cut T . By our assumption G' is equal to $G_0 \times \dots \times G_i$ and for $0 \leq k \leq i$, there is a family of (non-broken) metrics \mathfrak{W}_k on W_k , parametrized with G_k , such that $\mathfrak{W}|_{G'} = \mathfrak{W}_0 \times \dots \times \mathfrak{W}_i$. By definition, if η is a smooth perturbation for \mathfrak{W} , then there are perturbations η_k for the family of metrics \mathfrak{W}_k that determine $\eta|_{G'}$. We say η is admissible over G' , if it satisfies the following properties. Firstly we require η_k to be an admissible perturbation for \mathfrak{W}_k . Consider the restriction map:

$$r_{G'} : M_{\eta_0}(\mathfrak{W}_0) \times \dots \times M_{\eta_i}(\mathfrak{W}_i) \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \times \prod_{j=1}^i (\mathcal{R}(Z_j) \times \mathcal{R}(Z_j)) \quad (14)$$

that is defined by the restriction maps of the moduli spaces $M_{\eta_k}(\mathfrak{W}_k)$. Also, consider:

$$\begin{aligned} \Phi_\Delta : \prod_{k=1}^i \mathcal{R}(Z_k) &\rightarrow \prod_{k=1}^i (\mathcal{R}(Z_k) \times \mathcal{R}(Z_k)) \\ ([B_1], \dots, [B_i]) &\rightarrow ([B_1], [B_1], \dots, [B_i], [B_i]) \end{aligned} \quad (15)$$

As another assumption on the admissible perturbation η , we demand that $r_{G'}$ is transverse to Φ_Δ . Let $M_\eta(\mathfrak{W})|_{G'}$ be the subset of $M_\eta(\mathfrak{W})$ that is mapped to G' by the projection map π . The space

$M_\eta(\mathfrak{W})|_{G'}$ is equal to the following fiber product:

$$\begin{array}{ccc} M_\eta(\mathfrak{W})|_{G'} & \longrightarrow & \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \times \prod_i \mathcal{R}(Z_i) \\ \downarrow & & \downarrow (id, id, \Phi_\Delta) \\ M_{\eta_0}(\mathfrak{W}_0) \times \cdots \times M_{\eta_i}(\mathfrak{W}_i) & \xrightarrow{r_{G'}} & \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \times \prod_{j=1}^i (\mathcal{R}(Z_j) \times \mathcal{R}(Z_j)) \end{array} \quad (16)$$

Consequently, if η is an admissible perturbation, then $M_\eta(\mathfrak{W})|_{G'}$ is a smooth manifold of dimension:

$$\dim(G') - \frac{\chi(W) + \sigma(W)}{2} + \frac{b_1(Y_0) + b_1(Y_1)}{2}$$

For a smooth admissible perturbation η the moduli space is a cornered manifold of the expected dimension. The main step to show this claim is the following proposition:

Proposition 1.11. *Suppose T is a cut in a cobordism W with connected components T_1, \dots, T_i . Suppose also $W \setminus T = W_0 \sqcup \dots \sqcup W_i$, and \mathfrak{W}_i is a family of (non-broken) metrics on the cobordism W_k parametrized with G_k . These families define a family of metrics \mathfrak{W}' on $W \setminus T$ that is parametrized by $G' := G_0 \times \cdots \times G_i$. Assume that for each element of G' the product metric on the ends corresponding to T_i is induced by the same metric on T_i . Therefore, this family can be extended to a family of metrics \mathfrak{W} on the cobordism W parametrized with $[0, \infty]^i \times G'$. Let also η_k be an admissible perturbation for \mathfrak{W}_k and η' be the induced perturbation for \mathfrak{W}' . We can also pull back η' to produce a perturbation for the family \mathfrak{W} . We require that the restriction map $r_{G'}$ in (14) is transverse to the map:*

$$(id_{\mathcal{R}(Y_0) \times \mathcal{R}(Y_1)}, \Phi_\Delta) : \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \times \prod_{k=1}^i \mathcal{R}(Z_k) \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \times \prod_{k=1}^i (\mathcal{R}(Z_k) \times \mathcal{R}(Z_k)).$$

Then there exists $\tau \geq 1$ such that the followings hold:

- i) η is an admissible perturbation for the non-broken metrics on W parametrized by $(\tau, \infty)^i \times G'$ and the corresponding moduli space is diffeomorphic to $(\tau, \infty)^i \times M_\eta(\mathfrak{W})|_{G'}$. More generally, η defines an admissible perturbation for the family of metrics that lie over $H_1 \times \cdots \times H_i \times G'$ where H_j is either (τ, ∞) or $\{\infty\}$ for each j . The corresponding moduli space is diffeomorphic to $H_1 \times \cdots \times H_i \times M_\eta(\mathfrak{W})|_{G'}$.
- ii) The moduli space $M_\eta(\mathfrak{W})|_{(\tau, \infty)^i \times G'}$ is homeomorphic to $(\tau, \infty)^i \times M_\eta(\mathfrak{W})|_{G'}$.

Moreover, if a map $\Phi : X \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1)$ is given and $r_{G'}$ is transverse to (Φ, Φ_Δ) , then τ can be chosen large enough such that the map $r : M_\eta(\mathfrak{W}) \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1)$ restricted to the part of the moduli space that is parametrized by a set of the form $H_1 \times \cdots \times H_i \times G'$ is transverse to Φ . The fiber products:

$$M_\eta(\mathfrak{W})|_{(\tau, \infty)^i \times G'} \times_\Phi X \quad M_\eta(\mathfrak{W})|_{H_1 \times \cdots \times H_i \times G'} \times_\Phi X$$

are respectively homeomorphic and diffeomorphic to:

$$(\tau, \infty)^i \times (M_\eta(\mathfrak{W})|_{G'} \times_\Phi X) \quad H_1 \times \cdots \times H_i \times (M_\eta(\mathfrak{W})|_{G'} \times_\Phi X)$$

The techniques of [5, Chapter 4] can be used to prove this proposition and we leave the details of the proof for the reader.

Proposition 1.12. *For a family of metrics \mathfrak{W} parametrized by a compact cornered manifold G , there is an admissible perturbation η such that the following holds. For a face G' of G with codimension i , $M_\eta(\mathfrak{W})|_{G'}$ is a smooth manifold of dimension:*

$$\dim(G) - i - \frac{\chi(W) + \sigma(W)}{2} + \frac{b_1(Y_0) + b_1(Y_1)}{2}.$$

The total moduli space $M_\eta(\mathfrak{W})$ is a cornered C^0 -manifold. Moreover, there is a continuous map of cornered manifolds $\pi : M_\eta(\mathfrak{W}) \rightarrow G$ which maps a codimension i face of $M_\eta(\mathfrak{W})$ to a codimension i face of G by a smooth map. The perturbation η can be chosen such that for any face G' of G , inducing the decomposition $W_0 \coprod \dots \coprod W_i$, the restriction map $r_{G'}$ is transverse to a given map $\Phi_{G'} : X_{G'} \rightarrow \mathcal{R}(\partial W_0) \times \dots \times \mathcal{R}(\partial W_i)$.

Proof. Let G' be a 0-dimensional face of G that defines a broken metric on W , and $W_0 \coprod \dots \coprod W_i$ is the induced decomposition of W . Fix admissible perturbations η_0, \dots, η_i on W_0, \dots, W_i , and let η be the induced perturbation for the 0-dimensional face G' . Then use the argument in the second part of Lemma 1.9 to insure that $M_\eta(\mathfrak{W})|_{G'}$ is cut out transversally and $r_{G'}$ is transverse to $\Phi_{G'}$. Repeat this construction to extend η to all 0-dimensional faces of G . Use Lemma 1.9, Proposition 1.11, and Remark 1.10 to extend η , as an admissible perturbation, to the edges of G such that the required transversality assumptions hold. The same argument can be used to extend η inductively to all faces of G . Proposition 1.11 implies that $M_\eta(\mathfrak{W})$ is a cornered C^0 -manifold. This proposition also asserts the existence of the map π with the claimed properties. \square

The following example is extracted from [12]:

Example 1.13. The 4-manifold $U = D^2 \times S^2 \# \overline{\mathbb{C}P^2}$ satisfies $b_1(U) = b^+(U) = 0$ and can be considered as a cobordism from $Q = S^1 \times S^2$ to the empty set. There exists a disc D and a sphere S_0 in the connected summand $D^2 \times S^2$ such that the following assumptions hold. The disc D bounds $S^1 \times \{x\} \subset Q$ for an arbitrary point $x \in S^2$, the sphere S_0 intersects D in exactly one point, and it has self-intersection 0. Pick also an embedded sphere S_1 in the summand $\overline{\mathbb{C}P^2}$ of U that has self intersection -1 and is disjoint from D and S_0 . Let T_1 be the boundary of a regular neighborhood of S_1 that is diffeomorphic to S^3 . Consider a metric on W which is the standard product metric in a neighborhood of T_1 . We can construct a family of metrics parametrized with $[0, \infty]$ by stretching this metric along T_1 . In particular, the metric corresponding to ∞ is broken along the cut T_1 , i.e., it is a metric with cylindrical ends on $U \setminus T_1$ with connected components $U_0 \cong D^2 \times S^2 \setminus D^4$ and $U_1 \cong \overline{\mathbb{C}P^2} \setminus D^4$.

Next, consider the 2-sphere $S'_1 = S_1 \# S_0$. This sphere also has self-intersection -1 and its intersection with D is 1. By replacing S_1 with S'_1 , we can generate another family of metrics parametrized by $[-\infty, 0]$ where $-\infty$ corresponds to a broken metric with a cut along another copy of S^3 that is denoted by T'_1 . The 4-manifold $U \setminus T'_1$ has connected components $U'_0 \cong D^2 \times S^2 \setminus D^4$ and $U'_1 \cong \overline{\mathbb{C}P^2} \setminus D^4$. We can glue the two families to obtain a family of metrics \mathbb{U} parametrized by

$G = [-\infty, \infty]$. There is only one torsion spin^c structure on Q which is the spin structure \mathfrak{t}_0 on Q . Let \mathfrak{s} be a spin^c structure on U such that $\mathfrak{s}|_Q = \mathfrak{t}_0$. Any such spin^c structure \mathfrak{s} on U is uniquely determined with an odd integer which is the pairing of $c_1(\mathfrak{s})$ and S_1 . Suppose \mathfrak{s}_{2k+1} be the choice of the spin^c structure such that the above number is $2k+1$. Then $\overline{\mathfrak{s}_{2k+1}}$, the complex conjugate of \mathfrak{s}_{2k+1} , has pairing $-2k-1$ with S_1 and hence is equal to \mathfrak{s}_{-2k-1} . The spin^c structures \mathfrak{s}_{2k+1} and \mathfrak{s}_{-2k-1} have energy $(2k+1)^2$. Because G is contractible, it makes sense to talk about the spin^c connection \mathfrak{s}_{2k+1} for the family $\mathbb{U}|_{(-\infty, \infty)}$.

Since $b^+(U) = 0$, any choice of the perturbation gives a regular moduli space (Lemma 1.6). We will write $M(\mathbb{U}, \mathfrak{s}_{2k+1})$ for the moduli space with zero perturbation and the spin^c structure \mathfrak{s}_{2k+1} . For each $g \in G$ we have a unique connection \underline{A}^g on \mathfrak{s}_{2k+1} such that $F^{+g}(A^g) = 0$. Thus the moduli space $M(\mathbb{U}, \mathfrak{s}_{2k+1})$ is an interval for each choice of $k \in \mathbb{Z}$. Self-duality of the curvature of A^g implies that $\frac{1}{2\pi i} F(A^g)$ is a g -harmonic form representing $c_1(\mathfrak{s}_{2k+1})$. As g tends to ∞ , these forms converge to a harmonic 2-form on $U \setminus T_1$ that vanishes on U_0 and represents $(2k+1)PD(S_1)$ on U_1 . A similar discussion applies to $g = -\infty$.

Note that $\mathcal{R}(Q)$ is a circle and is parametrized by the holonomy of its elements around ∂D_0 which take the following values:

$$\text{hol}(\underline{B}, \partial D_0) \in \mathcal{R} = \left\{ \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{2\pi i \theta} \end{pmatrix} h_0 \mid \theta \in [0, 1] \right\}$$

where \underline{B} is a spin^c connection on \mathfrak{t}_0 , and $h_0 \in SU(2)$ is the holonomy of the spin connection around ∂D . By Gauss-Bonnet:

$$\det(\text{hol}(\underline{A}^g, \partial D)) = \exp \left(\int_D F(A^g) \right)$$

Therefore $\int_D F(A^\infty) = 0$ because $D \subset U_0$ and A^∞ is flat on U_0 . Similarly, $\int_D F(A^{-\infty}) = 2\pi i(2k+1)$ because S'_1 and D have intersection 1. As a result, $\det(\text{hol}(\underline{A}^g, \partial D))$ starts from 1 and traverses the unit circle an odd number of times. Thus the image of $r_k : M(\mathbb{U}|_{(-\infty, \infty)}, \mathfrak{s}_k) \rightarrow \mathcal{R}(Q)$ is given by a path between h_0 and $-h_0$ in \mathcal{R} that consists of $(2k+1)$ half-circles. For the complex conjugate spin^c structure, \mathfrak{s}_{-2k-1} , this path is the complex conjugate of the above path. Union of these two paths form a loop in \mathcal{R} with winding number $2k+1$. In summary, image of $M(\mathbb{U}, (2k+1)^2)$ in $\mathcal{R}(Q)$ is a loop with winding number $(2k+1)$.

1.3 Cobordism maps

Plane Floer homology of a 3-manifold Y , denoted by $\text{PFH}(Y)$, is defined to be the homology of the representation variety $\mathcal{R}(Y)$. In this subsection, we use the elements of the following Novikov ring as the coefficient ring:

$$\Lambda_0 := \left\{ \sum_{q_i \in \mathbb{Q}} a_i u^{q_i} \mid a_i \in \mathbb{Z}/2\mathbb{Z}, a_i \neq 0, q_i \in \mathbb{Q}^{\geq 0}, \lim_{i \rightarrow \infty} q_i = \infty \right\}$$

where u is a formal variable. If we relax the condition that q_i is non-negative, then the resulting field is denoted by Λ . The space $\mathcal{R}(Y)$ is a union of $|\text{Tor}(H_1(Y, \mathbb{Z}))|$ copies of $J(Y)$. Therefore, we

have:

$$\text{PFH}(Y) \cong H_*(J(Y), \Lambda_0)^{\oplus |\text{Tor}(H_1(Y, \mathbb{Z}))|} \cong \Lambda_0^{2^{b_1(Y)} \cdot |\text{Tor}(H_1(Y, \mathbb{Z}))|}$$

Later we will need to work with the lift of $\text{PFH}(Y)$ to the level of chain complexes. To fix one such chain complex, let f be a Morse-Smale function on the compact manifold $\mathcal{R}(Y)$ and define $(\text{PFC}(Y), d_p)$ to be the Morse complex for the homology of $\mathcal{R}(Y)$ associated with f . Therefore, the generators of $\text{PFC}(Y)$ is equal to the set of the critical points of f and the differential d_p maps an index i critical point of f to a linear combination of index $i - 1$ critical points. The definition of d_p involves counting unparameterized trajectories between critical points of f . Note that, however, because Λ_0 has characteristic 2, we do not need to orient the moduli spaces of trajectories between critical points. This Morse complex is called the *plane Floer complex* of Y . Equip $\text{PFC}(Y)$ with a grading that its value on a critical point α of f is equal to:

$$\text{deg}_p(\alpha) := 2 \cdot \text{ind}(\alpha) - b_1(Y)$$

where $\text{ind}(\alpha)$ stands for the Morse index of α .

For the remaining part of Section 1, we fix a family of metrics \mathfrak{W} on a cobordism $W : Y_0 \rightarrow Y_1$ which is parametrized by a compact cornered manifold G . As before, Y_0 is required to be connected. A typical face of codimension i of G is denoted by G' . This face parametrizes a family of broken metrics with a cut along a sub-manifold $T = T_1 \cup T_2 \cup \dots \cup T_i$. The restriction of \mathfrak{W} to this face is given by the family of metrics $\mathfrak{W}_0 \times \dots \times \mathfrak{W}_i$ on $W \setminus T = W_0 \coprod \dots \coprod W_i$. Here \mathfrak{W}_j is a family of (non-broken) metrics on W_j , parametrized by a manifold G_j . In particular, $G' = G_0 \times \dots \times G_i$. In order to set up the stage, fix Morse-Smale functions f_0 on $\mathcal{R}(Y_0)$, f_1 on $\mathcal{R}(Y_1)$, and h_j on $\mathcal{R}(T_j)$ for each T_j as above. By the results of the last subsection, there is an admissible perturbation η for the family of metrics \mathfrak{W} such that the following hypothesis holds:

Hypothesis 1. Let α and β be critical points of f_0 and f_1 , respectively. We will write U_α (respectively, S_β) for the unstable (respectively, stable) manifold of α (respectively, β). Let $\Phi(\alpha) : U_\alpha \hookrightarrow \mathcal{R}(Y_0)$ and $\Phi(\beta) : S_\beta \hookrightarrow \mathcal{R}(Y_1)$ be the inclusion maps. For a face G' of G with codimension i , suppose that the connected components of $T = \bigcup_{j=1}^i T_j$ is divided to two disjoint sets $\{T_{i_1}, T_{i_2}, \dots\}$ and $\{T_{j_1}, T_{j_2}, \dots\}$. For each i_k , consider critical points $\gamma_{i_k}, \gamma'_{i_k}$ of h_{i_k} and let $\Phi_{i_k} : S_{\gamma_{i_k}} \times U_{\gamma'_{i_k}} \hookrightarrow \mathcal{R}(T_{i_k}) \times \mathcal{R}(T_{i_k})$ denote the inclusion map. For each j_l , define $\Phi_{j_l} : \mathcal{R}(T_{j_l}) \times [0, \infty) \rightarrow \mathcal{R}(T_{j_l}) \times \mathcal{R}(T_{j_l})$ by $\Phi_{j_l}(z, t) = (\varphi_{j_l}^t(z), \varphi_{j_l}^{-t}(z))$ where $\varphi_{j_l}^t$ is the negative-gradient-flow of h_{j_l} on $\mathcal{R}(T_{j_l})$. Given these maps, we introduce:

$$\begin{aligned} \Phi : U_\alpha \times S_\beta \times \prod_{i_k} (S_{\gamma_{i_k}} \times U_{\gamma'_{i_k}}) \times \prod_{j_l} (\mathcal{R}(T_{j_l}) \times [0, \infty)) &\rightarrow \\ \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \times \prod_{i_k} (\mathcal{R}(T_{i_k}) \times \mathcal{R}(T_{i_k})) \times \prod_{j_l} (\mathcal{R}(T_{j_l}) \times \mathcal{R}(T_{j_l})) & \\ \Phi := (\Phi(\alpha), \Phi(\beta), \Phi_{i_1}, \dots, \Phi_{j_1}, \dots) & \end{aligned}$$

The smooth admissible perturbation η is such that:

$$r_{G'} : M_{\eta_0}(\mathfrak{W}_0) \times \dots \times M_{\eta_i}(\mathfrak{W}_i) \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(Y_1) \times \prod_{i_k} \mathcal{R}(T_{i_k}) \times \mathcal{R}(T_{i_k}) \times \prod_{j_l} \mathcal{R}(T_{j_l}) \times \mathcal{R}(T_{j_l})$$

is transverse to all such maps Φ . Also, η is chosen such that $M_\eta(\mathfrak{W}; e)$ is empty for negative values of e .

There are a few remarks about this hypothesis in order: firstly the domain of Φ is a cornered manifold. The transversality assumption means that the restriction of Φ to each face is transverse to $r_{G'}$. Secondly, for each face G' there are different but finitely many choices for Φ . This does not make any issue, as we can consider the union of the domains of all these different choices, and define a total map $\Phi_{G'}$ in order to use Proposition 1.12.

By Hypothesis 1 and Proposition 1.12, the following set:

$$M_\eta(\mathfrak{W}, \alpha, \beta) = \{z \in M_\eta(\mathfrak{W}) \mid (r_0(z), r_1(z)) \in U_\alpha \times S_\beta\} \quad (17)$$

is a cornered manifold and the restriction of π maps the faces of $M_\eta(\mathfrak{W}, \alpha, \beta)$ to the faces of G . We also have:

$$\begin{aligned} \dim(M_\eta(\mathfrak{W}, \alpha, \beta)) &= \dim(M_\eta(\mathfrak{W})) + \dim(U_\alpha \times S_\beta) - \dim(\mathcal{R}(Y_0) \times \mathcal{R}(Y_1)) \\ &= \frac{1}{2}(\deg_p(\alpha) - \deg_p(\beta) - \sigma(W) - \chi(W)) + \dim(G) \end{aligned} \quad (18)$$

Let $M_\eta(\mathfrak{W}, \alpha, \beta; e)$ be the space $M_\eta(\mathfrak{W}, \alpha, \beta) \cap M_\eta(\mathfrak{W}; e)$. This space is also a cornered manifold. Unlike $M_\eta(\mathfrak{W}; e)$, the moduli space $M_\eta(\mathfrak{W}, \alpha, \beta; e)$ is not necessarily a compact manifold. To see what might go wrong, consider a sequence $\{z_i\}_{i \in \mathbb{N}} \in M_\eta(\mathfrak{W}, \alpha, \beta; e)$. Since $M_\eta(\mathfrak{W}; e)$ is compact, this sequence is convergent to $z_0 \in M_\eta(\mathfrak{W}; e)$ after passing to a subsequence. The sequence $r_0(z_i)$ converges to $r_0(z_0)$. However, the unstable manifold of α is not necessarily compact and $r_0(z_0)$ might not lie on this set. Nevertheless, z_0 lives on the unstable manifold of a critical point α' that $\deg_p(\alpha') \leq \deg_p(\alpha)$ and equality holds if and only if $\alpha = \alpha'$. (In fact, there exists a broken trajectory from α to α' [18].) Similarly, z_0 is mapped by r_1 to the stable manifold of a critical point β' of f_1 such that $\deg_p(\beta') \geq \deg_p(\beta)$ and equality holds if and only if $\beta = \beta'$. In summary, $z_0 \in M_\eta(\mathfrak{W}, \alpha', \beta'; e)$ and the dimension of this moduli space is not greater than that of $M_\eta(\mathfrak{W}, \alpha, \beta; e)$. Furthermore, equality holds if and only if $\alpha' = \alpha$ and $\beta' = \beta$.

Let α and β be chosen such that $M_\eta(\mathfrak{W}, \alpha, \beta)$ is 0-dimensional, i.e.:

$$\deg_p(\beta) - \deg_p(\alpha) = 2 \dim(G) - \sigma(W) - \chi(W) \quad (19)$$

The discussion of the previous paragraph implies that in this special case, $M_\eta(\mathfrak{W}, \alpha, \beta; e)$ is compact. Furthermore, the transversality assumptions imply that the elements of $M_\eta(\mathfrak{W}, \alpha, \beta)$ live over the interior of G . The 0-dimensional space $M_\eta(\mathfrak{W}, \alpha, \beta)$ is subset of a *secondary moduli space* $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. The space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ is defined over an *open fattening* \mathbf{G} of G , i.e., there is a map $\tilde{\pi} : \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta) \rightarrow \mathbf{G}$, and $\tilde{\pi}^{-1}(G) = M_\eta(\mathfrak{W}, \alpha, \beta)$.

In order to construct \mathbf{G} , firstly note that G is the union of all products $(0, \infty]^i \times G'$ when i is an integer number and G' is a codimension i face of G . The set:

$$\{\infty\} \times \dots \times \{\infty\} \times \underbrace{(0, \infty)}_{k^{\text{th}} \text{ factor}} \times \{\infty\} \times \dots \times \{\infty\} \times G' \quad (20)$$

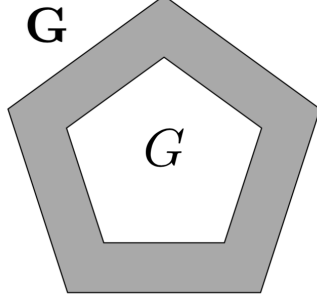


Figure 1: An open fattening of the white pentagon G is given by adding the shaded region.

can be regarded as an open subset of a codimension $i - 1$ face G'' in G . We can arrange the standard neighborhoods $(0, \infty]^i \times G'$ and $(0, \infty]^{i-1} \times G''$ of these two faces such that:

$$(t_1, \dots, t_i, g) \in (0, \infty]^i \times G' \quad (t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_i, (t_k, g)) \in (0, \infty]^{i-1} \times G'' \quad (21)$$

represent the same points in G , where $t_k \in (0, \infty)$ and (t_k, g) is considered as a point in (20).

Let $\overline{(0, \infty)}$ be the result of gluing the intervals $(0, \infty]$ and $[0, \infty)$ by identifying the ∞ end of the first interval (the *primary* interval) with the 0 end of the second interval (the *secondary* interval). The cornered manifold \mathbf{G} is the union of all the products of the form $\overline{(0, \infty)}^i \times G'$ with the following equivalence relation. For faces G' and G'' we identify

$$\overline{(0, \infty)}^{k-1} \times (0, \infty) \times \overline{(0, \infty)}^{i-k} \times G'$$

in which the k^{th} factor is a primary interval, with an open subset of $\overline{(0, \infty)}^{i-1} \times G''$ using (21). The manifold \mathbf{G} is non-compact and can be compactified to a cornered manifold that is homeomorphic to G . This compactification is given by a similar construction except that one replaces the secondary interval $[0, \infty) \subset \overline{(0, \infty)}$ with the closed interval $[0, \infty]$. If the product $G_1 \times G_2$ is a face of G , then the corresponding face of the compactification of \mathbf{G} is the same as $\mathbf{G}_1 \times \mathbf{G}_2$. A contraction map $\rho : \mathbf{G} \rightarrow G$ can be defined in the following way. The subset $\overline{(0, \infty)}^i \times G' \subset \mathbf{G}$ can be decomposed to 2^n sets of the following form:

$$H = A_{l_1} \times \dots \times A_{l_i} \times G' \quad (22)$$

where each l_k is either 1 or 2 and determines whether the k^{th} interval is primary or secondary, i.e., $A_1 = (0, \infty]$ and $A_2 = [0, \infty)$. Define:

$$\rho : H \rightarrow (0, \infty]^i \times G'$$

$$(g, t_1, t_2, \dots, t_i) \rightarrow (g, \epsilon_{l_1}(t_1), \epsilon_{l_2}(t_2), \dots, \epsilon_{l_i}(t_i))$$

where $\epsilon_1 : A_1 \rightarrow (0, \infty]$ is the identity map and $\epsilon_2 : A_2 \rightarrow (0, \infty]$ is the constant ∞ map. These maps together, for different involved choices, define the contraction map $\rho : \mathbf{G} \rightarrow G$.

The moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ will be produced by constructing moduli spaces over the subsets $H \subset \mathbf{G}$ of the form of (22), and then putting them together. We denote the moduli space over H by $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$. This space can be defined without any assumption on the dimension of $M_\eta(\mathfrak{W}, \alpha, \beta)$. Without loss of generality, assume that in the definition of H we have $l_1 = \dots = l_k = 1$, $l_{k+1} = \dots = l_i = 2$. The subset $I = (0, \infty]^k \times \{\infty\}^{i-k} \times G'$ of G parametrizes a family of metrics \mathfrak{W}' on $W' := W \setminus \bigcup_{k+1 \leq j \leq i} T_j$ with $i - k + 1$ connected components. The perturbation η produces a perturbation η' for the family of metrics \mathfrak{W}' and hence we can form the moduli space $M_{\eta'}(\mathfrak{W}', \alpha, \beta)$. Each 3-manifold T_j , $k + 1 \leq j \leq i$, appears both as the incoming end of one of the components of W' , and also as an outgoing end of another component of W' . Therefore, we have the restriction maps $r_j^{in} : M_{\eta'}(\mathfrak{W}', \alpha, \beta) \rightarrow \mathcal{R}(T_j)$ and $r_j^{out} : M_{\eta'}(\mathfrak{W}', \alpha, \beta) \rightarrow \mathcal{R}(T_j)$. The space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ is defined to be the fiber-product of the following diagram:

$$\begin{array}{ccc} \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H & \longrightarrow & \prod_{k+1 \leq j \leq i} \mathcal{R}(T_j) \times [0, \infty) \\ \downarrow & & \downarrow \Phi \\ M_{\eta'}(\mathfrak{W}', \alpha, \beta) & \xrightarrow{\prod_{k+1 \leq j \leq i} r_j^{in} \times r_j^{out}} & \prod_{k+1 \leq j \leq i} \mathcal{R}(T_j) \times \mathcal{R}(T_j) \end{array} \quad (23)$$

where Φ is defined as:

$$\Phi([\underline{B}_{k+1}], t_{k+1}, \dots, ([\underline{B}_i], t_i) \rightarrow (\varphi_{k+1}^{t_{k+1}}([\underline{B}_{k+1}]), \varphi_{k+1}^{-t_{k+1}}([\underline{B}_{k+1}]), \dots, \varphi_i^{t_i}([\underline{B}_i]), \varphi_i^{-t_i}([\underline{B}_i]))$$

The map φ_j^t is the negative-gradient-flow of the Morse-Smale functions h_j . Hypothesis 1 implies that $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ is a smooth cornered manifold. This set consists of the following points:

$$\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H = \{([\underline{A}], g, t_{k+1}, \dots, t_i) \mid ([\underline{A}], g) \in M_{\eta'}(\mathfrak{W}', \alpha, \beta), \varphi_j^{t_j}(r_j^{out}([\underline{A}])) = \varphi_j^{-t_j}(r_j^{in}([\underline{A}]))\} \quad (24)$$

We can also define $\tilde{\pi} : \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H \rightarrow H$ by:

$$\tilde{\pi}([\underline{A}], g, t_{k+1}, \dots, t_i) = (g, t_{k+1}, \dots, t_i)$$

The dimension of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ is equal to that of $M_\eta(\mathfrak{W}, \alpha, \beta)$. Thus in the case that $M_\eta(\mathfrak{W}, \alpha, \beta)$ is zero dimensional, $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ is 0-dimensional and $\tilde{\pi}^{-1}(\partial H)$ is empty. Therefore, we can put together the moduli spaces $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$, for different choices of H , to form a zero dimensional moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ and a map $\tilde{\pi} : \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta) \rightarrow \mathbf{G}$. The energy functional can be extended to $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ and the spaces $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)$ is defined to be the elements of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ with energy equal to e .

Lemma 1.14. *The moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)$ is compact.*

Proof. Suppose $H = A_{l_1} \times \dots \times A_{l_i} \times G'$ is as in (22), and $H' \subseteq H$ is the set $A'_{l_1} \times \dots \times A'_{l_i} \times G'$ such that $A'_{l_j} = [1, \infty)$ if A_{l_j} is a primary interval, and $A'_{l_j} = A_{l_j}$ if A_{l_j} is secondary. It suffices to show that for each choice of H , $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)|_{H'} := \pi^{-1}(H') \cap \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)$ is compact. For the simplicity of exposition, we assume that H has the above special form, i.e., the first k

intervals are primary and the remaining ones are secondary. Let $\{([\underline{A}^j], g^j, t_{k+1}^j, \dots, t_i^j)\}_{j \in \mathbb{N}}$ be a sequence in $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)|_{H'}$. This sequence after passing to a subsequence is convergent to $([\underline{A}^0], g^0, t_{k+1}^0, \dots, t_i^0)$ with $t_l^0 \in [1, \infty]$, and $([\underline{A}^0], g^0) \in M_{\eta'}(\mathfrak{W}', \alpha', \beta'; e)$ for critical points α' of f_0 and β' of f_1 . Furthermore, $\deg_p(\alpha') \leq \deg_p(\alpha)$ and $\deg_p(\beta') \geq \deg_p(\beta)$ and the equalities hold if and only if $\alpha' = \alpha$ and $\beta' = \beta$. Suppose $t_l^0 = \infty$ for some l . There is a trajectory $\gamma_l : [0, t_l^j] \rightarrow \mathcal{R}(T_l)$ of the flow φ_l^t that $\gamma_l(0) = r_l^{\text{out}}([\underline{A}^j])$ and $\gamma_l(t_l^j) = r_l^{\text{in}}([\underline{A}^j])$. Since t_l^j tends to $t_l^0 = \infty$, according to [18] there is a broken trajectory that starts from $r_l^{\text{out}}([\underline{A}^0])$ and terminates at $r_l^{\text{in}}([\underline{A}^0])$. In particular, $(r_l^{\text{in}}([\underline{A}^0]), r_l^{\text{out}}([\underline{A}^0])) \in U_{\alpha_l} \times S_{\beta_l}$ where α_l and β_l are two critical points of h_l , and $\dim(U_{\alpha_l}) + \dim(S_{\beta_l}) \leq \dim(\mathcal{R}(T_l))$. The point $([\underline{A}^0], g^0, t_{k+1}^0, \dots, t_i^0)$ lives in the following fiber product:

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \prod_{k+1 \leq l \leq i} \mathfrak{R}_l \\ \downarrow & & \downarrow \Phi' \\ M_{\eta'}(\mathfrak{W}', \alpha', \beta') & \xrightarrow{\prod_{k+1 \leq l \leq i} r_l^{\text{in}} \times r_l^{\text{out}}} & \prod_{k+1 \leq l \leq i} \mathcal{R}(T_l) \times \mathcal{R}(T_l) \end{array} \quad (25)$$

If $t_l \in [0, \infty)$, $\mathfrak{R}_l = \mathcal{R}(T_l) \times [0, \infty)$, otherwise $\mathfrak{R}_l = U_{\alpha_l} \times S_{\beta_l}$ is chosen as above. The smooth map Φ is defined as in Hypothesis 1. Hypothesis 1 implies that, $\prod_{k+1 \leq l \leq i} r_l^{\text{in}} \times r_l^{\text{out}}$ is transverse to Φ . The inequalities:

$$\dim(\mathfrak{R}_l) \leq \dim(\mathcal{R}(T_l) \times [0, \infty)) \quad \dim(M_{\eta'}(\mathfrak{W}', \alpha', \beta')) \leq \dim(M_{\eta'}(\mathfrak{W}', \alpha, \beta)) \quad (26)$$

imply that for a choice of η as above:

$$\dim(\mathfrak{M}) \leq \dim(\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_{H'})$$

But the dimension of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_{H'}$ is zero. Therefore, \mathfrak{M} is non-empty only if the inequalities in (26) become equalities. That is to say, $\alpha' = \alpha$, $\beta' = \beta$, and all t_l^0 are finite and hence $([\underline{A}^0], g^0, t_{k+1}^0, \dots, t_i^0) \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)|_{H'}$. Therefore, $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)|_{H'}$ is compact. \square

Now we are ready to define the map $f_{\mathfrak{W}}^\eta : \text{PFC}(Y_0) \rightarrow \text{PFC}(Y_1)$. Let α be a critical point of f_0 and $\langle \alpha \rangle$ be the corresponding generator of $\text{PFC}(Y_0)$:

$$f_{\mathfrak{W}}^\eta(\langle \alpha \rangle) = \sum_{\beta \in \text{Crit}(f_1)} \left(\sum_{p \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)} u^{e(p)} \right) \langle \beta \rangle \quad (27)$$

Here β ranges over all critical points of f_1 that $M_\eta(\mathfrak{W}, \alpha, \beta)$ is 0-dimensional. Lemma 1.14 insures that the above sum is a well-defined element of Λ_0 . Note that this is the first place that we need to work with the Novikov ring Λ_0 rather than, say, $\mathbb{Z}/2\mathbb{Z}$. We call the coefficient of $\langle \beta \rangle$ in the above expression, the *matrix entry* for the pair (α, β) and denote it by $f_{\mathfrak{W}}^\eta(\alpha, \beta)$. As a result of (19):

$$\deg_p(f_{\mathfrak{W}}^\eta) = 2 \dim(G) - \sigma(W) - \chi(W) \quad (28)$$

In order to study the homotopic properties of $f_{\mathfrak{W}}^\eta$, we construct secondary moduli spaces when $M_\eta(\mathfrak{W}, \alpha, \beta)$ is 1-dimensional. In this case, the space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ is a 1-manifold whose boundary lies over the codimension 1 faces of H via the projection map $\tilde{\pi}$. If H and H' are two sets

of the form (22) that share a face, then the moduli spaces over them match with each other. In the case that the common face has codimension at least 2, this is clear because the moduli spaces over the common face are empty and hence equal. To prove the case of codimension 1, let H be as above and H' be the following subset of \mathbf{G} :

$$A_{l'_1} \times \cdots \times A_{l'_i} \times G_b$$

where $l'_1 = l_1 = 1, \dots, l'_{k-1} = l_{k-1} = 1, l'_k = 2, l'_{k+1} = l_{k+1} = 2, \dots, l'_i = l_i = 2$. Then the moduli spaces over the common face are equal to:

$$\begin{aligned} (\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H)|_{H \cap H'} &= (\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_{H'})|_{H \cap H'} = \{([\underline{A}], g, t_{k+1}, \dots, t_i) \mid ([\underline{A}], g) \in M_{\eta'}(\mathfrak{W}', \alpha, \beta), \\ &g \in \underbrace{[0, \infty] \times \cdots \times [0, \infty]}_{k-1} \times \{\infty\} \times G_b, r_k^{out}([\underline{A}]) = r_k^{in}([\underline{A}]), \varphi_j^{t_j}(r_j^{out}([\underline{A}])) = \varphi_j^{-t_j}(r_j^{in}([\underline{A}]))\} \end{aligned}$$

By gluing together these moduli spaces, form the 1-dimensional secondary moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. As in the 0-dimensional case, the energy functional can be extended to $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$, and the set of elements of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ with energy e is denoted again by $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)$. The 1-dimensional manifold $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ is a union of open intervals and simple closed curves. An open interval can be compactified by adding two endpoints, and a punctured neighborhood of these endpoints are called an *open end* of the interval.

Let G^1, \dots, G^m be the codimension 1 faces of G such that each G^i parametrizes a family of metrics with a cut along a connected 3-manifold T_i . There are two possibilities for each T_i . Removing T_i decomposes W to either cobordisms $W_1^i : Y_0 \rightarrow T_i$ and $W_2^i : T_i \rightarrow Y_1$ or cobordisms $W_1^i : Y_0 \rightarrow T_i \sqcup Y_1$ and $W_2^i : T_i \rightarrow \emptyset$. In each of these cases, there are families of metrics $\mathfrak{W}_1^i, \mathfrak{W}_2^i$ over the cobordisms W_1^i, W_2^i , parametrized by G_1^i, G_2^i , such that $G^i = G_1^i \times G_2^i$ and the restriction of \mathfrak{W} to the face G^i is given by $\mathfrak{W}_1^i \times \mathfrak{W}_2^i$.

Lemma 1.15. *Let $M_\eta(\mathfrak{W}, \alpha, \beta)$ be a 1-dimensional moduli space. Then the open ends of the moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ can be identified with the following set:*

$$\begin{aligned} &\bigcup_{i, \gamma \in \text{Crit}(h_i)} \widetilde{M}_\eta(\mathfrak{W}_1^i, \alpha, \gamma) \times \widetilde{M}_\eta(\mathfrak{W}_2^i, \gamma, \beta) \cup \\ &\bigcup_{\substack{\gamma \in \text{Crit}(f_0) \\ \text{ind}(\alpha) - \text{ind}(\gamma) = 1}} \mathcal{M}(\alpha, \gamma) \times \widetilde{M}_\eta(\mathfrak{W}, \gamma, \beta) \quad \cup \quad \bigcup_{\substack{\gamma \in \text{Crit}(f_1) \\ \text{ind}(\gamma) - \text{ind}(\beta) = 1}} \widetilde{M}_\eta(\mathfrak{W}, \alpha, \gamma) \times \mathcal{M}(\gamma, \beta) \end{aligned} \quad (29)$$

where $\mathcal{M}(\alpha, \gamma)$ (respectively, $\mathcal{M}(\gamma, \beta)$) is the set of unparameterized Morse trajectories from α to γ (respectively, from γ to β). Furthermore, adding the endpoints of the moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)$ makes this moduli space compact.

Proof. The proof of this lemma is standard and consists of two parts. Firstly we need to show that for each point in (29) there is an open end of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. In the second part, we need to show that after adding the endpoints in (29) the moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta; e)$ is compact. The latter

can be proved in the similar way as in the proof of Lemma 1.14. In order to prove the first part, fix a codimension 1 face G^i of G and a critical point γ of the function h_i . Define:

$$\mathcal{M}(\mathcal{R}(T_i), \mathcal{R}(T_i)) := \{(z_1, z_2) \in \mathcal{R}(T_i) \times \mathcal{R}(T_i) \mid \exists t \in \mathbb{R}^{\geq 0} \phi_i^t(z_1) = \phi_i^{-t}(z_2)\}. \quad (30)$$

According to [18], $\mathcal{M}(\mathcal{R}(T_i), \mathcal{R}(T_i))$ can be compactified to a cornered manifold and the set $U_\gamma \times S_\gamma$ is a codimension 1 face of this compactification. Roughly speaking, this face can be thought as the case that $t = \infty$ and $\phi_i^t(z_1) = \phi_i^{-t}(z_2) = \gamma$. Let \mathcal{M}_γ be the union of $\mathcal{M}(\mathcal{R}(T_i), \mathcal{R}(T_i))$ and $U_\gamma \times S_\gamma$. This space as a manifold with boundary can be embedded in $\mathcal{R}(T_i) \times \mathcal{R}(T_i)$ in a natural way. By Hypothesis 1, the restriction map $r : M_\eta(\mathfrak{W}_1^i) \times M_\eta(\mathfrak{W}_2^i) \rightarrow \mathcal{R}(Y_0) \times \mathcal{R}(T_i) \times \mathcal{R}(T_i) \times \mathcal{R}(Y_1)$ is transverse to the sub-manifolds $\mathcal{R}(Y_0) \times \mathcal{M}_\gamma \times \mathcal{R}(Y_1)$. Note that $r^{-1}(\mathcal{M}(\mathcal{R}(T_i), \mathcal{R}(T_i)))$ is an open subset of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. Therefore, $r^{-1}(\mathcal{M}_\gamma)$ is a 1-dimensional manifold that its interior is a subset of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ and its boundary is equal to $\widetilde{M}_\eta(\mathfrak{W}_1^i, \alpha, \gamma) \times \widetilde{M}_\eta(\mathfrak{W}_2^i, \gamma, \beta)$. This verifies that the first term in (30) parametrizes part of the open ends of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. A similar argument shows that the other two terms parametrize other open ends and these three sets of the open ends are mutually disjoint. \square

Lemma 1.16. *For the family of metrics G as above, we have:*

$$\sum_i f_{\mathfrak{W}_1^i \times \mathfrak{W}_2^i}^\eta = d \circ f_{\mathfrak{W}}^\eta + f_{\mathfrak{W}}^\eta \circ d \quad (31)$$

If $W \setminus T_i$ is the composition of the cobordisms $W_1^i : Y_0 \rightarrow T_i$ and $W_2^i : T_i \rightarrow Y_1$, then $f_{\mathfrak{W}_1^i \times \mathfrak{W}_2^i}^\eta = f_{\mathfrak{W}_1^i}^\eta \circ f_{\mathfrak{W}_2^i}^\eta$. If $W \setminus T_i$ is the union of the cobordisms $W_1^i : Y_0 \rightarrow T_i \coprod Y_1$ and $W_2^i : T_i \rightarrow \emptyset$, then:

$$f_{\mathfrak{W}_1^i}^\eta : \text{PFC}(Y_0) \rightarrow \text{PFC}(T_i \coprod Y_1) = \text{PFC}(T_i) \otimes \text{PFC}(Y_1) \quad f_{\mathfrak{W}_2^i}^\eta : \text{PFC}(T_i) \rightarrow \Lambda_0$$

In this case, $f_{\mathfrak{W}_1^i \times \mathfrak{W}_2^i}^\eta$ is equal to $(f_{\mathfrak{W}_2^i}^\eta \otimes id) \circ f_{\mathfrak{W}_1^i}^\eta$

Proof. By Lemma 1.15 elements of the set (29) form the boundary of a 1-dimensional manifold. In particular, they can be divided into pairs where elements of each pair consists of open ends of an interval. Elements of each pair have the same energy, because they are connected by a path in $M_\eta(\mathfrak{W})$. Therefore, the weighted sum of the points is zero. Weighted number of the points in the first, second, and third sets of (29), respectively, correspond to the coefficients of $\langle \beta \rangle$ in $\sum_{i=1}^k f_{\mathfrak{W}_1^i \times \mathfrak{W}_2^i}^\eta(\langle \alpha \rangle)$, $f_{\mathfrak{W}}^\eta \circ d(\langle \alpha \rangle)$, and $d \circ f_{\mathfrak{W}}^\eta(\langle \alpha \rangle)$. Thus vanishing of the summation of these numbers verifies the above relation. \square

In the case that G has only one metric g , Lemma 1.16 asserts that $f_{\mathfrak{W}}^\eta$ is a chain map. We will write $\text{PFC}(W)$ for $f_{\mathfrak{W}}^\eta$ in this special case, and the resulting map at the level of homology will be denoted by $\text{PFH}(W)$. Standard arguments show that:

Proposition 1.17. *The map $\text{PFH}(W) : \text{PFH}(Y_0) \rightarrow \text{PFH}(Y_1)$ is independent of the perturbation and the metric on W . If $W = W_0 \circ W_1$ then $\text{PFH}(W) = \text{PFH}(W_1) \circ \text{PFH}(W_0)$.*

1.4 Oriented Moduli Spaces and a Local Coefficient System

In the previous subsections, plane Floer homology is defined as a functor from the category of 3-manifolds and cobordisms between them to the category of \mathbb{Z} -graded Λ_0 -modules. In this section, this functor will be lifted to the category of \mathbb{Z} -graded modules over the following ring:

$$\tilde{\Lambda}_0 := \left\{ \sum_{i=1}^{\infty} a_i u^{q_i} \mid a_i \in \mathbb{Z}, q_i \in \mathbb{Q}^{\geq 0}, \lim_{i \rightarrow \infty} q_i = \infty \right\} \quad (32)$$

In the above definition, if we allow the exponents q_i to be negative, then the resulting Novikov ring is denoted by $\tilde{\Lambda}$. To define such lift of plane Floer homology, it is necessary to change slightly the definition of morphisms in our topological category. In the new topological category, a morphism between two 3-manifolds is a cobordism with a choice of a homology orientation. Composition of two morphisms is also given by the composition of the underlying cobordisms equipped with the composition of homology orientations. As before, an important feature of the cobordism maps is the possibility of extending their definition to the more general case of family of metrics.

Firstly we set our conventions for orienting the elements of the real K -group of a topological space. For a topological space G , a line bundle can be associated with each element of $KO(G)$ that is called the *orientation bundle*. This line bundle for $O = [E_1] - [E_2] \in KO(G)$ is defined as $l_O = \Lambda^{\max}(E_1) \otimes (\Lambda^{\max} E_2)^*$. Different representations of this element of $KO(G)$ gives rise to naturally isomorphic line bundles (more precisely, $\mathbb{Z}/2\mathbb{Z}$ -bundles). As an example, the determinant bundle of a family of metrics is the orientation bundle of the index bundle. If $O = [E_1] - [E_2]$ and $P = [F_1] - [F_2]$ are two elements of $KO(G)$ then $l_O \otimes l_P$ and l_{O+P} are isomorphic and we fix this isomorphism to be:

$$(e_1 \wedge e_2^*) \otimes (f_1 \wedge f_2^*) \in l_O \otimes l_P \rightarrow (-1)^{\dim(E_2) \cdot \dim(F_1)} (e_1 \wedge f_1) \wedge (e_2^* \wedge f_2^*) \in l_{O+P}$$

where $e_1 \in \Lambda^{\max} E_1$, and so on. In what follows, this convention is used to trivialize l_{O+P} when the line bundles l_O and l_P are trivialized. Note that, the isomorphism depends on the order of O and P , and reversing this order changes the isomorphism by $(-1)^{\text{rk}(O) \cdot \text{rk}(P)}$.

For a 3-manifold Y , fix a Morse-Smale function f on $\mathcal{R}(Y)$ and define $(\widetilde{\text{PFC}}(Y), \tilde{d}_p)$ to be the Morse complex of f with coefficients in $\tilde{\Lambda}_0$ (cf., for example, [9]). Because $\tilde{\Lambda}_0$ is a characteristic zero ring, to define this chain complex, one needs to work with oriented moduli spaces of gradient flow lines between the critical points. For the convenience of the reader, we briefly recall the definition of the Morse complex in this case. For a critical point α of the Morse function f , let \mathcal{K}_α^+ and \mathcal{K}_α^- be the tangent bundles of the stable and the unstable manifolds of α . Because these manifolds are contractible, the tangent bundles are trivial. Define $o(\alpha)$ to be the abelian group generated by the two different orientations of \mathcal{K}_α^+ with the relation that these two generators are reverse of each other. Clearly, this group has rank 1. However, it does not have a distinguished generator. With this convention, $\widetilde{\text{PFC}} = \bigoplus_{\alpha \in \text{Crit}(f)} o(\alpha) \otimes_{\mathbb{Z}} \tilde{\Lambda}_0$. In order to define the differential, we orient the moduli spaces of flow lines in the following way. For two distinct critical points α and β , let $\check{\mathcal{M}}(\alpha, \beta)$ be the set of parametrized flow lines from α to β . By the aid of inclusion maps of $\check{\mathcal{M}}(\alpha, \beta)$ in the stable and the unstable manifolds, $[\mathcal{K}_\beta^+] - [\mathcal{K}_\alpha^+]$ can be regarded as an element of $KO(\check{\mathcal{M}}(\alpha, \beta))$, and is canonically isomorphic to $[T\check{\mathcal{M}}(\alpha, \beta)]$. The translation action of \mathbb{R} on

$\widetilde{\mathcal{M}}(\alpha, \beta)$ lets us to view this manifolds as a fiber bundle over $\mathcal{M}(\alpha, \beta)$ with the fiber \mathbb{R} . Thus, as an identity in $KO(\widetilde{\mathcal{M}}(\alpha, \beta))$:

$$[\mathbb{R}] + [q^*(T\mathcal{M}(\alpha, \beta))] = [\mathcal{K}_\beta^+] - [\mathcal{K}_\alpha^+]$$

where $q : \widetilde{\mathcal{M}}(\alpha, \beta) \rightarrow \mathcal{M}(\alpha, \beta)$ is the quotient map. Therefore, given elements of $o(\alpha)$ and $o(\beta)$, we can orient $T\mathcal{M}(\alpha, \beta)$. In the case that $\mathcal{M}(\alpha, \beta)$ is 0-dimensional, the orientation assigns a sign to each gradient flow line and the signed count of the points of $\mathcal{M}(\alpha, \beta)$ is used to define the differential of the Morse complex. The homology of the complex $(\widetilde{\text{PFC}}(Y), \widetilde{d}_P)$, denoted by $\widetilde{\text{PFH}}(Y)$, is the oriented plane Floer homology of Y .

Remark 1.18. The standard convention to orient the moduli space $\mathcal{M}(\alpha, \beta)$ uses orientations of \mathcal{K}_α^- and \mathcal{K}_β^- rather than \mathcal{K}_α^+ and \mathcal{K}_β^+ . In fact, the standard orientation gives rise to a Morse homology group that is canonically isomorphic to singular homology groups. However, note that if M is a compact manifold and f is a Morse-Smale function, then the stable manifolds (respectively, unstable manifolds) of f are the same as unstable manifolds (respectively, stable manifolds) of $-f$. Therefore, the Morse complex for the homology of M , defined by f and the orientations of the stable manifolds of f , is the same as the Morse complex for the cohomology of M , defined by $-f$ and the orientations of the unstable manifolds of $-f$. This is a manifestation of Poincaré duality at the level of Morse complexes. Thus $\widetilde{\text{PFH}}(Y)$ is in fact naturally isomorphic to the cohomology of $\mathcal{R}(Y)$, and in order to identify it with the homology of $\mathcal{R}(Y)$, one needs to fix an orientation of the representation variety.

Let \mathfrak{W} be a family of metrics on a cobordism $W : Y_0 \rightarrow Y_1$ parametrized by a cornered manifold G . Let also G^1, \dots, G^m be the set of open faces of codimension 1 in G such that $G^i = G_1^i \times G_2^i$ and the family of metrics $\mathfrak{W}|_{G^i}$ has the form $\mathfrak{W}_1^i \times \mathfrak{W}_2^i$. Here \mathfrak{W}_j^i is a family of metrics parametrized by G_j^i . We assume that for each i, j , an orientation of $\text{ind}(\mathfrak{W}_j^i; G_j^i)$ and also an orientation of $\text{ind}(\mathfrak{W}; G)$ are fixed such that the following holds. The orientation of $\text{ind}(\mathfrak{W}; G)$ induces an orientation of $\text{ind}(\mathfrak{W}|_{G^i}; G^i) \in KO(G^i)$ which is required to be the same as the orientation given by the sum $\text{ind}(\mathfrak{W}_1^i; G_1^i) + \text{ind}(\mathfrak{W}_2^i; G_2^i)$. Furthermore, we assume that G and all G_j^i 's are oriented. Thus G^i can be either oriented as a codimension 1 face of G by the outward-normal-first convention or as the product $G_1^i \times G_2^i$ by the first-summand-first convention. We demand these two orientations to be the same. We will write $\text{ind}(W; \mathbf{G})$ for the pull-back of the index bundle $\text{ind}(W; G)$ to the fattening \mathbf{G} by the contraction map.

The following lemma explains how to orient $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$:

Lemma 1.19. *Suppose \mathfrak{W} is a family of metrics as above and η is an admissible perturbation for this family that satisfies Hypothesis 1. Let also α, β be critical points of the Morse functions f_0, f_1 such that $M_\eta(\mathfrak{W}, \alpha, \beta)$ is either 0- or 1-dimensional. Then the orientation double cover of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ is canonically isomorphic to the pull-back of the orientation bundle of:*

$$-[\mathcal{K}_\alpha^+] + \text{ind}(\mathfrak{W}; \mathbf{G}) + [\mathcal{K}_\beta^+] + [T\mathbf{G}] \quad (33)$$

by the projection map $\pi : \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta) \rightarrow \mathbf{G}$.

In (33), $[\mathcal{K}_\alpha^+]$ and $[\mathcal{K}_\beta^+]$ are given by the pull-back of arbitrary constant maps from G to the stable manifolds of α and β . These elements of $KO(\mathbf{G})$ are well-defined because \mathcal{K}_α^+ and \mathcal{K}_β^+ are bundles over contractible spaces.

Proof. Firstly consider the case that $M_\eta(\mathfrak{W}, \alpha, \beta)$ is 0-dimensional. Fix $p \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$, and without loss of generality assume that $p \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ for the same choice of H that produces (24). Recall that this set is described as:

$$\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H = \{([\underline{A}], g, t_{k+1}, \dots, t_i) \mid ([\underline{A}], g) \in M_{\eta'}(\mathfrak{W}', \alpha, \beta), \varphi_j^{t_j}(r_j^{out}([\underline{A}])) = \varphi_j^{-t_j}(r_j^{in}([\underline{A}]))\}$$

where the definitions of W' , \mathfrak{W}' and $M_{\eta'}(\mathfrak{W}', \alpha, \beta)$ are given in the discussion before Diagram (23). Schematically, the set $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ is equal to $F^{-1}(C)$ for a Fredholm map of Banach manifolds $F : B_1 \rightarrow B_2$ and a finite dimensional smooth sub-manifold $C \subset B_2$. Hypothesis 1 asserts that F is transversal to C and the tangent space of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ at $p = ([\underline{A}], g, t_{k+1}, \dots, t_i)$ is given by the kernel of the composition of $D_p F$ and the projection map to the the normal bundle of C at the point $F(p)$. More explicitly, the tangent space $T_p \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ is given by the kernel of a map D that has the following form:

$$\begin{aligned} D : L_{k, \text{ext}}^2(W', i\Lambda^1) \oplus T_g G' \oplus \mathbb{R}^{i-k} &\rightarrow \\ &\rightarrow L_{k-1, \delta}^2(W', i\Lambda^0)_0 \oplus L_{k-1, \delta}^2(W', i\Lambda^+) \oplus \mathcal{K}_\alpha^+ \oplus C(T_{k+1}) \oplus \dots \oplus C(T_i) \oplus \mathcal{K}_\beta^- \\ D &= \begin{pmatrix} -d^* \oplus d^+ & * & * \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \end{pmatrix} \end{aligned}$$

Here $C(T_j)$ denotes the space of harmonic 1-forms on T_j . Although the terms which are represented by $*$ above can be spelled out in more detail, the only point that we need about them is that for each of them either the domain or the codomain is finite dimensional. The direct sum $D \oplus id_{\mathcal{K}_\beta^+}$ of D and the identity map of \mathcal{K}_β^+ has the same kernel as D , and there is a homotopy from this operator through the set of Fredholm operators to the following one:

$$D' = A_\alpha \oplus \mathcal{D}_{W'} \oplus B_\beta \oplus C_G$$

where $A_\alpha : \{0\} \rightarrow \mathcal{K}_\alpha^+$, $B_\beta : \mathcal{K}_\beta^+ \rightarrow \{0\}$ and $C_G : T_{(g, t_{k+1}, \dots, t_i)} \mathbf{G} \rightarrow \{0\}$. Orientation of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$ at p is determined by an orientation of the kernel (and hence the index) of D . The homotopy to D' shows the set of orientations of the index of D is canonically isomorphic to the set of orientations of the index of D' . The latter can be identified with the orientation of:

$$-[\mathcal{K}_\alpha^+] + \text{ind}(\mathfrak{W}; \mathbf{G}) + [\mathcal{K}_\beta^+] + [T\mathbf{G}]$$

at (g, t_{k+1}, \dots, t_i) . It is also easy to check that these identifications for different choices of H are compatible.

In the case that $M_\eta(\mathfrak{W}, \alpha, \beta)$ is 1-dimensional, the elements of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ that lie over the interior of H can be oriented analogously, and this orientation can be extended to the boundary of

H in the obvious way. It is again straightforward to check that these orientations are compatible and define an orientation of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. \square

Remark 1.20. In the context of Lemma 1.19 consider the special case that G has only one element g . Let L be a complex line bundle whose restriction to the ends of $W : Y_0 \rightarrow Y_1$ is trivialized. Let also A be a connection on L such that the induced connections on Y_0 and Y_1 are trivial and $F^{+g}(A) = 0$. Tensor product with the connection A , defines a map from $\widetilde{M}_\eta(W^g, \alpha, \beta)$ to itself. The proof of Lemma 1.19 shows that this map is orientation preserving.

Suppose that $M_\eta(\mathfrak{W}, \alpha, \beta)$ is 0-dimensional. In the previous subsection, the matrix entry $f_{\mathfrak{W}}^\eta(\alpha, \beta)$ was defined by the weighted count of the points of the moduli space $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. To lift this matrix entry element to a $\tilde{\Lambda}_0$ -module homomorphism $\tilde{f}_{\mathfrak{W}}^\eta(\alpha, \beta) : o(\alpha) \otimes \tilde{\Lambda}_0 \rightarrow o(\beta) \otimes \tilde{\Lambda}_0$, fix a generator for each of $o(\alpha)$ and $o(\beta)$ (i.e. an orientation for each of \mathcal{K}_α^+ and \mathcal{K}_β^+). We can orient (33) using our convention to orient the direct sums. According to Lemma 1.19, this determines an orientation of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$, i.e., a sign assignment to the points of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. For $p \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$, define $\epsilon_p : o(\alpha) \rightarrow o(\beta)$ to be the map that sends the generator of $o(\alpha)$ to the generator of $o(\beta)$ if the sign of p is positive. Otherwise ϵ_p maps the generator of $o(\alpha)$ to reverse of the generator of $o(\beta)$. Define:

$$\tilde{f}_{\mathfrak{W}}^\eta(\alpha, \beta) := \sum_{p \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)} \epsilon_p \otimes u^{e(p)}. \quad (34)$$

Lemma 1.21. *With our conventions as above:*

$$\sum_i \tilde{f}_{\mathfrak{W}_1^i \times \mathfrak{W}_2^i}^\eta = \tilde{d}_p \circ \tilde{f}_{\mathfrak{W}}^\eta - (-1)^{\dim(G)} \tilde{f}_{\mathfrak{W}}^\eta \circ \tilde{d}_p \quad (35)$$

Proof. This lemma is the oriented version of Lemma 1.16 and the proof is very similar. Lemma 1.19 explains how to orient $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ when it is 1-dimensional. The terms in (35) are in correspondence with the oriented boundary of such 1-dimensional manifolds. We leave it to the reader to check that our conventions produce the signs in (35). \square

Now let $W : Y_0 \rightarrow Y_1$ be a cobordism with a homology orientation, and g be a metric on W . By Lemma 1.21, a $\tilde{\Lambda}_0$ -module homomorphism $\widetilde{\text{PFH}}(W) : \widetilde{\text{PFH}}(Y_0) \rightarrow \widetilde{\text{PFH}}(Y_1)$ can be defined by $\tilde{f}_{W^g}^\eta$. An analogue of Lemma 1.17 holds for $\widetilde{\text{PFH}}$, namely, $\widetilde{\text{PFH}}$ defines a functor from the category of 3-manifolds and cobordisms with homology orientations to the category of $\tilde{\Lambda}_0$ -modules.

To characterize the map $\widetilde{\text{PFH}}(W)$, we shall recall the definition of *pull-up-push-down* map. In the following diagram, suppose M, R_0, R_1 are smooth closed manifolds and r_0, r_1 are smooth maps:

$$\begin{array}{ccc} & M & \\ r_0 \swarrow & & \searrow r_1 \\ R_0 & & R_1 \end{array} \quad (36)$$

The pull-up-push-down map from $H^*(R_0, \mathbb{Z})$ to $H^*(R_1, \mathbb{Z})$ is given by:

$$F_M := \text{PD}_{R_1}^{-1} \circ (r_1)_* \circ \text{PD}_M \circ (r_0)^*$$

where PD_X stands for the Poincaré duality isomorphism from $H^*(X, \mathbb{Z})$ to $H_*(X, \mathbb{Z})$. If f_0, f_1 are respectively Morse functions on R_0, R_1 , then the pull-up-push-down map can be lifted to the level of Morse complexes by mapping a critical point α of f_0 to:

$$\sum_{\beta \in \text{Cirt}(f_1)} f_M(\alpha, \beta) \beta$$

Here $f_M(\alpha, \beta)$ is the signed count of the points of:

$$\{z \in M \mid (r_0(z), r_1(z)) \in U_\alpha \times S_\beta\}$$

where the signs of the points of this set are determined by an orientation of:

$$-[\mathcal{K}_\alpha^+] + [TM] - [r_1^*TR_1] + [\mathcal{K}_\beta^+]$$

The involved moduli space in the definition of $\widetilde{\text{PFH}}(W)$ is $M_\nu(W)$ for a generic choice of a 2-form ν . Lemma 1.6 implies that $\widetilde{\text{PFC}}(W)$ (and hence $\widetilde{\text{PFH}}(W)$) vanishes if $b^+(W) > 0$. On the other hand, if $b^+(W) = 0$ and \mathfrak{s} is a spin^c structure on W , with $\mathfrak{s}|_{Y_i} = \mathfrak{t}_i$, then the diagram:

$$\begin{array}{ccc} & M_\nu(W, \mathfrak{s}) & \\ r_0 \swarrow & & \searrow r_1 \\ \mathcal{R}(Y_0, \mathfrak{t}_0) & & \mathcal{R}(Y_1, \mathfrak{t}_1) \end{array} \quad (37)$$

can be identified with:

$$\begin{array}{ccc} & J(W) & \\ i_0 \swarrow & & \searrow i_1 \\ J(Y_0) & & J(Y_1) \end{array} \quad (38)$$

where i_0 and i_1 are induced by the inclusion maps of Y_0 and Y_1 in W (Remark 1.7). Although the identification of $\mathcal{R}(Y_i, \mathfrak{t}_i)$ with $J(Y_i)$ is canonical only up to translation, the identification of the homology groups of these spaces is canonical because the action of translation on the homology of a torus is trivial. Therefore, the part of $\widetilde{\text{PFH}}(W)$, that is induced by the moduli space $M_\nu(W, \mathfrak{s})$, maps the summand $H^*(\mathcal{R}(Y_0, \mathfrak{t}_0); \tilde{\Lambda}_0)$ of $\widetilde{\text{PFH}}(Y_0)$ by:

$$F_{J(W)} \otimes id_{\tilde{\Lambda}_0} : H^*(\mathcal{R}(Y_0, \mathfrak{t}_0); \tilde{\Lambda}_0) = H^*(J(Y_0)) \otimes \tilde{\Lambda}_0 \rightarrow H^*(\mathcal{R}(Y_1, \mathfrak{t}_1); \tilde{\Lambda}_0) = H^*(J(Y_1)) \otimes \tilde{\Lambda}_0$$

and maps the summands, corresponding to the other torsion spin^c structures of Y_0 , to zero. We will write $D(W; Y_0, Y_1)(\mathfrak{t}_0, \mathfrak{t}_1)$ for this map. In summary we have:

Proposition 1.22. *Suppose $W : Y_1 \rightarrow Y_2$ is a cobordism:*

i) If $b^+(W) > 0$, then $\widetilde{\text{PFH}}(W) = 0$.

ii) If $b^+(W) = 0$, then:

$$\widetilde{\text{PFH}}(W) = \sum_{\mathfrak{s}: \mathfrak{t}_0 \rightarrow \mathfrak{t}_1} u^{-c_1(\mathfrak{s}) \cdot c_1(\mathfrak{s})} D(W; Y_0, Y_1)(\mathfrak{t}_0, \mathfrak{t}_1)$$

where $\mathfrak{s}: \mathfrak{t}_0 \rightarrow \mathfrak{t}_1$ denotes a spin^c structure \mathfrak{s} on W , with $\mathfrak{s}|_{Y_0} = \mathfrak{t}_0$ and $\mathfrak{s}|_{Y_1} = \mathfrak{t}_1$ being torsion spin^c structures.

For the purpose of proving the existence of exact triangles in the next section, we need to extend the definition of plane Floer homology again. Suppose Y is a connected and closed 3-manifold and ζ is an embedded closed 1-manifold. We shall assign $\widetilde{\text{PFH}}(Y, \zeta)$ to this pair that is equal to $\widetilde{\text{PFH}}(Y)$ in the case that ζ is empty. This extension is defined with the aid of a local coefficient system Γ_ζ defined on $\mathcal{R}(Y)$. For a cobordism of the pairs $(W, Z): (Y_0, \zeta_0) \rightarrow (Y_1, \zeta_1)$, we also define a map $\widetilde{\text{PFH}}(W; Z): \widetilde{\text{PFH}}(Y_0; \zeta_0) \rightarrow \widetilde{\text{PFH}}(Y_1, \zeta_1)$. More generally, cobordism maps can be defined for the family of metrics over the pair (W, Z) (Definition 1.25).

Fix a pair (Y, ζ) and let α be the equivalence class of a spin^c connection on Y under the action of the gauge group. A triple $(W, Z, [\underline{A}])$ fills (Y, ζ, α) , if W is a compact oriented 4-manifold such that $\partial W = Y$, Z is a compact, oriented and embedded surface in W with $\partial Z = \zeta$, and $[\underline{A}]$ is the equivalence class of a spin^c connection \underline{A} on W with $[\underline{A}]|_{\partial W} = \alpha$. For technical convenience, we assume that the restriction of \underline{A} to a collar neighborhood of ∂W is given by the pull-back of a representative of α .

Lemma 1.23. *Let $(W, Z, [\underline{A}])$ fills (Y, ζ, α) and fix a framing for the 1-manifold ζ . Use the framing to define:*

$$\iota(W, Z, [\underline{A}]) := \frac{1}{2}(Z \cdot Z - \frac{i}{2\pi} \int_Z F(A)) \quad (39)$$

This function is constant under the homotopies of the connection $[\underline{A}]$ that fix $[\underline{A}]|_Y$. If $(W', Z', [\underline{A}'])$ is another triple that fills (Y, ζ, α) , then $\iota(W, Z, [\underline{A}]) - \iota(W', Z', [\underline{A}'])$ is an integer independent of the choice of the framing.

Proof. Glue the reverse of $(W', Z', [\underline{A}'])$ to $(W, Z, [\underline{A}])$ in order to form $(W \# W', Z \# Z', [\underline{A} \# \underline{A}'])$:

$$\iota(W, Z, [\underline{A}]) - \iota(W', Z', [\underline{A}']) = \frac{1}{2}(Z \# Z' \cdot Z \# Z' - \frac{i}{2\pi} \int_Z F(A \# A')) = \frac{1}{2}(Z \# Z' \cdot Z \# Z' - c_1(\mathfrak{s} \# \mathfrak{s}') [Z \# Z'])$$

where $\mathfrak{s} \# \mathfrak{s}'$ is the spin^c structure that carries $[\underline{A} \# \underline{A}']$. Since $c_1(\mathfrak{s} \# \mathfrak{s}') \equiv \omega_1(W \# W') \pmod{2}$, Wu's formula asserts that $\iota(W, Z, [\underline{A}]) - \iota(W', Z', [\underline{A}'])$ is an integer independent of the framing. Also, this topological interpretation implies that ι is constant under a homotopy that fixes the restriction of $[\underline{A}]$ to the boundary. \square

Let $\Gamma_\zeta(\alpha)$ be the abelian group generated by all the triples that fill (Y, ζ, α) with the relation that for any two triples $(W, Z, [\underline{A}]), (W', Z', [\underline{A}'])$ we have $(W, Z, [\underline{A}]) = \pm(W', Z', [\underline{A}'])$ where the sign is positive or negative depending on whether $\iota(W, Z, [\underline{A}]) - \iota(W', Z', [\underline{A}'])$ is even or odd. Take a cobordism of the pair $(W, Z): (Y_0, \zeta_0) \rightarrow (Y_1, \zeta_1)$ and let $[\underline{A}]$ be the equivalence class of a spin^c

connection on W such that its restrictions to Y_0, Y_1 are equal to α_0, α_1 , respectively. Then we can define a map of the local coefficient systems:

$$\Gamma_Z([\underline{A}]) : \Gamma_{\zeta_0}(\alpha_0) \rightarrow \Gamma_{\zeta_1}(\alpha_1)$$

$$\Gamma_Z([\underline{A}])([W_0, Z_0, [\underline{A}_0]]) = (-1)^{\iota(W_0 \# W \# W_1, Z_0 \# Z \# Z_1, [\underline{A}_0 \# \underline{A} \# \underline{A}_1])} [W_1, Z_1, [\underline{A}_1]]$$

where $(W_0, Z_0, [\underline{A}_0]), (W_1, Z_1, [\underline{A}_1])$ are fillings of $(Y_0, \zeta_0, \alpha_0), (Y_1, \zeta_1, \alpha_1)$, respectively. Here, ι is defined as in (39). This definition is well-defined and does not depend on the choice of the generators of $\Gamma_{\zeta_0}(\alpha_0)$ and $\Gamma_{\zeta_1}(\alpha_1)$.

Suppose $\{\alpha_t\}$ is a path of spin^c connections. This path determines a spin^c connection (in temporal gauge) on $I \times Y$. Thus for any 1-manifold $\zeta \subset Y$, we can define an isomorphism $\Gamma_{I \times \zeta}(\{\alpha_t\}) : \Gamma_{\zeta}(\alpha_0) \rightarrow \Gamma_{\zeta}(\alpha_1)$. This map depends only on the homotopy class of the path. Thus Γ_{ζ} defines a local coefficient system on $\mathcal{R}(Y)$. The plane Floer homology of the pair (Y, ζ) , denoted by $\widetilde{\text{PFH}}(Y, \zeta)$, is the homology of $\mathcal{R}(Y)$ with coefficients in Γ_{ζ} . The Morse complex that realizes this homology group is denoted by $\widetilde{\text{PFC}}(Y, \zeta)$ (cf. [9, Section 2]). In particular, for each critical point α , this complex has a generator of the form $\Gamma_{\zeta}(\alpha) \otimes o(\alpha) \otimes \tilde{\Lambda}_0$.

Lemma 1.24. *If ζ and ζ' are 1-dimensional embedded sub-manifolds in a 3-manifold Y , inducing the same elements of $H_1(Y; \mathbb{Z}/2\mathbb{Z})$, then the local coefficient systems Γ_{ζ} and $\Gamma_{\zeta'}$ are isomorphic. In particular, $\widetilde{\text{PFH}}(Y, \zeta) \cong \widetilde{\text{PFH}}(Y, \zeta')$.*

Proof. Let γ be a closed loop in $\mathcal{R}(Y)$. This loop induces a spin^c connection $[\underline{A}]$ on $S^1 \times Y$. Since $S^1 \times \zeta, S^1 \times \zeta'$ have zero self-intersection and $S^1 \times Y$ is spin, the difference between the following numbers is even:

$$\frac{1}{2}((S^1 \times \zeta) \cdot (S^1 \times \zeta) - \frac{i}{2\pi} \int_{S^1 \times \zeta} F(A)) \quad \frac{1}{2}((S^1 \times \zeta') \cdot (S^1 \times \zeta') - \frac{i}{2\pi} \int_{S^1 \times \zeta'} F(A))$$

Therefore, Γ_{ζ} and $\Gamma_{\zeta'}$ are isomorphic local coefficient systems. \square

Definition 1.25. A family of non-broken metrics $(\mathfrak{W}, \mathfrak{Z})$ on the pair (W, Z) and parametrized by G consists of a family of non-broken metrics \mathfrak{W} over W and an embedded manifold $\mathfrak{Z} \subset \mathfrak{W}$ such that \mathfrak{Z} is a fiber bundle over G with fiber Z .

Given a pair (W, Z) , let T be a cut in W transversal to Z and ξ be the embedded 1-manifold $Z \cap T$. Let also $\bigcup_{j=1}^l T_j$ be the decomposition of T to the connected components, and $\xi_j = \xi \cap T_j$. Removing T decomposes W and Z to $W_0 \coprod \dots \coprod W_l$ and $Z_0 \coprod \dots \coprod Z_l$ such that $Z_j \subset W_j$. Fix the labeling such that the pair (T_j, ξ_j) is equal to the incoming end of (W_j, Z_j) and an outgoing end of $(W_{o(j)}, Z_{o(j)})$:

Definition 1.26. A broken metric on (W, Z) with a cut along (T, ξ) is given by a metric on W_j for each j such that the following holds. After an appropriate identification of the incoming end of (W_j, Z_j) (respectively the outgoing end of $(W_{o(j)}, Z_{o(j)})$ in correspondence with (T_j, ξ_j)) with $\mathbb{R}^{\leq 0} \times (T_j, \xi_j)$ (respectively with $\mathbb{R}^{\geq 0} \times (T_j, \xi_j)$), the metric on $\mathbb{R}^{\leq 0} \times T_j \subset W_j$ (respectively $\mathbb{R}^{\geq 0} \times T_j \subset W_{o(j)}$) is given by the product metric. Furthermore, we require that these metrics are given by the same metrics on T_j .

A broken metric as above can be used to define a family of (possibly broken) metrics on (W, Z) that is parametrized with $[0, \infty]^i$. As in the case of the family of metrics on a cobordism W , we can combine the definitions 1.25 and 1.26 to define a family of (possibly broken) metrics on a pair (W, Z) .

A family of metrics $(\mathfrak{W}, \mathfrak{Z})$ on the pair $(W, Z) : (Y_0, \zeta_0) \rightarrow (Y_1, \zeta_1)$ gives rise to a map between the chain complexes $\widehat{\text{PFC}}(Y_0, \zeta_0), \widehat{\text{PFC}}(Y_1, \zeta_1)$. To that end, we need to extend the definition of $\Gamma_Z([\underline{A}])$ to the case that $[\underline{A}]$ is an element of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$. Fix $p \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)$ and without loss of generality assume that $p = ([\underline{A}], g, t_{k+1}, \dots, t_i)$ is an element of $\widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)|_H$, defined in (24). Then \underline{A} is a spin^c connection on $W' = W \setminus \bigcup_{j=k+1}^n T_j$. For each j , the flow φ_j determines a path from $r_j^{\text{out}}([\underline{A}])$ to $r_j^{\text{in}}([\underline{A}])$. Use the same methods as in subsection 1.2 to glue these paths and \underline{A} to form a spin^c connection on W . This connection can be used to define a homomorphism $\Gamma_Z([\underline{A}]) : \Gamma_{\zeta_0}(\alpha) \rightarrow \Gamma_{\zeta_1}(\beta)$. This map does not depend on the choices involved in gluing and hence is well-defined. We can define a map $\tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta : \widehat{\text{PFC}}(Y_0, \zeta_0) \rightarrow \widehat{\text{PFC}}(Y_1, \zeta_1)$ with the following matrix entry:

$$\begin{aligned} \tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta(\alpha, \beta) &: \Gamma_{\zeta_0}(\alpha) \otimes o(\alpha) \otimes \tilde{\Lambda}_0 \rightarrow \Gamma_{\zeta_1}(\beta) \otimes o(\beta) \otimes \tilde{\Lambda}_0 \\ \tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta(\alpha, \beta) &:= \sum_{p \in \widetilde{M}_\eta(\mathfrak{W}, \alpha, \beta)} \Gamma_Z(p) \otimes \epsilon_p \otimes u^{e(p)} \end{aligned} \quad (40)$$

This map has degree:

$$\deg_p(\tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta) = 2 \dim(G) - \sigma(W) - \chi(W) \quad (41)$$

Because the map Γ_Z is locally constant, an analogue of (35) holds for the map $\tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta$:

$$\sum_i \tilde{f}_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i) \times (\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta = \tilde{d}_p \circ \tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta - (-1)^{\dim(G)} \tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta \circ \tilde{d}_p \quad (42)$$

As a result of (42), we have a well-defined map $\widehat{\text{PFH}}(W, Z) : \widehat{\text{PFH}}(Y_0, \zeta_0) \rightarrow \widehat{\text{PFH}}(Y_1, \zeta_1)$.

Remark 1.27. As in the unoriented case, it is more convenient to write down $\tilde{f}_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i) \times (\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta$ in the left hand side of (35) in terms of $\tilde{f}_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i)}^\eta$ and $\tilde{f}_{(\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta$. However, due to possible incompatibilities in orienting moduli spaces, more care is needed. If $W \setminus Z_i$ is the composition of the cobordisms $W_1^i : Y_0 \rightarrow Z_i$ and $W_2^i : Z_i \rightarrow Y_1$, then:

$$f_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i) \times (\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta = (-1)^{\dim(G_1^i) \times \dim(G_2^i)} f_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i)}^\eta \circ f_{(\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\beta.$$

Next, let $W \setminus Z_i$ be the union of the cobordisms $W_1^i : Y_0 \rightarrow Z_i \coprod Y_1$ and $W_2^i : Z_i \rightarrow \emptyset$. If $\dim(G_2^i)$ is an even number then a similar formula holds:

$$f_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i) \times (\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta = (-1)^{\dim(G_1^i) \times \dim(G_2^i)} (f_{(\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta \otimes id) \circ f_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i)}^\eta.$$

In the case that $\dim(G_2^i)$ is an odd number, we need to introduce $s : \widehat{\text{PFC}}(Y, \zeta) \rightarrow \widehat{\text{PFC}}(Y, \zeta)$ in the following way:

$$s(\alpha) := (-1)^{b_1(Y) - \text{ind}(\alpha)} \alpha = (-1)^{\frac{\deg_p(\alpha) - b_1(Y)}{2}} \alpha$$

where α is a generator of $\widehat{\text{PFC}}(Y, \zeta)$. The map s commutes with the cobordism maps up to a sign. For example, if $(\mathfrak{W}, \mathfrak{Z})$ is a family of metrics parametrized with G on the cobordism $(W, Z) : (Y_0, \Gamma_0) \rightarrow (Y_1, \Gamma_1)$, with Y_0 and Y_1 being connected, then:

$$s \circ \tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta = (-1)^{\text{ind}(\mathcal{D}_W) + \dim(G)} \tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta \circ s. \quad (43)$$

For an odd dimensional family of metrics G_2^i , we have:

$$f_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i) \times (\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta = (-1)^{\dim(G_1^i) \times \dim(G_2^i)} (f_{(\mathfrak{W}_2^i, \mathfrak{Z}_2^i)}^\eta \otimes s) \circ f_{(\mathfrak{W}_1^i, \mathfrak{Z}_1^i)}^\eta.$$

In the following, we study the cobordism map $\tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta$ in two special cases. Suppose Z is an embedded sphere in $\overline{\mathbb{C}P^2}$ with self-intersection -1 . Let D^4 be a disc neighborhood of a point away from Z and consider $R := \overline{\mathbb{C}P^2} \setminus D^4$. Equip the pair (R, Z) , as a cobordism from (S^3, \emptyset) to the empty pair, with a metric that is cylindrical on the end. Let $\widehat{\text{PFC}}(R, Z) : \widehat{\text{PFC}}(S^3) \rightarrow \tilde{\Lambda}_0$ be the cobordism map induced by an arbitrary homology orientation of R and the zero perturbation. The representation variety $\mathcal{R}(S^3)$ consists of only one point α and $b^+(R) = b^1(R) = 0$. Therefore, the zero perturbation satisfies Hypothesis 1, and the moduli space $M(R)$ consists of one spin^c connection for each choice of spin^c structure. Any spin^c structure \mathfrak{s} on R is uniquely determined by the pairing of $c_1(\mathfrak{s})$ and Z which is an odd number. Let \mathfrak{s}_{2k+1} be the spin^c structure that this pairing is equal to $2k+1$ and \underline{A}_{2k+1} is the corresponding element of $M(R)$. The spin^c connections $\underline{A}_{2k+1}, \underline{A}_{-2k-1}$ have the same energy and $\Gamma_Z(\underline{A}_{2k+1}) = -\Gamma_Z(\underline{A}_{-2k-1})$. Furthermore, by Remark 1.20 the orientation of the moduli space at these two points match with each other. Consequently, we have the following result, which is our main motivation to introduce plane Floer homology for cobordism pairs:

Lemma 1.28. *The cobordism map $\widehat{\text{PFC}}(R, Z)$ is equal to zero.*

Remark 1.29. Removing a ball neighborhood of a point in $Z \subset \overline{\mathbb{C}P^2}$ produces a cobordism of the pairs $(R, Z \setminus D^2)$ from (S^3, S^1) to the empty pair. A similar argument as above shows that $\widehat{\text{PFC}}(R, Z \setminus D^2) = 0$.

Next, consider the cobordism U in Example 1.13 from Q to the empty 3-manifold. Extend U to a cobordism of pairs $(U, Z) : (Q, \emptyset) \rightarrow (\emptyset, \emptyset)$ by defining $Z := S_1$. The family of metrics \mathfrak{U} in Example 1.13 determines a family of metrics $(\mathfrak{U}, \mathfrak{Z})$ on the pair (U, Z) . Fix a Morse-Smale function on $\mathcal{R}(Q)$ with two critical points α of index 0 and α' of index 1 such that neither of them is equal to h_0 (defined in Example 1.13) or $-h_0$. Also fix a homology orientation for U and a generator for $o(\alpha)$. By the characterization of moduli spaces in Example 1.13, the cobordism map $f_{\mathfrak{U}, \mathfrak{Z}} : \widehat{\text{PFC}}(Q) \rightarrow \tilde{\Lambda}_0$, induced by the zero perturbation, sends the corresponding generator of $o(\alpha)$ to:

$$c = \sum_{i=0}^{\infty} \pm (2k+1) u^{(2k+1)^2} \in \tilde{\Lambda}_0 \quad (44)$$

for an appropriate choices of the signs. The ring element c can be regarded as an element of $\tilde{\Lambda}$ and independent of what the signs are it is invertible in $\tilde{\Lambda}$. On the other hand, $f_{\mathfrak{U}, \mathfrak{Z}}$ maps the summand $o(\alpha')$ to zero.

Lemma 1.30. Consider $(V := S^1 \times D^3, \emptyset)$ as a cobordism from (Q, \emptyset) to the empty pair. Then there exists a homology orientation of V such that:

$$\tilde{f}_{\mathfrak{U}, \mathfrak{Z}} = c \cdot \widetilde{\text{PFC}}(V)$$

Proof. Since $M(V) = \mathcal{R}(Q)$ and the outgoing end of V is empty, the cobordism map $\widetilde{\text{PFC}}(V)$, for appropriate choices of homology orientations of V and generators of $o(\alpha)$, maps the generator of $o(\alpha)$ to $1 \in \tilde{\Lambda}_0$ and the summand $o(\alpha')$ to zero. This verifies the claim of the lemma. \square

Remark 1.31. Consider two 2-spheres S_1 and S_2 . Then the cobordism U of Example 1.13 is diffeomorphic to the plumbing of two disc bundles with Euler number -1 over these spheres. This diffeomorphism can be chosen such that S_1 is in correspondence with the previously defined S_1 , and S_2 is sent to S'_1 with an orientation reversing map. Let \hat{Z} be the embedded surface in U that is union of a fiber of the disc bundle over S_1 and a fiber of the disc bundle over S_2 . The family of metrics \mathfrak{U} can be lifted to a family of metrics $(\mathfrak{U}, \hat{\mathfrak{Z}})$ over the pair (U, \hat{Z}) . The intersection $\check{\zeta} = \hat{Z} \cap Q$ is equal to $S^1 \times \{p, q\} \subset Q \cong S^1 \times S^2$ where $p, q \in S^2$ and the induced orientations on these two circles are reverse of each other. Since the orientations of these two circles are different, there is an embedded annulus \check{Z} in V such that its intersection with the boundary of V , identified with Q , is equal to $\check{\zeta}$. Therefore, (V, \check{Z}) is another cobordism from $(Q, \check{\zeta})$ to the empty pair. The proof of the following relation is analogous to that of Lemma 1.30:

$$\tilde{f}_{\mathfrak{U}, \hat{\mathfrak{Z}}} = c \cdot \widetilde{\text{PFC}}(V, \check{\zeta})$$

2 Link Surgery Cube Complex

Suppose (C_1, d_1) and (C_2, d_2) are two (not necessarily \mathbb{Z} -graded) chain complexes. A map $f : C_1 \rightarrow C_2$ is an anti-chain map if:

$$f \circ d_1 + d_2 \circ f = 0$$

Composition of two maps which are either chain map or anti-chain map is again either a chain map or an anti-chain map. Two anti-chain maps $f_1, f_2 : C_1 \rightarrow C_2$ are chain homotopy equivalent if there is a map $h : C_1 \rightarrow C_2$ such that:

$$f_2 - f_1 = h \circ d_1 - d_2 \circ h$$

A chain homotopy of chain maps is also defined in the standard way. Define $\mathfrak{C}(R)$ to be the category whose objects are the collection of chain complexes over a fixed ring R . A morphism from (C_1, d_1) to (C_2, d_2) is represented by either a chain map or an anti-chain map and two such morphisms represent the same elements if they are chain homotopy equivalent.

Given a general category \mathcal{C} , we can define the *isomorphism equivalence relation* on the objects of \mathcal{C} . That is to say, two objects are equivalent if there is an isomorphism between them. The equivalence class of an object with respect to this relation is called the *isomorphism class* of that object. In the case of the category $\mathfrak{C}(R)$, the isomorphism class of an object is also called the chain homotopy type of the corresponding chain complex.

Given a pair (Y, ζ) , the chain complex $\widetilde{\text{PFC}}(Y, \zeta)$ was associated with this pair in the previous section. We use $\tilde{\Lambda}$ as the coefficient ring in this section and hence $\widetilde{\text{PFC}}(Y, \zeta)$ is an object of $\mathfrak{C}(\tilde{\Lambda})$. We shall show that, for any framed link L in $Y \setminus \zeta$, one can construct a new chain complex that has the same chain homotopy type as $\widetilde{\text{PFC}}(Y, \zeta)$. In subsection 2.1, starting with the pair (Y, ζ) and L , various families of metrics are introduced. These families of metrics are used in subsection 2.2 to construct the new chain complex and to show that this chain complex has the same chain homotopy type as $\widetilde{\text{PFC}}(Y, \zeta)$.

2.1 Framed Links and Families of Metrics

Fix the lexicographical ordering on the lattice \mathbb{Z}^n which is induced by the standard ordering of integer numbers. For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, define:

$$|\mathbf{m}|_1 := \sum_{i=1}^n m_i \quad |\mathbf{m}|_\infty = \max\{m_1, \dots, m_n\}$$

Definition 2.1. A *cube* of dimension d in \mathbb{Z}^n consists of a pair (\mathbf{m}, \mathbf{n}) such that $\mathbf{m} > \mathbf{n}$, $|\mathbf{m} - \mathbf{n}|_\infty = 1$ and $|\mathbf{m} - \mathbf{n}|_1 = d$. A 1-dimensional (respectively, 2-dimensional) cube is called an edge (respectively, a face) of \mathbb{Z}^n . The set of edges and faces of \mathbb{Z}^n are denoted by $\mathcal{E}(n)$ and $\mathcal{F}(n)$.

Definition 2.2. A *double-cube* in \mathbb{Z}^n is a pair (\mathbf{m}, \mathbf{n}) such that $\mathbf{m} > \mathbf{n}$, $|\mathbf{m} - \mathbf{n}|_\infty = 2$, and the entries of \mathbf{m}, \mathbf{n} differ by 2 in exactly one instance. A *triple-cube* in \mathbb{Z}^n is a pair (\mathbf{m}, \mathbf{n}) such that $\mathbf{m} > \mathbf{n}$, $|\mathbf{m} - \mathbf{n}|_\infty = 3$. Furthermore, we require that the entries of \mathbf{m}, \mathbf{n} differ by 3 in exactly one instance and otherwise they differ by at most 1. Dimension of a double-cube (respectively, triple-cube) is equal to $|\mathbf{m} - \mathbf{n}|_1 - 1$ (respectively, $|\mathbf{m} - \mathbf{n}|_1 - 2$).

Suppose (Y, ζ) is a pair of a 3-manifold and an embedded closed 1-manifold. Let $L = \bigcup_{1 \leq i \leq n} K_i$ be an oriented framed link with n connected components in $Y \setminus \zeta$. The complement of a regular neighborhood of L in Y , denoted by Y^L , is a 3-manifold with boundary $\bigcup_{1 \leq i \leq n} \tau_i$ where τ_i is the torus boundary of a neighborhood of K_i . The complement of a regular neighborhood of only the i^{th} component is denoted by Y^{K_i} . Since K_i is framed, τ_i has well-defined oriented longitude and meridian and we will denote them by λ_i and μ_i . In particular, the orientation of $\tau_i = \partial Y^{K_i}$ determined by the outward-normal-first convention agrees with the orientation determined by the ordered pair $\{\mu_i, \lambda_i\}$. Let γ_i^0, γ_i^1 , and γ_i^2 be oriented curves in τ_i in the homology class of $\lambda_i, -(\lambda_i + \mu_i)$, and μ_i . We extend these curves to the family $\{\gamma_i^j\}_{j \in \mathbb{Z}}$ by requiring that this family is 3-periodic in j . Given $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, the closed 3-manifold $Y_{\mathbf{m}}$ is defined to be the result of the Dehn filling of Y^L along the curves $\gamma_i^{m_i}$. We will write $D_{\mathbf{m}, i}$ for the solid torus which is glued to Y^L along τ_i . Note that $\gamma_i^{m_i-1}$ defines a framing of this solid torus. The 1-manifold $\zeta_{\mathbf{m}}$ is equal to the union of ζ and $\bigcup_i \zeta_{\mathbf{m}, i}$, where $\zeta_{\mathbf{m}, i}$ is the core of $D_{\mathbf{m}, i}$ if $m_i \equiv 0 \pmod{3}$, and is empty otherwise. Fix a metric on Y^L which is the standard product metric in the union $\bigcup_{1 \leq i \leq n} \tau_i$, and extend this to $Y_{\mathbf{m}}$ by the standard metric on $D_{\mathbf{m}, i}$.

For each pair (\mathbf{m}, \mathbf{n}) that $\mathbf{m} \geq \mathbf{n}$, a cobordism of the pairs $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}}) : (Y_{\mathbf{m}}, \zeta_{\mathbf{m}}) \rightarrow (Y_{\mathbf{n}}, \zeta_{\mathbf{n}})$ can be defined as follows. Firstly let (\mathbf{m}, \mathbf{n}) be a 1-dimensional cube and the i^{th} entry be the

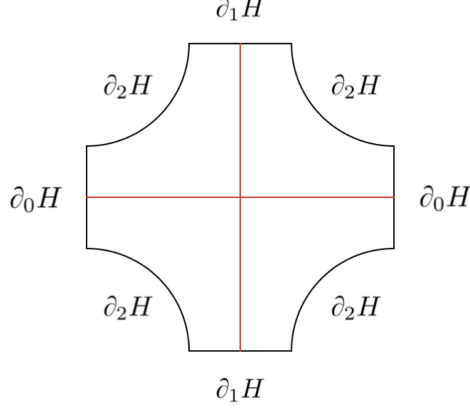


Figure 2: A schematic picture of a 2-handle; the horizontal red line represents the core of the 2-handle and the vertical red line is the representative of the co-core

only coordinate that \mathbf{m} and \mathbf{n} differs. In this case, define $W_{\mathbf{m}}^{\mathbf{n}}$ to be the elementary cobordism from $Y_{\mathbf{m}}$ to $Y_{\mathbf{n}}$ given by attaching a 2-handle along $D_{\mathbf{m},i}$. More concretely, this cobordism can be constructed as the union of $[0, 1] \times Y^{K_i}$ and the space:

$$H = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|, |z_2| \leq 1, |z_1| \cdot |z_2| \leq \frac{1}{2}\}$$

Boundary of H is the union of the following sets (Figure 2):

$$\partial_0 H = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = 1, |z_2| \leq \frac{1}{2}\} \quad \partial_1 H = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq \frac{1}{2}, |z_2| = 1\}$$

$$\partial_2 H = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|, |z_2| \leq 1, |z_1| \cdot |z_2| = \frac{1}{2}\}$$

With the aid of the negative gradient flow of the Morse function $|z_1|^2 - |z_2|^2$, the set $\partial_2 H$ can be identified with the product of the interval $[0, 1]$ and the 2-dimensional torus. The space $[0, 1] \times \partial Y^{K_i}$ can be identified with the same space using the closed curves $\gamma_i^{m_i}$ and $\gamma_i^{m_i-1}$ in ∂Y^{K_i} . The cobordism $W_{\mathbf{m}}^{\mathbf{n}}$ is given by gluing $[0, 1] \times Y^{K_i}$ to H along these parts of their boundaries and then adding the cylindrical ends. Thus $W_{\mathbf{m}}^{\mathbf{n}}$ is equal to the union of $\mathbb{R} \times Y^{K_i}$ and \bar{H} where $\bar{H} := H \cup \mathbb{R}^{\leq 0} \times \partial_0 H \cup \mathbb{R}^{\geq 0} \times \partial_1 H$. The 2-dimensional cobordism $Z_{\mathbf{m}}^{\mathbf{n}} : \zeta_{\mathbf{m}} \rightarrow \zeta_{\mathbf{n}}$ is determined by the value of $m_i = n_i + 1 \pmod{3}$. If $m_i \equiv 0 \pmod{3}$, then $\zeta_{\mathbf{n},i}$ is empty and $Z_{\mathbf{m}}^{\mathbf{n}}$ is the union of $\mathbb{R} \times \zeta_{\mathbf{n}}$ and the following cobordism from $\zeta_{\mathbf{m},i}$ to the empty set:

$$\{(z_1, 0) \mid |z_1| \leq 1\} \cup \mathbb{R}^{\leq 0} \times \{(z_1, 0) \mid |z_1| = 1\} \subset H \cup \mathbb{R}^{\leq 0} \times \partial_0 H \quad (45)$$

If $m_i \equiv 1 \pmod{3}$, then $\zeta_{\mathbf{m}}$ is empty and $Z_{\mathbf{m}}^{\mathbf{n}}$ is the union of $\mathbb{R} \times \zeta_{\mathbf{m}}$ and the following cobordism from the empty set to $\zeta_{\mathbf{n},i}$:

$$\{(0, z_2) \mid |z_2| \leq 1\} \cup \mathbb{R}^{\geq 0} \times \{(0, z_2) \mid |z_2| = 1\} \subset H \cup \mathbb{R}^{\geq 0} \times \partial_1 H. \quad (46)$$

. Finally, if $m_i \equiv 2 \pmod{3}$, then $\zeta_{\mathbf{m},i}$ and $\zeta_{\mathbf{n},i}$ are both empty and $Z_{\mathbf{m}}^{\mathbf{n}}$ is defined to be $\mathbb{R} \times \zeta_{\mathbf{m}}$

For a general pair (\mathbf{m}, \mathbf{n}) , we can find a sequence $\mathbf{m} = \mathbf{k}_0 > \mathbf{k}_1 > \dots > \mathbf{k}_{l-1} > \mathbf{k}_l = \mathbf{n}$ such that $|\mathbf{k}_i - \mathbf{k}_{i+1}|_1 = 1$. In this case we define $W_{\mathbf{m}}^{\mathbf{n}} = W_{\mathbf{k}_0}^{\mathbf{k}_1} \circ W_{\mathbf{k}_1}^{\mathbf{k}_2} \circ \dots \circ W_{\mathbf{k}_{l-1}}^{\mathbf{k}_l}$, $Z_{\mathbf{m}}^{\mathbf{n}} = Z_{\mathbf{k}_0}^{\mathbf{k}_1} \circ Z_{\mathbf{k}_1}^{\mathbf{k}_2} \circ \dots \circ Z_{\mathbf{k}_{l-1}}^{\mathbf{k}_l}$. It is clear that the diffeomorphism type of $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$ is independent of how we choose \mathbf{k}_i 's.

Lemma 2.3. *For each pair (\mathbf{m}, \mathbf{n}) with $\mathbf{m} \geq \mathbf{n}$, there exists a homology orientation $o(\mathbf{m}, \mathbf{n})$ of the cobordism $W_{\mathbf{m}}^{\mathbf{n}}$ such that if $\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}$, then the composition of $o(\mathbf{m}, \mathbf{k})$ and $o(\mathbf{k}, \mathbf{n})$ is equal to $o(\mathbf{m}, \mathbf{n})$. Furthermore, if $o'(\mathbf{m}, \mathbf{n}) \in o(W_{\mathbf{m}}^{\mathbf{n}})$ is another series of homology orientations that has the same property, then there exists $\mathfrak{z} : \mathbb{Z}^n \rightarrow \{\pm 1\}$, such that:*

$$o'(\mathbf{m}, \mathbf{n}) = \mathfrak{z}(\mathbf{m})\mathfrak{z}(\mathbf{n})o(\mathbf{m}, \mathbf{n}). \quad (47)$$

The function \mathfrak{z} is unique up to an overall sign.

Proof. The composition of homology orientations is associative and hence the first part of this lemma can be proved with the same argument as in [11, Lemma 6.1]. To prove the second part of the lemma, note that \mathbf{u} , defined as the difference of o and o' , is a map from the set of $(\mathbf{m}, \mathbf{n}) \in \mathbb{Z}^n \times \mathbb{Z}^n$, with $\mathbf{m} \geq \mathbf{n}$, to $\mathbb{Z}/2\mathbb{Z}$ that satisfies:

$$\mathbf{u}(\mathbf{m}, \mathbf{n}) = \mathbf{u}(\mathbf{m}, \mathbf{k}) + \mathbf{u}(\mathbf{k}, \mathbf{n})$$

for $\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}$. This identity implies that \mathbf{u} induces a simplicial co-cycle on \mathbb{R}^n with respect to the simplicial structure spelled out in [11]. Since \mathbb{R}^n is contractible, there is a map $\mathfrak{z} : \mathbb{Z}^n \rightarrow \mathbb{Z}/2\mathbb{Z}$ such that:

$$\mathbf{u}(\mathbf{m}, \mathbf{n}) = \mathfrak{z}(\mathbf{m}) - \mathfrak{z}(\mathbf{n})$$

that is to say \mathfrak{z} satisfies (47). □

The pairs $(Y_{\mathbf{m}}, \zeta_{\mathbf{m}})$ and $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$ are 3-periodic. That is to say, for any $\mathbf{k} \in \mathbb{Z}^n$ we have the natural identifications $(Y_{\mathbf{m}+3\mathbf{k}}, \zeta_{\mathbf{m}+3\mathbf{k}}) = (Y_{\mathbf{m}}, \zeta_{\mathbf{m}})$ and $(W_{\mathbf{m}+3\mathbf{k}}^{\mathbf{n}+3\mathbf{k}}, Z_{\mathbf{m}+3\mathbf{k}}^{\mathbf{n}+3\mathbf{k}}) = (W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$. However, Lemma 2.3 does not guarantee that the homology orientations for the cobordisms $W_{\mathbf{m}}^{\mathbf{n}}$ and $W_{\mathbf{m}+3\mathbf{k}}^{\mathbf{n}+3\mathbf{k}}$ are the same. Nevertheless, as a consequence of Lemma 2.3, there exists $s_{\mathbf{k}} : \mathbb{Z}^n \rightarrow \{\pm 1\}$ such that $o(\mathbf{m} + 3\mathbf{k}, \mathbf{n} + 3\mathbf{k}) = s_{\mathbf{k}}(\mathbf{m})s_{\mathbf{k}}(\mathbf{n})o(\mathbf{m}, \mathbf{n})$.

The rest of this subsection is devoted to define families of metrics on $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$, in the case that (\mathbf{m}, \mathbf{n}) represents a cube, double-cube, or a triple-cube. An important ingredient of the proof of exact triangles in the next subsection is the existence of such families of metrics. Almost the same families of metrics are described in [3] which in turn are based on those of [12] (cf. also [17]). Families of orbifold metrics with similar formal properties play a similar role in [11].

We start with the case that (\mathbf{m}, \mathbf{n}) is a d -dimensional cube in \mathbb{Z}^n and construct a family of metrics of dimension $d - 1$. For the sake of simplicity of exposition, we assume that $\mathbf{m} = (1, \dots, 1) + \mathbf{n}$. The cobordism $W_{\mathbf{m}}^{\mathbf{n}}$ in this case is produced by gluing 2-handles along disjoint curves in $Y_{\mathbf{m}}$. This cobordism can be decomposed to the union $\mathbb{R} \times Y^L \cup \bar{H}_1 \cup \dots \cup \bar{H}_n$. Here \bar{H}_i is a copy of \bar{H} corresponding to the i^{th} component of L . The cobordism $Z_{\mathbf{m}}^{\mathbf{n}}$ also decomposes to the union $\mathbb{R} \times \zeta \cup Z_1 \cup \dots \cup Z_n$ where $\mathbb{R} \times \zeta \subset \mathbb{R} \times Y^L$ and $Z_i \subset \bar{H}_i$. The product metric on $\mathbb{R} \times Y^L$ and

the standard metrics on \bar{H}_i determine a metric on $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$. Given $t_i \in \mathbb{R}$ for each $1 \leq i \leq n$, we can compose the gluing map of \bar{H}_i to $\mathbb{R} \times Y^L$ with a translation in the \mathbb{R} direction and produce a new metric on $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$. This family of metrics is parameterized by \mathbb{R}^n . The fibers of this family over (t_1, \dots, t_n) and $(t_1 + t, \dots, t_n + t)$ can be naturally identified with each other. Thus by considering only the metrics which are parametrized by $\mathbb{R}^{n-1} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid \sum t_i = 0\}$, we can come up with a more effective family of metrics. This family can be compactified to a family $(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, \mathfrak{Z}_{\mathbf{m}}^{\mathbf{n}})$ by allowing the difference of some of the coordinates to be equal to ∞ . More precisely, $(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, \mathfrak{Z}_{\mathbf{m}}^{\mathbf{n}})$ is parametrized by a polyhedron $G_{\mathbf{m}}^{\mathbf{n}}$ such that a face of codimension i of $G_{\mathbf{m}}^{\mathbf{n}}$ is determined by a partition of $\{1, \dots, n\}$ to $i + 1$ (labeled) sets $A_0 = \{a_1^0, \dots, a_{j_0}^0\}, \dots, A_i = \{a_1^i, \dots, a_{j_i}^i\}$. Roughly speaking, this face parametrizes the set of points (t_1, \dots, t_n) that the entries with indices in A_{l+1} have larger values than those with indices in A_l and in fact the difference is ∞ . In order to give a more detailed definition and for the sake of simplicity of exposition, assume that:

$$A_0 = \{1, \dots, j_0\} \quad A_1 = \{j_0 + 1, \dots, j_0 + j_1\} \quad \dots \quad A_i = \left\{ \left(\sum_{l=0}^{i-1} j_l \right) + 1, \dots, n = \sum_{l=0}^i j_l \right\}$$

and for $1 \leq l \leq i$ define \mathbf{k}_l to be:

$$\mathbf{k}_l := \left(\underbrace{1, \dots, 1}_{j_0 + \dots + j_{l-1}}, \underbrace{0, \dots, 0}_{j_l + \dots + j_i} \right) + \mathbf{n}$$

Then closure of the corresponding face of $G_{\mathbf{m}}^{\mathbf{n}}$ can be identified with $G_{\mathbf{m}}^{\mathbf{k}_i} \times G_{\mathbf{k}_i}^{\mathbf{k}_{i-1}} \times \dots \times G_{\mathbf{k}_1}^{\mathbf{n}}$. A point in this face has the form $((t_1, \dots, t_{j_0}), (t_{j_0+1}, \dots, t_{j_0+j_1}), \dots, (t_{(\sum_{l=0}^{i-1} j_l)+1}, \dots, t_n))$ such that $t_{j_0+\dots+j_{l-1}+1} + \dots + t_{j_0+\dots+j_l} = 0$ for each l . Intersection of a neighborhood of this point and the interior of $G_{\mathbf{m}}^{\mathbf{n}}$ parametrizes the normalized version of the points of the following form:

$$(t'_1, \dots, t'_{j_0}, t'_{j_0+1} + M_1, \dots, t'_{j_0+j_1} + M_1, t'_{j_0+j_1+1} + M_1 + M_2, \dots, \\ t'_{(\sum_{l=0}^{i-1} j_l)+1} + M_1 + \dots + M_i, \dots, t'_n + M_1 + \dots + M_i)$$

where t'_k is a real number close to t_k and M_j 's are large real numbers. Here the normalization of the above element of \mathbb{R}^n is produced by subtracting a constant real number from all the entries such that the summation of the entries is equal to zero. A point in the interior of this face parametrizes a broken metric on $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$ with a cut along $(Y_{\mathbf{k}_1}, \zeta_{\mathbf{k}_1}) \cup \dots \cup (Y_{\mathbf{k}_i}, \zeta_{\mathbf{k}_i})$. In particular, a face of co-dimension 1 of $G_{\mathbf{m}}^{\mathbf{n}}$ has the form $G_{\mathbf{m}}^{\mathbf{k}} \times G_{\mathbf{k}}^{\mathbf{n}}$ where $\mathbf{k} = \mathbf{I} + \mathbf{n}$, $\mathbf{I} \in \{0, 1\}^n$, and $\mathbf{k} \neq \mathbf{m}, \mathbf{n}$. The restriction of the family of metrics to this face is $(\mathfrak{W}_{\mathbf{m}}^{\mathbf{k}}, \mathfrak{Z}_{\mathbf{m}}^{\mathbf{k}}) \times (\mathfrak{W}_{\mathbf{k}}^{\mathbf{n}}, \mathfrak{Z}_{\mathbf{k}}^{\mathbf{n}})$.

Lemma 2.4. *The polyhedron $G_{\mathbf{m}}^{\mathbf{n}}$ can be oriented for all cubes (\mathbf{m}, \mathbf{n}) such that for each co-dimension 1 face $G_{\mathbf{m}}^{\mathbf{k}} \times G_{\mathbf{k}}^{\mathbf{n}}$ of $G_{\mathbf{m}}^{\mathbf{n}}$ the orientation given by the outward-normal-first convention differs from the product orientation by $(-1)^{\dim(G_{\mathbf{m}}^{\mathbf{k}}) \cdot \dim(G_{\mathbf{k}}^{\mathbf{n}}) + |\mathbf{n}|_1}$. Moreover, in the case that (\mathbf{m}, \mathbf{n}) is an edge, the orientation of the point $G_{\mathbf{m}}^{\mathbf{n}}$ is given by the following sign:*

$$\text{sgn}(\mathbf{m}, \mathbf{n}) := (-1)^{\sum_{i=1}^{i_0-1} m_i} \quad (48)$$

where i_0 is the unique place that \mathbf{m} and \mathbf{n} differ.

Proof. In [11], the polyhedrons $G_{\mathbf{m}}^{\mathbf{n}}$ are oriented such that the boundary orientation on the codimension 1 face $G_{\mathbf{m}}^{\mathbf{k}} \times G_{\mathbf{k}}^{\mathbf{n}}$ of $G_{\mathbf{m}}^{\mathbf{n}}$ differs from the product orientation by $(-1)^{|\mathbf{m}-\mathbf{k}|_1-1}$. As it is shown there, this orientation can be fixed such that the orientation of $G_{\mathbf{m}}^{\mathbf{n}}$ for an edge is given by the following sign:

$$(-1)^{\sum_{i=i_0}^n m_i}$$

where i_0 is defined above. In order to produce the desired orientation, multiply this orientation of $G_{\mathbf{m}}^{\mathbf{n}}$ by $(-1)^{\text{sgn}'(\mathbf{m}, \mathbf{n})}$, where:

$$\text{sgn}'(\mathbf{m}, \mathbf{n}) := \frac{|\mathbf{m} - \mathbf{n}|_1 (|\mathbf{m} - \mathbf{n}|_1 - 1)}{2} + |\mathbf{m}|_1$$

□

If (\mathbf{m}, \mathbf{n}) is a d -dimensional double-cube, then a family of metrics on $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$, parametrized by a polyhedron $N_{\mathbf{m}}^{\mathbf{n}}$, can be constructed in a similar way. We explain the construction of this family in the special case that $\mathbf{m} = (2, 1, \dots, 1) + \mathbf{n}$. The general case can be treated similarly. The cobordism $W_{\mathbf{m}}^{\mathbf{n}}$ can be decomposed as $\mathbb{R} \times Y^L \cup D \cup \bar{H}_2 \cup \dots \cup \bar{H}_n$, where D is produced by gluing two 2-handles H_1^1, H_1^2 along $\partial_1 H_1^1 = \partial_0 H_1^2$ and then adding the cylindrical ends $\mathbb{R}^{\leq 0} \times \partial_0 H_1^1$ and $\mathbb{R}^{\geq 0} \times \partial_1 H_1^2$. Note that ∂D is the same as $\mathbb{R} \times \partial Y^{K_1}$. An elementary but important observation is that our choices of framings produce a 2-sphere S with self-intersection -1 which is the union of the co-core disc of H_1 (i.e. $\{(z_1, z_2) \in H_1^1 \mid z_1 = 0\}$) and the core of H_2 (i.e. $\{(z_1, z_2) \in H_1^2 \mid z_2 = 0\}$). Consequently, a regular neighborhood of S denoted by R is diffeomorphic to $\mathbb{C}P^2 \setminus D^4$. We will write T for the boundary of R . The space D can be equipped with a family of metrics that is parametrized by \mathbb{R} . For $\tau \in \mathbb{R}^{\geq 0}$, insert the cylinder $[-\tau, \tau] \times \partial_1 H_1^1 = [-\tau, \tau] \times \partial_0 H_1^2$ between H_1^1 and H_1^2 and then glue the cylindrical ends. For $\tau \in \mathbb{R}^{\leq 0}$, insert the cylinder $[\tau, -\tau] \times T$ between R and its complement. In summary, for positive values of τ , we stretch along the solid torus $\partial_1 H_1^1 = \partial_1 H_1^2$ and for negative values of τ , we stretch along the boundary of R . For this family of metrics, pin down an isomorphism between the boundary of D with $\mathbb{R} \times \partial Y^{K_1}$ in the following way. For $\tau \in \mathbb{R}^{\geq 0}$, let $[-\tau, \tau] \times \partial(\partial_1 H_1^1)$, inside the inserted cylinder, be identified with $[-\tau, \tau] \times \partial Y^{K_1}$ and extend this identification in an obvious way. For $\tau \in \mathbb{R}^{\leq 0}$, the corresponding metrics form a constant family on the boundary of D and let the isomorphism be the same as the isomorphism for $\tau = 0$. For a given $(t_1, \dots, t_n) \in \mathbb{R}^n$, glue the handle H_i to $\mathbb{R} \times Y^L$ such that the gluing map is determined by t_i . Furthermore, glue the metric corresponding to t_1 on D to $\mathbb{R} \times Y^L$ with the above identification. As in the case of cubes, this family of metrics can be compactified to a family of (possibly broken) metrics $\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}$ parametrized by a polyhedron $N_{\mathbf{m}}^{\mathbf{n}}$.

The 2-dimensional cobordism $Z_{\mathbf{m}}^{\mathbf{n}} : \zeta_{\mathbf{m}} \rightarrow \zeta_{\mathbf{n}}$ can be decomposed as the union $\mathbb{R} \times \zeta \cup Z \cup Z_2 \cup \dots \cup Z_n$ where $\mathbb{R} \times \zeta \subset \mathbb{R} \times Y^L$, $Z \subset D$, $Z_2 \subset \bar{H}_2$, and so on. For $i \geq 2$, $Z_i \subset \bar{H}_i$ has the same form as in the case of cubes; it is either empty, or of the form (45), or of the form (46). There are three possibilities for Z depending on $m_1 \bmod 3$. If $m_i \equiv 0 \bmod 3$, then Z is a disc cobordism from $\zeta_{\mathbf{m},1}$ to the empty set. The disc intersects S in exactly one point. Therefore, Z intersects boundary of R in a circle. On the other hand, the intersection with $\partial_1 H_1^1 = \partial_0 H_1^2$ is empty. A similar discussion applies to the case that $m_i \equiv 2 \bmod 3$. If $m_i \equiv 1 \bmod 3$, then $Z = S$ and intersection of Z and $\partial_1 H_1^1 = \partial_0 H_1^2$ is a circle while $S \cap \partial R = \emptyset$. In each of these cases, extend the family of metrics on D in the previous paragraph to a family of metrics for the pair (D, Z) by

inserting $[-\tau, \tau] \times (Z \cap \partial_1 H_1^1)$ (respectively, $[\tau, -\tau] \times (Z \cap \partial R)$) in the inserted $[-\tau, \tau] \times \partial_1 H_1^1$ (respectively, $[\tau, -\tau] \times \partial R$) for $\tau \geq 0$ (respectively, $\tau \leq 0$). Use this family of metrics to extend $\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}$ to a family of metrics $(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, \mathfrak{Z}_{\mathbf{m}}^{\mathbf{n}})$ on the pair $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$, parametrized by $N_{\mathbf{m}}^{\mathbf{n}}$. Codimension 1 faces of this polyhedron parametrized the following broken metrics:

- broken metrics along the cut $(Y_{\mathbf{k}}, \zeta_{\mathbf{k}})$ where $\mathbf{m} > \mathbf{k} > \mathbf{n}$. This case can be divided to three different cases:
 - (a) $\mathbf{k} = (0, k_2, \dots, k_n)$; this face can be identified with the family parametrized by $N_{\mathbf{m}}^{\mathbf{k}} \times G_{\mathbf{k}}^{\mathbf{n}}$
 - (b) $\mathbf{k} = (1, k_2, \dots, k_n)$; this face can be identified with the family parametrized by $G_{\mathbf{m}}^{\mathbf{k}} \times G_{\mathbf{k}}^{\mathbf{n}}$
 - (c) $\mathbf{k} = (2, k_2, \dots, k_n)$; this face can be identified with the family parametrized by $G_{\mathbf{m}}^{\mathbf{k}} \times N_{\mathbf{k}}^{\mathbf{n}}$
- broken metrics along the cut $(T, Z \cap T)$. This face will be denoted by $I_{\mathbf{m}}^{\mathbf{n}}$

The family of broken metrics in the second bullet is determined by families of metrics on the two connected components of $(W_{\mathbf{m}}^{\mathbf{n}} \setminus T, Z \setminus T)$. The family of metrics on the component $(R, Z \cap R)$ is either the same as the one used in Lemma 1.28 or that of Remark 1.29.

Suppose an orientation of the polyhedrons $G_{\mathbf{m}}^{\mathbf{n}}$, satisfying the assumption of Lemma 2.4 is fixed. Then the following lemma sets our convention for the orientation of $N_{\mathbf{m}}^{\mathbf{n}}$:

Lemma 2.5. *The polyhedron $N_{\mathbf{m}}^{\mathbf{n}}$ can be oriented for all double-cubes (\mathbf{m}, \mathbf{n}) such that for each co-dimension 1 face $A_{\mathbf{m}}^{\mathbf{k}} \times B_{\mathbf{k}}^{\mathbf{n}}$ of $N_{\mathbf{m}}^{\mathbf{n}}$, in which $A, B \in \{G, N\}$, the orientation given by the outward-normal-first convention differs from the product orientation by $(-1)^{\dim(A_{\mathbf{m}}^{\mathbf{k}}) \cdot \dim(B_{\mathbf{k}}^{\mathbf{n}}) + |\mathbf{n}|_1}$.*

Proof. As in Lemma 2.4, we can change the orientation of $N_{\mathbf{m}}^{\mathbf{n}}$, defined in [11], to produce the desired orientation of the polyhedron $N_{\mathbf{m}}^{\mathbf{n}}$. \square

For a d -dimensional triple-cube (\mathbf{m}, \mathbf{n}) , a $(d+1)$ -dimensional family of metrics parametrized by a polyhedron $K_{\mathbf{m}}^{\mathbf{n}}$ is defined in an analogous way. For the simplicity of the exposition, we explain the construction in the special case that $\mathbf{m} = (3, 1, \dots, 1) + \mathbf{n}$. There is a decomposition $\mathbb{R} \times Y^L \cup F \cup \bar{H}_2 \cup \dots \cup \bar{H}_n$ of the cobordism $W_{\mathbf{m}}^{\mathbf{n}}$ with F being the result of composing $\mathbb{R}^{\leq 0} \times \partial_0 H_1^1$, the handles H_1^1, H_1^2, H_1^3 , and finally $\mathbb{R}^{\geq 0} \times \partial_1 H_1^3$. There is also a decomposition of $Z_{\mathbf{m}}^{\mathbf{n}}$ as the union $\mathbb{R} \times \zeta \cup B \cup Z_2 \cup \dots \cup Z_n$ where $\mathbb{R} \times \zeta \subset \mathbb{R} \times Y^L$, $B \subset F$, $Z_2 \subset \bar{H}_2$, and so on.

The co-core of the handle H_1^i and the core of H_1^{i+1} produces a 2-sphere S_i for $i = 1, 2$ (S_1 is what we called S before). Regular neighborhoods of S_1, S_2 , denoted by R_1, R_2 , are diffeomorphic to $\bar{\mathbb{C}P}^2 \setminus D^4$. We will also write T_1, T_2 for the boundaries of R_1, R_2 . The spheres S_1 and S_2 intersect each other in exactly one point, and hence a neighborhood of their union, denoted by U , is diffeomorphic to $D^2 \times S^2 \# \bar{\mathbb{C}P}^2$. The boundary of U is also denoted by Q . We assume that U contains R_1 and R_2 . In the list:

$$E_1 = \partial_1 H_1^2 = \partial_0 H_1^3 \quad E_2 = \partial_1 H_1^1 = \partial_0 H_1^2 \quad E_3 = T_2 \quad E_4 = Q \quad E_5 = T_1$$

each 3-manifold E_i is disjoint from E_{i-1} and E_{i+1} where $E_0 = E_5$ and $E_6 = E_1$. In fact removing E_i disconnects F , and E_{i+1}, E_{i-1} are both in the same connected component of $F \setminus E_i$. For each

i , construct a family of metrics on F that are broken along E_i , and are parametrized by $[-\infty, \infty]$, in the following way. Firstly remove E_i and then for $\tau \in [0, \infty]$ (respectively, $\tau \in [-\infty, 0]$) remove E_{i+1} (respectively, E_{i-1}) and insert $[-\tau, \tau] \times E_{i+1}$ (respectively, $[\tau, -\tau] \times E_{i-1}$). The metrics corresponding to ∞ and $-\infty$ are broken along E_{i+1} and E_{i-1} , respectively. Furthermore, this family of metrics is constant on the connected component of $F \setminus E_i$ that does not contain E_{i-1} and E_{i+1} . Example 1.13 determines one such family for $i = 4$. As in the case of double-cubes, these families can be lifted to families of metrics on the pair (F, B) .

These five families of metrics consist of broken metrics parametrized by the edges of a pentagon, denoted by \mathfrak{p} . Extend these metrics into the interior of \mathfrak{p} in an arbitrary way. This extended family on (F, B) , and the metrics on $(\mathbb{R} \times Y^L, \mathbb{R} \times \zeta)$, (\tilde{H}_i, Z_i) generate a family of metrics on $(W_{\mathfrak{m}}^{\mathfrak{n}}, Z_{\mathfrak{m}}^{\mathfrak{n}})$ parametrized by $\mathbb{R}^{n-1} \times \mathfrak{p}$. This family can be compactified by adding broken metrics. The resulting family, $(\mathfrak{W}_{\mathfrak{m}}^{\mathfrak{n}}, \mathfrak{Z}_{\mathfrak{m}}^{\mathfrak{n}})$, is parametrized by a polyhedron $K_{\mathfrak{m}}^{\mathfrak{n}}$. Codimension 1 faces of this polyhedron parametrize the following families of metrics:

- broken metrics along the cut $(Y_{\mathfrak{k}}, \zeta_{\mathfrak{k}})$ where $\mathfrak{m} > \mathfrak{k} > \mathfrak{n}$. This case can be divided to four different sub-cases:
 - (a) $\mathfrak{k} = (0, k_2, \dots, k_n)$; this face can be identified with the family parametrized by $K_{\mathfrak{m}}^{\mathfrak{k}} \times G_{\mathfrak{k}}^{\mathfrak{n}}$
 - (b) $\mathfrak{k} = (1, k_2, \dots, k_n)$; this face can be identified with the family parametrized by $N_{\mathfrak{m}}^{\mathfrak{k}} \times G_{\mathfrak{k}}^{\mathfrak{n}}$
 - (c) $\mathfrak{k} = (2, k_2, \dots, k_n)$; this face can be identified with the family parametrized by $G_{\mathfrak{m}}^{\mathfrak{k}} \times N_{\mathfrak{k}}^{\mathfrak{n}}$
 - (d) $\mathfrak{k} = (3, k_2, \dots, k_n)$; this face can be identified with the family parametrized by $G_{\mathfrak{m}}^{\mathfrak{k}} \times K_{\mathfrak{k}}^{\mathfrak{n}}$
- broken metrics along the cut $(Q, B \cap Q)$. This face will be denoted by $Q_{\mathfrak{m}}^{\mathfrak{n}}$
- broken metrics along the cut $(T_1, B \cap T_1)$. This face will be denoted by $(I_1)_{\mathfrak{m}}^{\mathfrak{n}}$
- broken metrics along the cut $(T_2, B \cap T_2)$. This face will be denoted by $(I_2)_{\mathfrak{m}}^{\mathfrak{n}}$

Orientation of the polyhedrons $K_{\mathfrak{m}}^{\mathfrak{n}}$ can be fixed such that these orientations are compatible with the orientations of $G_{\mathfrak{m}}^{\mathfrak{n}}$ and $N_{\mathfrak{m}}^{\mathfrak{n}}$ from Lemmas 2.4 and 2.5 in the following way:

Lemma 2.6. *The polyhedron $K_{\mathfrak{m}}^{\mathfrak{n}}$ can be oriented for all triple-cubes $(\mathfrak{m}, \mathfrak{n})$ such that for each co-dimension 1 face $A_{\mathfrak{m}}^{\mathfrak{k}} \times B_{\mathfrak{k}}^{\mathfrak{n}}$ of $K_{\mathfrak{m}}^{\mathfrak{n}}$, in which $A, B \in \{G, N, K\}$, the orientation given by the outward-normal-first convention differs from the product orientation by $(-1)^{\dim(A_{\mathfrak{m}}^{\mathfrak{k}}) \cdot \dim(B_{\mathfrak{k}}^{\mathfrak{n}}) + |\mathfrak{n}|_1}$.*

The families broken along T_1 and T_2 are similar to the family parametrized by $I_{\mathfrak{m}}^{\mathfrak{n}}$ in the case of double-cubes. The family of metrics broken along Q is determined by an $(|\mathfrak{m} - \mathfrak{n}|_1 - 3)$ -dimensional family of metrics on $(W_{\mathfrak{m}}^{\mathfrak{n}} \setminus U, Z_{\mathfrak{m}}^{\mathfrak{n}} \setminus U)$, denoted by $\tilde{Q}_{\mathfrak{m}}^{\mathfrak{n}}$ and a family on $(U, B \cap U)$ parametrized by a closed interval \tilde{I} . The former family is the compactification of a family parametrized by \mathbb{R}^{n-1} such that a typical element in the family is given by gluing $\mathbb{R} \times Y^L$, with the product metric, to the translations of H_i 's, with the standard metrics, and $F \setminus U$, with a fixed metric. The family of metrics on $(U, B \cap U)$ is one of the families introduced in Example 1.13 and Remark 1.31. Note that an orientation of $\tilde{Q}_{\mathfrak{m}}^{\mathfrak{n}}$ can be fixed such that the product orientation on $\tilde{Q}_{\mathfrak{m}}^{\mathfrak{n}} \times \tilde{I}$ agrees with the boundary orientation, induced by the orientation of $K_{\mathfrak{m}}^{\mathfrak{n}}$ from Lemma 2.6.

Let V be a copy of $S^1 \times D^3$ and C is empty or the surface introduced in Remark 1.31, depending on whether $(U, B \cap U)$ is the pair of Example 1.13 or Remark 1.31. Glue a copy of (V, C) to $(W_{\mathbf{m}}^{\mathbf{n}} \setminus U, Z_{\mathbf{m}}^{\mathbf{n}} \setminus U)$, along the boundary component Q , in order to form a pair $(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}, \widetilde{Z}_{\mathbf{m}}^{\mathbf{n}})$. Note that Gluing (V, C) to $(F \setminus U, B \setminus U) \subset (W_{\mathbf{m}}^{\mathbf{n}} \setminus U, Z_{\mathbf{m}}^{\mathbf{n}} \setminus U)$ gives the product pair $(\mathbb{R} \times \partial_0 H_1^1, \mathbb{R} \times (\partial_0 H_1^1 \cap \zeta_{\mathbf{m}}))$. Therefore, $(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}, \widetilde{Z}_{\mathbf{m}}^{\mathbf{n}})$ is diffeomorphic to $(W_{\mathbf{m}'}^{\mathbf{n}}, Z_{\mathbf{m}'}^{\mathbf{n}})$ where $\mathbf{m}' = (m_1 - 3, m_2, \dots, m_n)$. Fix homology orientations of U and V such that the assertion of Lemma 1.31 holds. Then these homology orientations and the homology orientation of $W_{\mathbf{m}}^{\mathbf{n}}$ can be utilized to define a homology orientation for $\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}$ by requiring:

$$o(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}) = o(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}} \setminus V, V) \quad o(W_{\mathbf{m}}^{\mathbf{n}}) = o(W_{\mathbf{m}}^{\mathbf{n}} \setminus U, U) \quad o(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}} \setminus V) = o(W_{\mathbf{m}}^{\mathbf{n}} \setminus U)$$

Lemma 2.7. *Suppose (\mathbf{m}, \mathbf{n}) is a triple-cube and $\mathbf{m} \geq \mathbf{k}, \mathbf{k}' \geq \mathbf{n}$ such that (\mathbf{m}, \mathbf{k}) and $(\mathbf{k}', \mathbf{n})$ are triple-cubes, i.e., (\mathbf{k}, \mathbf{n}) and $(\mathbf{m}, \mathbf{k}')$ are cubes. Then the composition $\widetilde{W}_{\mathbf{m}}^{\mathbf{k}} \circ W_{\mathbf{k}}^{\mathbf{n}}$ and $W_{\mathbf{m}}^{\mathbf{k}'} \circ \widetilde{W}_{\mathbf{k}'}^{\mathbf{n}}$ are diffeomorphic to $\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}$. Furthermore, the homology orientations of these cobordisms are related via the following relations:*

$$o(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}) = (-1)^{\text{ind}(\mathcal{D}_{W_{\mathbf{k}}^{\mathbf{n}}})} o(\widetilde{W}_{\mathbf{m}}^{\mathbf{k}}, W_{\mathbf{k}}^{\mathbf{n}}) = o(W_{\mathbf{m}}^{\mathbf{k}}, \widetilde{W}_{\mathbf{k}}^{\mathbf{n}}) \quad (49)$$

Proof. The claim about the diffeomorphisms follows from the fact that $W_{\mathbf{m}}^{\mathbf{n}}$ and the compositions $W_{\mathbf{m}}^{\mathbf{k}} \circ W_{\mathbf{k}}^{\mathbf{n}}$ and $W_{\mathbf{m}}^{\mathbf{k}'} \circ W_{\mathbf{k}'}^{\mathbf{n}}$ are diffeomorphic to each other. The relationship between the homology orientations of $W_{\mathbf{m}}^{\mathbf{n}}$ and $W_{\mathbf{m}}^{\mathbf{k}} \circ W_{\mathbf{k}}^{\mathbf{n}}$ can be verified as follows:

$$\begin{aligned} o(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}, U, U) &= o(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}} \setminus V, V, U, U) = o(W_{\mathbf{m}}^{\mathbf{n}} \setminus U, U, V, U) \\ &= o(W_{\mathbf{m}}^{\mathbf{n}}, V, U) = o(W_{\mathbf{m}}^{\mathbf{k}}, W_{\mathbf{k}}^{\mathbf{n}}, V, U) \\ &= o(W_{\mathbf{m}}^{\mathbf{k}} \setminus U, U, W_{\mathbf{k}}^{\mathbf{n}}, V, U) \\ &= (-1)^{\text{ind}(\mathcal{D}_{W_{\mathbf{k}}^{\mathbf{n}}})} o(\widetilde{W}_{\mathbf{m}}^{\mathbf{k}} \setminus V, V, W_{\mathbf{k}}^{\mathbf{n}}, U, U) \\ &= (-1)^{\text{ind}(\mathcal{D}_{W_{\mathbf{k}}^{\mathbf{n}}})} o(\widetilde{W}_{\mathbf{m}}^{\mathbf{k}}, W_{\mathbf{k}}^{\mathbf{n}}, U, U) \implies \\ o(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}) &= (-1)^{\text{ind}(\mathcal{D}_{W_{\mathbf{k}}^{\mathbf{n}}})} o(\widetilde{W}_{\mathbf{m}}^{\mathbf{k}}, W_{\mathbf{k}}^{\mathbf{n}}) \end{aligned}$$

Here we use $\text{ind}(\mathcal{D}_U) = 0$ and $\text{ind}(\mathcal{D}_V) = 1$. The other identity in (49) can be checked similarly. \square

The family of metrics on $(W_{\mathbf{m}}^{\mathbf{n}} \setminus U, Z_{\mathbf{m}}^{\mathbf{n}} \setminus U)$, parametrized by $\widetilde{Q}_{\mathbf{m}}^{\mathbf{n}}$, and a fixed metric with a cylindrical end on (V, C) determine a family of broken metrics on $(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}, \widetilde{Z}_{\mathbf{m}}^{\mathbf{n}})$ parametrized by $\widehat{Q}_{\mathbf{m}}^{\mathbf{n}}$. However, we will write $\widehat{Q}_{\mathbf{m}}^{\mathbf{n}}$ for the parametrizing set of this family in order to distinguish between the families. Assume that again $\mathbf{m} = \mathbf{n} + (3, 1, \dots, 1)$. Then an element of this family is a metric on:

$$(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}} = \mathbb{R} \times Y^L \cup \mathbb{R} \times \partial_0 H_1^1 \cup \bar{H}_2 \cup \dots \cup \bar{H}_n, \widetilde{Z}_{\mathbf{m}}^{\mathbf{n}} = \mathbb{R} \times \zeta \cup \mathbb{R} \times (\partial_0 H_1^1 \cap \zeta_{\mathbf{m}}) \cup Z_2 \cup \dots \cup Z_n)$$

where the metric on $(\mathbb{R} \times \partial_0 H_1^1, \mathbb{R} \times (\partial_0 H_1^1 \cap \zeta_{\mathbf{m}}))$ is broken along Q . Replacing this broken metric with the standard product metric constructs another family of metrics on $(\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}, \widetilde{Z}_{\mathbf{m}}^{\mathbf{n}})$, parametrized with the set $\widehat{Q}_{\mathbf{m}}^{\mathbf{n}}$. To emphasize the difference, we will denote the parametrizing set by $\widehat{Q}_{\mathbf{m}}^{\mathbf{n}}$. By

construction, it is clear that one can interpolate between the families parametrized by \bar{Q}_m^n and \hat{Q}_m^n ; that is to say, there exists a family parametrized by $I \times \bar{Q}_m^n$ such that $\{0\} \times \bar{Q}_m^n$ agrees with \hat{Q}_m^n and $\{1\} \times \bar{Q}_m^n$ is equal to \bar{Q}_m^n . The orientation of \hat{Q}_m^n determines an orientation of \bar{Q}_m^n which can be used to equip $I \times \bar{Q}_m^n$ with the product orientation. Also, the codimension 1 faces of $I \times \bar{Q}_m^n$ are given as follows:

- broken metrics along the cut (Y_k, ζ_k) where $m > k > n$ and $k = (0, k_2, \dots, k_n)$; this face can be identified with $(I \times \bar{Q}_m^k) \times G_k^n$
- broken metrics along the cut (Y_k, ζ_k) where $m > k > n$ and $k = (3, k_2, \dots, k_n)$; this face can be identified with $G_m^k \times (I \times \bar{Q}_k^n)$
- the family of metrics \bar{Q}_m^n
- the family of metrics \hat{Q}_m^n

Lemma 2.8. *For any triple-cube (m, n) , the fixed orientation of $I \times \bar{Q}_m^n$ produces the following orientation of its faces:*

- the boundary orientation on $(I \times \bar{Q}_m^k) \times G_k^n$ differs from the product orientation by

$$(-1)^{\dim(K_m^k) \cdot \dim(G_k^n) + |n|_1 + 1 + |k-n|_1} = (-1)^{\dim(I \times \bar{Q}_m^k) \cdot \dim(G_k^n) + |n|_1}$$

- the boundary orientation on $G_m^k \times (I \times \bar{Q}_k^n)$ differs from the product orientation by

$$(-1)^{\dim(G_m^k) \cdot (\dim(K_k^n) + 1) + |n|_1 + 1 + |m-k|_1} = (-1)^{\dim(G_m^k) \cdot \dim(I \times \bar{Q}_k^n) + |n|_1 + 1 + |m-k|_1}$$

- the boundary orientation on $\bar{Q}_m^n = \{1\} \times \bar{Q}_m^n$ is the same as the fixed orientation of \bar{Q}_m^n ;
- the boundary orientation on $\hat{Q}_m^n = \{0\} \times \bar{Q}_m^n$ is the reverse of the fixed orientation of \bar{Q}_m^n .

Proof. It suffices to prove the same claim when $I \times \bar{Q}_m^n$ in the statement of the lemma is replaced with $\tilde{I} \times \tilde{Q}_m^n$. The polyhedrons $\tilde{Q}_m^n \times \tilde{I}$ and $K_m^k \times G_k^n$, with $k = (0, k_2, \dots, k_n)$, are codimension 1 faces of K_m^n . Intersection of these two polyhedrons is $(\tilde{Q}_m^k \times \tilde{I}) \times G_k^n$ which has codimension 1 with respect to $\tilde{Q}_m^n \times \tilde{I}$ and $K_m^k \times G_k^n$. The polyhedrons $\tilde{Q}_m^n \times \tilde{I}$ and $K_m^k \times G_k^n$ inherit boundary orientations from K_m^k which in turn induce two orientations on $(\tilde{Q}_m^k \times \tilde{I}) \times G_k^n$. These two orientations differ by a sign. Another way to orient $(\tilde{Q}_m^k \times \tilde{I}) \times G_k^n$ is to consider it as the product of $\tilde{Q}_m^k \times \tilde{I}$ and G_k^n . This orientation is the same as the boundary orientation by considering $(\tilde{Q}_m^k \times \tilde{I}) \times G_k^n$ as a codimension 1 face of $K_m^k \times G_k^n$ where the latter is equipped with the product orientation. Therefore, Lemma 2.5 states that the product and the boundary orientations of $(\tilde{Q}_m^k \times \tilde{I}) \times G_k^n$, as a face of $\tilde{Q}_m^n \times \tilde{I}$ differ by $(-1)^{\dim(K_m^k) \cdot \dim(G_k^n) + |n|_1 + 1}$. This proves the claim about the orientation of the faces in the first bullet. A similar argument verifies the claim for the faces in the second bullet. The assertions about the orientations of the faces in the last two bullets are trivial. \square

2.2 Exact Triangles

Suppose a pair (Y, ζ) and a framed link L is fixed as in the previous subsection. Then we can form the pairs $(Y_{\mathbf{m}}, \zeta_{\mathbf{m}})$ and the cobordisms of the pairs $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$. Use Lemma 2.3 to fix homology orientations for the cobordisms $W_{\mathbf{m}}^{\mathbf{n}}$. For each representation variety $\mathcal{R}(Y_{\mathbf{m}})$, fix a Morse-Smale function $f_{\mathbf{m}}$ such that $f_{\mathbf{m}+3\mathbf{k}} = f_{\mathbf{m}}$ for any $\mathbf{k} \in \mathbb{Z}^n$. For a cube, double-cube, or a triple-cube (\mathbf{m}, \mathbf{n}) , we constructed a family of metrics $(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, \mathfrak{Z}_{\mathbf{m}}^{\mathbf{n}})$ on the pair $(W_{\mathbf{m}}^{\mathbf{n}}, Z_{\mathbf{m}}^{\mathbf{n}})$. Since the parametrizing sets for these families are contractible, the homology orientation for $W_{\mathbf{m}}^{\mathbf{n}}$ determines an orientation of $\text{ind}(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, G_{\mathbf{m}}^{\mathbf{n}})$, $\text{ind}(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, N_{\mathbf{m}}^{\mathbf{n}})$ or $\text{ind}(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, K_{\mathbf{m}}^{\mathbf{n}})$ depending on whether (\mathbf{m}, \mathbf{n}) is a cube, double-cube, or a triple-cube.

Fix a perturbation $\eta_{\mathbf{m}}^{\mathbf{n}}$ for the family $(\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, \mathfrak{Z}_{\mathbf{m}}^{\mathbf{n}})$ such that Hypothesis 1 holds. Thus the map $\tilde{f}_{\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}, \mathfrak{Z}_{\mathbf{m}}^{\mathbf{n}}}^{\eta_{\mathbf{m}}^{\mathbf{n}}}$ is well-defined. To simplify our notation, we denote this cobordism map by $\tilde{f}_{G_{\mathbf{m}}^{\mathbf{n}}}$, $\tilde{f}_{N_{\mathbf{m}}^{\mathbf{n}}}$ or $\tilde{f}_{K_{\mathbf{m}}^{\mathbf{n}}}$ when (\mathbf{m}, \mathbf{n}) is a cube, double-cube, or triple-cube, respectively. We also require that the induced perturbation for the families $I_{\mathbf{m}}^{\mathbf{n}}$, $(I_1)_{\mathbf{m}}^{\mathbf{n}}$, $(I_2)_{\mathbf{m}}^{\mathbf{n}}$ and $Q_{\mathbf{m}}^{\mathbf{n}}$, are equal to zero on R , R_1 , R_2 and U , respectively. We also assume that $\eta_{\mathbf{m}}^{\mathbf{n}}$ is equal to $\eta_{\mathbf{m}+3\mathbf{k}}^{\mathbf{n}}$ for any $\mathbf{k} \in \mathbb{Z}^n$.

Given a cube (\mathbf{m}, \mathbf{n}) , an application of (42) with the aid of Lemmas 2.3 and 2.4 to the family of metrics parametrized by $G_{\mathbf{m}}^{\mathbf{n}}$ gives:

$$\sum_{\mathbf{m} > \mathbf{k} > \mathbf{n}} (-1)^{|\mathbf{n}|_1} \tilde{f}_{G_{\mathbf{k}}^{\mathbf{n}}} \circ \tilde{f}_{G_{\mathbf{m}}^{\mathbf{k}}} = d_{\mathbf{n}} \circ \tilde{f}_{G_{\mathbf{m}}^{\mathbf{n}}} + (-1)^{|\mathbf{m}-\mathbf{n}|_1} \tilde{f}_{G_{\mathbf{m}}^{\mathbf{n}}} \circ d_{\mathbf{m}}$$

where $d_{\mathbf{k}}$ is the differential on the complex $\widetilde{\text{PFC}}(Y_{\mathbf{k}}, \zeta_{\mathbf{k}})$. If we set the convention that $\tilde{f}_{G_{\mathbf{k}}^{\mathbf{k}}}$ is equal to $(-1)^{|\mathbf{k}|_1+1} d_{\mathbf{k}}$, then the above identity can be rewritten as:

$$\sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} \tilde{f}_{G_{\mathbf{k}}^{\mathbf{n}}} \circ \tilde{f}_{G_{\mathbf{m}}^{\mathbf{k}}} = 0 \quad (50)$$

This relation states that the following is a chain complex:

$$\left(C_{\mathbf{m}}^{\mathbf{n}} := \bigoplus_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} \widetilde{\text{PFC}}(Y_{\mathbf{k}}, \zeta_{\mathbf{k}}), d_{\mathbf{m}}^{\mathbf{n}} := (\tilde{f}_{G_{\mathbf{k}}^{\mathbf{k}'}})_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{k}' \geq \mathbf{n}} \right)$$

For $\mathbf{m} = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ and $i \in \mathbb{Z}$, let $\mathbf{m}(i) := (i, m_1, \dots, m_{n-1}) \in \mathbb{Z}^n$ be given as the juxtaposition of i and \mathbf{m} . Given a cube (\mathbf{m}, \mathbf{n}) in \mathbb{Z}^{n-1} and $i \in \mathbb{Z}$, define $g_{\mathbf{m}(i)}^{\mathbf{n}(i)} : C_{\mathbf{m}(i)}^{\mathbf{n}(i)} \rightarrow C_{\mathbf{m}(i-1)}^{\mathbf{n}(i-1)}$ by the following matrix presentation:

$$g_{\mathbf{m}(i)}^{\mathbf{n}(i)} := (\tilde{f}_{G_{\mathbf{k}(i)}^{\mathbf{k}'(i-1)}})_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{k}' \geq \mathbf{n}}$$

By (50), $g_{\mathbf{m}(i)}^{\mathbf{n}(i)}$ is an anti-chain map, namely, we have the following identity:

$$d_{\mathbf{m}(i-1)}^{\mathbf{n}(i-1)} \circ g_{\mathbf{m}(i)}^{\mathbf{n}(i)} + g_{\mathbf{m}(i)}^{\mathbf{n}(i)} \circ d_{\mathbf{m}(i)}^{\mathbf{n}(i)} = 0$$

In general, given an anti-chain map $g : (C, d) \rightarrow (C', d')$ we can form the mapping cone chain complex $\text{Cone}(g)$ defined in the following way:

$$\text{Cone}(g) := (C \oplus C', \begin{pmatrix} d & 0 \\ g & d' \end{pmatrix})$$

For the anti-chain map $g_{\mathbf{m}^{(i)}}^{\mathbf{n}^{(i)}}$, the chain complex $\text{Cone}(g_{\mathbf{m}^{(i)}}^{\mathbf{n}^{(i)}})$ is clearly the same as $(C_{\mathbf{m}^{(i)}}^{\mathbf{n}^{(i-1)}}, d_{\mathbf{m}^{(i)}}^{\mathbf{n}^{(i-1)}})$. The heart of this paper lies in the following theorem:

Theorem 2.9. *The chain complexes $(C_{\mathbf{m}^{(i)}}^{\mathbf{n}^{(i)}}, d_{\mathbf{m}^{(i)}}^{\mathbf{n}^{(i)}})$ and $\text{Cone}(g_{\mathbf{m}^{(i-1)}}^{\mathbf{n}^{(i-1)}})$ have the same chain homotopy type.*

An immediate corollary of Theorem 2.9 is that the chain complex $(C_{\mathbf{m}}^{\mathbf{n}}, d_{\mathbf{m}}^{\mathbf{n}})$ is a new chain complex for the plane Floer homology of an appropriate pair:

Corollary 2.10. *Let (\mathbf{m}, \mathbf{n}) be a cube and $\mathbf{p} = 2\mathbf{m} - \mathbf{n}$. Then the chain complexes $(\widetilde{\text{PFC}}(Y_{\mathbf{p}}, \zeta_{\mathbf{p}}), d_{\mathbf{p}})$ and $(C_{\mathbf{m}}^{\mathbf{n}}, d_{\mathbf{m}}^{\mathbf{n}})$ have the same chain homotopy type.*

Proof. For the simplicity of the exposition, assume that $\mathbf{p} = (2, \dots, 2)$, $\mathbf{m} = (1, \dots, 1)$, $\mathbf{n} = (0, \dots, 0)$. By Theorem 2.9, $(C_{\mathbf{m}}^{\mathbf{n}}, d_{\mathbf{m}}^{\mathbf{n}})$ and $(C_{\mathbf{m}'}^{\mathbf{n}'}, d_{\mathbf{m}'}^{\mathbf{n}'})$ have the same chain homotopy type when $\mathbf{m}' = (2, 1, \dots, 1)$, $\mathbf{n}' = (2, 0, \dots, 0)$. By repeating this application of Theorem 2.9, we can show that $(\widetilde{\text{PFC}}(Y_{\mathbf{p}}, \zeta_{\mathbf{p}}), d_{\mathbf{p}})$ and $(C_{\mathbf{m}}^{\mathbf{n}}, d_{\mathbf{m}}^{\mathbf{n}})$ have the same chain homotopy type. \square

Consider the grading on $(C_{\mathbf{m}}^{\mathbf{n}}, d_{\mathbf{m}}^{\mathbf{n}})$ that is defined to be equal to $|\mathbf{k}|_1$ on the summand $\widetilde{\text{PFC}}(Y_{\mathbf{k}}, \zeta_{\mathbf{k}})$ of $C_{\mathbf{m}}^{\mathbf{n}}$. The differential $d_{\mathbf{m}}^{\mathbf{n}}$ does not increase the grading. Therefore, we can use this filtration to produce a spectral sequence:

Corollary 2.11. *There is a spectral sequence with the first page $\bigoplus_{\mathbf{k} \in \{0,1\}^n} \widetilde{\text{PFH}}(Y_{\mathbf{k}}, \zeta_{\mathbf{k}})$ that abuts to $\widetilde{\text{PFH}}(Y, \zeta)$. The differential in the first page can be identified with $\bigoplus (-1)^{\text{sgn}(\mathbf{k}, \mathbf{k}')} \widetilde{\text{PFH}}(W_{\mathbf{k}}^{\mathbf{k}'}, Z_{\mathbf{k}}^{\mathbf{k}'})$. Here the sum is over all the edges of the cube $\{0, 1\}^n$ and $\text{sgn}(\mathbf{k}, \mathbf{k}')$ is defined in (48).*

Proof. The definition of the filtration implies the first part. The second part is the result of Lemma 2.4. \square

The following algebraic lemma underlies our strategy of proving Theorem 2.9 (cf. [16, 11]):

Lemma 2.12. *For $i \in \mathbb{Z}$, let (C_i, d_i) be a chain complex and $g_i : C_i \rightarrow C_{i-1}$ be an anti-chain map. For each i , suppose there are $n_i : C_i \rightarrow C_{i-2}$ and $k_i : C_i \rightarrow C_{i-3}$ such that:*

$$g_{i-1} \circ g_i + d_{i-2} \circ n_i + n_i \circ d_i = 0 \tag{51}$$

and

$$n_{i-1} \circ g_i + g_{i-2} \circ n_i + d_{i-3} \circ k_i + k_i \circ d_i = q_i \tag{52}$$

where $q_i : C_i \rightarrow C_{i-3}$ is an isomorphism. Then C_i and $\text{Cone}(g_{i-1})$ have the same chain homotopy type.

Proof. The following maps:

$$\phi_i := (g_i, n_i) : C_i \rightarrow \text{Cone}(g_{i-1}) \quad \psi_i := \begin{pmatrix} n_{i-1} \\ g_{i-2} \end{pmatrix} : \text{Cone}(g_{i-1}) \rightarrow C_{i-3} \quad (53)$$

are anti-chain maps. We also have:

$$\phi_i \circ \phi_i \sim_h q_i \quad \phi_i \circ \psi_{i+3} \sim_h \begin{pmatrix} q_{i+2} & 0 \\ n_i \circ n_{i+2} + k_{i+1} \circ g_{i+2} + g_{i-1} \circ k_{i+2} & q_{i+1} \end{pmatrix} \quad (54)$$

where the homotopies are given by the following maps:

$$k_i \quad \begin{pmatrix} k_{i+2} & n_{i+1} \\ 0 & k_{i+1} \end{pmatrix} \quad (55)$$

The homotopies in (54) show that $\psi_i \circ \phi_i$ and $\phi_i \circ \psi_{i+3}$ are homotopic to isomorphisms. Consequently, $\phi_i : C_i \rightarrow \text{Cone}(g_{i-1})$ is a homotopy equivalence. \square

Proof of Theorem 2.9. In order to utilize Lemma 2.12, we need to construct maps $n_{\mathbf{m}(i)}^{\mathbf{n}(i)} : C_{\mathbf{m}(i)}^{\mathbf{n}(i)} \rightarrow C_{\mathbf{m}(i-2)}^{\mathbf{n}(i-2)}$ and $k_{\mathbf{m}(i)}^{\mathbf{n}(i)} : C_{\mathbf{m}(i)}^{\mathbf{n}(i)} \rightarrow C_{\mathbf{m}(i-3)}^{\mathbf{n}(i-3)}$ such that the constructed maps satisfy the relations (57) and (52). For each $i \in \mathbb{Z}$, the pair $(\mathbf{m}(i), \mathbf{n}(i-2))$ determines a double-cube in \mathbb{Z}^n , and hence we can consider the family of metrics parametrized by $N_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}$ on the pair cobordism $(W_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}, Z_{\mathbf{m}(i)}^{\mathbf{n}(i-2)})$. This family defines the map:

$$f_{N_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}} : \widetilde{\text{PFC}}(Y_{\mathbf{m}(i)}, \zeta_{\mathbf{m}(i)}) \rightarrow \widetilde{\text{PFC}}(Y_{\mathbf{n}(i-2)}, \zeta_{\mathbf{n}(i-2)})$$

Identity (42), Lemma 2.5 and the characterization of codimension 1 faces of $N_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}$ imply that:

$$\begin{aligned} \tilde{f}_{I_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}} + \sum_{\mathbf{m} \geq \mathbf{k} > \mathbf{n}} (-1)^{|\mathbf{n}|+i} \tilde{f}_{G_{\mathbf{k}(i-2)}^{\mathbf{n}(i-2)}} \circ \tilde{f}_{N_{\mathbf{m}(i)}^{\mathbf{k}(i-2)}} + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} (-1)^{|\mathbf{n}|+i} \tilde{f}_{G_{\mathbf{k}(i-1)}^{\mathbf{n}(i-2)}} \circ \tilde{f}_{G_{\mathbf{m}(i)}^{\mathbf{k}(i-1)}} + \\ + \sum_{\mathbf{m} > \mathbf{k} \geq \mathbf{n}} (-1)^{|\mathbf{n}|+i} \tilde{f}_{N_{\mathbf{k}(i)}^{\mathbf{n}(i-2)}} \circ \tilde{f}_{G_{\mathbf{m}(i)}^{\mathbf{k}(i)}} = d_{\mathbf{n}(i-2)} \circ \tilde{f}_{N_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}} - (-1)^{|\mathbf{m}-\mathbf{n}|+1} \tilde{f}_{\eta}^{\tilde{N}_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}} \circ d_{\mathbf{m}(i)} \end{aligned}$$

According to Lemma 1.28, the cobordism map $\tilde{f}_{I_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}}$ vanishes. We can use this and the convention that $\tilde{f}_{G_{\mathbf{k}}^{\mathbf{k}}} = (-1)^{|\mathbf{k}|+1} d_{\mathbf{k}}$ to simplify slightly:

$$\sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} \tilde{f}_{G_{\mathbf{k}(i-2)}^{\mathbf{n}(i-2)}} \circ \tilde{f}_{N_{\mathbf{m}(i)}^{\mathbf{k}(i-2)}} + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} \tilde{f}_{G_{\mathbf{k}(i-1)}^{\mathbf{n}(i-2)}} \circ \tilde{f}_{G_{\mathbf{m}(i)}^{\mathbf{k}(i-1)}} + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} \tilde{f}_{N_{\mathbf{k}(i)}^{\mathbf{n}(i-2)}} \circ \tilde{f}_{G_{\mathbf{m}(i)}^{\mathbf{k}(i)}} = 0 \quad (56)$$

If we define a map $n_{\mathbf{m}(i)}^{\mathbf{n}(i)} : C_{\mathbf{m}(i)}^{\mathbf{n}(i)} \rightarrow C_{\mathbf{m}(i-2)}^{\mathbf{n}(i-2)}$ as:

$$n_{\mathbf{m}(i)}^{\mathbf{n}(i)} := (\tilde{f}_{N_{\mathbf{k}(i)}^{\mathbf{k}'(i-2)}})_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{k}' \geq \mathbf{n}}$$

then (56) implies:

$$g_{\mathbf{m}(i-1)}^{\mathbf{n}(i-1)} \circ g_{\mathbf{m}(i)}^{\mathbf{n}(i)} + d_{\mathbf{m}(i-2)}^{\mathbf{n}(i-2)} \circ n_{\mathbf{m}(i)}^{\mathbf{n}(i)} + n_{\mathbf{m}(i)}^{\mathbf{n}(i)} \circ d_{\mathbf{m}(i)}^{\mathbf{n}(i)} = 0 \quad (57)$$

We can repeat the above construction for the family of metrics parametrized by $K_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}$ and define the map:

$$\tilde{f}_{K_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} : \widetilde{\text{PFC}}(Y_{\mathbf{m}(i)}, \zeta_{\mathbf{m}(i)}) \rightarrow \widetilde{\text{PFC}}(Y_{\mathbf{n}(i-3)}, \zeta_{\mathbf{n}(i-3)})$$

The description of the boundary of $K_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}$ provides us with the following relation:

$$\begin{aligned} & f_{(I_1)_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} + f_{(I_2)_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} + f_{Q_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} + (-1)^{|\mathbf{n}|_1+i-3} \left(\sum_{\mathbf{m} \geq \mathbf{k} > \mathbf{n}} f_{G_{\mathbf{k}(i-3)}^{\mathbf{n}(i-3)}} \circ f_{K_{\mathbf{m}(i)}^{\mathbf{k}(i-3)}} + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} f_{G_{\mathbf{k}(i-2)}^{\mathbf{n}(i-3)}} \circ f_{N_{\mathbf{m}(i)}^{\mathbf{k}(i-2)}} \right) \\ & + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} f_{N_{\mathbf{k}(i-1)}^{\mathbf{n}(i-3)}} \circ f_{G_{\mathbf{m}(i)}^{\mathbf{k}(i-1)}} + \sum_{\mathbf{m} > \mathbf{k} \geq \mathbf{n}} f_{K_{\mathbf{k}(i)}^{\mathbf{n}(i-3)}} \circ f_{G_{\mathbf{m}(i)}^{\mathbf{k}(i)}} \Big) = d_{\mathbf{n}(i-3)} \circ f_{K_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} - (-1)^{|\mathbf{m}-\mathbf{n}|_1+2} f_{K_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} \circ d_{\mathbf{m}(i)} \end{aligned}$$

Again, the first two terms in the left hand side are zero by Lemma 1.28. Therefore, the above relation can be rewritten as follows:

$$\begin{aligned} & \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} f_{G_{\mathbf{k}(i-3)}^{\mathbf{n}(i-3)}} \circ f_{K_{\mathbf{m}(i)}^{\mathbf{k}(i-3)}} + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} f_{G_{\mathbf{k}(i-2)}^{\mathbf{n}(i-3)}} \circ f_{N_{\mathbf{m}(i)}^{\mathbf{k}(i-2)}} + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} f_{N_{\mathbf{k}(i-1)}^{\mathbf{n}(i-3)}} \circ f_{G_{\mathbf{m}(i)}^{\mathbf{k}(i-1)}} + \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} f_{K_{\mathbf{k}(i)}^{\mathbf{n}(i-3)}} \circ f_{G_{\mathbf{m}(i)}^{\mathbf{k}(i)}} \\ & = (-1)^{|\mathbf{n}|_1+i} f_{Q_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} \end{aligned} \quad (58)$$

Define $\bar{k}_{\mathbf{m}(i)}^{\mathbf{n}(i)}, \bar{q}_{\mathbf{m}(i)}^{\mathbf{n}(i)} : C_{\mathbf{m}(i)}^{\mathbf{n}(i)} \rightarrow C_{\mathbf{m}(i-3)}^{\mathbf{n}(i-3)}$ as follows:

$$\bar{k}_{\mathbf{m}(i)}^{\mathbf{n}(i)} := (\tilde{f}_{K_{\mathbf{k}(i)}^{\mathbf{k}'(i-3)}})_{\mathbf{m} \geq \mathbf{k} > \mathbf{k}' \geq \mathbf{n}} \quad \bar{q}_{\mathbf{m}(i)}^{\mathbf{n}(i)} := ((-1)^{|\mathbf{k}'|_1+i} \tilde{f}_{Q_{\mathbf{k}(i)}^{\mathbf{k}'(i-3)}})_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{k}' \geq \mathbf{n}}$$

then (58) gives rise to the following identity:

$$d_{\mathbf{m}(i-3)}^{\mathbf{n}(i-3)} \circ \bar{k}_{\mathbf{m}(i)}^{\mathbf{n}(i)} + g_{\mathbf{m}(i-2)}^{\mathbf{n}(i-2)} \circ n_{\mathbf{m}(i)}^{\mathbf{n}(i)} + n_{\mathbf{m}(i-1)}^{\mathbf{n}(i-1)} \circ g_{\mathbf{m}(i)}^{\mathbf{n}(i)} + \bar{k}_{\mathbf{m}(i)}^{\mathbf{n}(i)} \circ d_{\mathbf{m}(i)}^{\mathbf{n}(i)} = \bar{q}_{\mathbf{m}(i)}^{\mathbf{n}(i)}$$

As the final step of the proof, we show that $\bar{q}_{\mathbf{m}(i)}^{\mathbf{n}(i)}$ is homotopy equivalent to an isomorphism from $C_{\mathbf{m}(i)}^{\mathbf{n}(i)}$ to $C_{\mathbf{m}(i-3)}^{\mathbf{n}(i-3)}$. Using Lemma 1.30 and Remark 1.27, it can be shown:

$$\tilde{f}_{Q_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} = (-1)^{|\mathbf{m}-\mathbf{n}|_1} c \cdot s \circ \tilde{f}_{\bar{Q}_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} \quad (59)$$

where $\tilde{f}_{\bar{Q}_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}}$ is the map for the pair $(\widetilde{W}_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}, \check{Z}_{\mathbf{m}(i)}^{\mathbf{n}(i-3)})$. Furthermore, the family of metrics $I \times \bar{Q}_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}$ defines a cobordism map and with the aid of (42) and Lemmas 2.7 and 2.8, we can derive the following identity:

$$\begin{aligned} & \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} (-1)^{|\mathbf{n}|_1+i+1+\text{ind}(\mathcal{D}(W_{\mathbf{k}(i-3)}^{\mathbf{n}(i-3)}))} f_{G_{\mathbf{k}(i-3)}^{\mathbf{n}(i-3)}} \circ f_{I \times \bar{Q}_{\mathbf{m}(i)}^{\mathbf{k}(i-3)}} + (-1)^{|\mathbf{n}|_1+|\mathbf{m}-\mathbf{k}|_1+i} f_{I \times \bar{Q}_{\mathbf{k}(i)}^{\mathbf{n}(i-3)}} \circ f_{G_{\mathbf{m}(i)}^{\mathbf{k}(i)}} \\ & = f_{\bar{Q}_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} - f_{\bar{Q}_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}} \end{aligned}$$

Post-composing with s and using (43) let us to rewrite above identity as:

$$\begin{aligned} \sum_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}} f_{G_{\mathbf{k}^{(i-3)}}^{\mathbf{n}(i-3)}} \circ ((-1)^{|\mathbf{m}-\mathbf{k}|} c \cdot s \circ f_{I \times \bar{Q}_{\mathbf{m}^{(i)}}^{\mathbf{k}(i-3)}}) + ((-1)^{|\mathbf{k}-\mathbf{n}|} c \cdot s \circ f_{I \times \bar{Q}_{\mathbf{k}^{(i)}}^{\mathbf{n}(i-3)}}) \circ f_{G_{\mathbf{m}^{(i)}}^{\mathbf{k}(i)}} &= \\ &= (-1)^{|\mathbf{m}|_1+i} c \cdot s \circ f_{\hat{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}} - (-1)^{|\mathbf{m}|_1+i} c \cdot s \circ f_{\bar{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}} \end{aligned} \quad (60)$$

Define $\tilde{k}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} : C_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} \rightarrow C_{\mathbf{m}^{(i-3)}}^{\mathbf{n}(i-3)}$ and $q_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} : C_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} \rightarrow C_{\mathbf{m}^{(i-3)}}^{\mathbf{n}(i-3)}$ as follows:

$$\begin{aligned} \tilde{k}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} &:= ((-1)^{|\mathbf{k}-\mathbf{k}'|} c \cdot s \circ f_{I \times \bar{Q}_{\mathbf{k}^{(i)}}^{\mathbf{k}'(i-3)}})_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{k}' \geq \mathbf{n}} \\ q_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} &:= ((-1)^{|\mathbf{k}|_1+i} c \cdot s \circ f_{\hat{Q}_{\mathbf{k}^{(i)}}^{\mathbf{k}'(i-3)}})_{\mathbf{m} \geq \mathbf{k} \geq \mathbf{k}' \geq \mathbf{n}} \end{aligned}$$

Identity (60) gives rise to the following chain homotopy:

$$d_{\mathbf{m}^{(i-3)}}^{\mathbf{n}(i-3)} \circ \tilde{k}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} + \tilde{k}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} \circ d_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} = q_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} - \bar{q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)}$$

Finally, with $k_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} = \bar{k}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} + \tilde{k}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)}$ we have:

$$d_{\mathbf{m}^{(i-3)}}^{\mathbf{n}(i-3)} \circ k_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} + g_{\mathbf{m}^{(i-2)}}^{\mathbf{n}(i-2)} \circ n_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} + n_{\mathbf{m}^{(i-1)}}^{\mathbf{n}(i-1)} \circ g_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} + k_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} \circ d_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)} = q_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)}$$

which is the analogue of (52), if we can show that $q_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)}$ is an isomorphism. If $\mathbf{m} \neq \mathbf{n}$, the family of metrics parametrized by $\hat{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}$ admits a faithful \mathbb{R} -action. Therefore, in this case we can arrange the involved perturbation terms in the definition of $\hat{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}$ such that the dimension of the non-empty moduli spaces is at least one. Thus if $\mathbf{m} \neq \mathbf{n}$, then with these perturbation terms $\tilde{f}_{\hat{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}} = 0$. If $\mathbf{m} = \mathbf{n}$, the family of metrics $\hat{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}$ consists of the product metric on $(\widetilde{W}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}, \widetilde{Z}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}) \cong (I \times Y_{\mathbf{m}}, I \times \zeta_{\mathbf{m}})$. However, the fixed homology orientation of this cobordism and the orientation of the point $\hat{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}$ might not be the trivial ones. Consequently, $\tilde{f}_{\hat{Q}_{\mathbf{m}^{(i)}}^{\mathbf{n}(i-3)}}$ is equal to $\pm c \cdot s$. Therefore, $q_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)}$ is indeed an isomorphism of the complexes $C_{\mathbf{m}^{(i)}}^{\mathbf{n}(i)}$ and $C_{\mathbf{m}^{(i-3)}}^{\mathbf{n}(i-3)}$. \square

3 Classical Links

A classical link is a link embedded in S^3 (or equivalently \mathbb{R}^3). Let K be an oriented classical link. Then we can form $\Sigma(K)$, the branched double cover of S^3 branched along K . Clearly, $\widehat{\text{PFH}}(\Sigma(K))$ is an invariant of K that is functorial in the following sense. For classical links K_0 and K_1 , let $S : K_0 \rightarrow K_1$ be a (not necessarily orientable) link cobordism embedded in $I \times S^3$ such that $S \cap (\{i\} \times S^3) = K_i$ for $i \in \{0, 1\}$. Then $\Sigma(S)$, the branched double cover of $I \times S^3$ branched along S , is a cobordism from $\Sigma(K_0)$ to $\Sigma(K_1)$. Therefore, we can consider the cobordism map $\widehat{\text{PFH}}(\Sigma(S)) : \widehat{\text{PFH}}(\Sigma(K_0)) \rightarrow \widehat{\text{PFH}}(\Sigma(K_1))$. In subsection 3.1, we will explore the basic properties of this functor defined on the category of classical links and cobordisms between them. In particular,

we shall explain how the exact triangles in subsection 2.2 give rise to a generalization of unoriented skein exact triangle for this link invariant. We shall use these exact triangles in subsection 3.2 to define *plane knot homology*, a stronger version of the plane Floer homology of the branched double covers.

3.1 Plane Floer Homology of Branched Double Covers

Suppose $S : K_0 \rightarrow K_1$ is a cobordism of classical links, $\zeta_0 \subset \Sigma(K_0)$ and $\zeta_1 \subset \Sigma(K_1)$ are embedded 1-manifolds, and $Z : \zeta_0 \rightarrow \zeta_1$ is an embedded cobordism in $W := \Sigma(S)$. Given a family of metrics $(\mathfrak{W}, \mathfrak{Z})$ on (W, Z) , consider the cobordism map $\tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta$. The following lemma shows how the degree of this map can be computed in terms of the topological invariants of S , K_0 , and K_1 :

Lemma 3.1. *The degree of the map $\tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta : \widehat{\text{PFC}}(\Sigma(K_0), \zeta_0) \rightarrow \widehat{\text{PFC}}(\Sigma(K_1), \zeta_1)$ with respect to deg_p is given by:*

$$\text{deg}_p(\tilde{f}_{\mathfrak{W}, \mathfrak{Z}}^\eta) = \chi(S) + \frac{1}{2}S \cdot S - (\sigma(K_1) - \sigma(K_0)) + 2 \dim(G) \quad (61)$$

In particular, the degree of the map $\widehat{\text{PFH}}(\Sigma(S), Z)$ is equal to:

$$\chi(S) + \frac{1}{2}S \cdot S - (\sigma(K_1) - \sigma(K_0)).$$

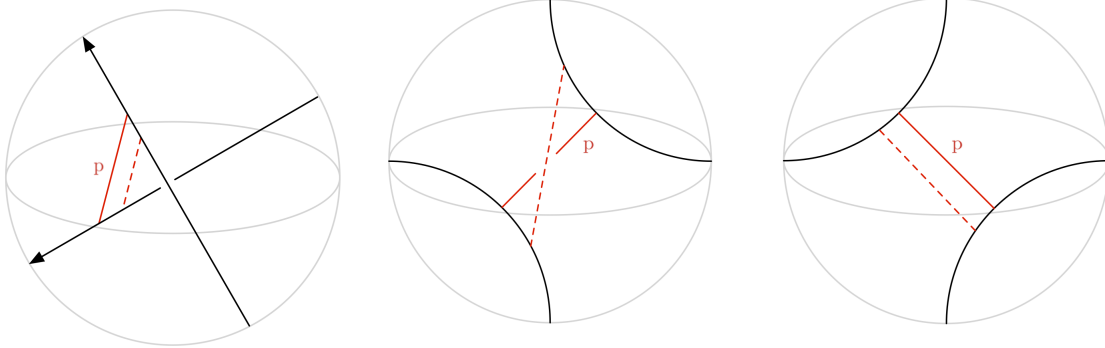
Here $\sigma(K_i)$ denotes the signature of the link K_i and the self-intersection $S \cdot S$ is computed with respect to the Seifert framing on K_0 and K_1 .

Note that the self-intersection $S \cdot S$ is defined even when S is not orientable. To that end, firstly fix a Seifert surface for the link K_i and move this link on this surface to produce \tilde{K}_i , a parallel copy of K_i . Then perturb S to a cobordism $\tilde{S} : \tilde{K}_0 \rightarrow \tilde{K}_1$ that is transversal to S . The self-intersection number $S \cdot S$ is given by the signed count of the intersection points of S and \tilde{S} . To determine the sign of each intersection point, fix an arbitrary orientation of S in a neighborhood of the intersection point. This induces a local orientation of \tilde{S} and the sign of the intersection point is determined by comparing the sum of these local orientations on S , \tilde{S} and the orientation of the ambient space $I \times S^3$.

Proof. For the cobordism $S : K_0 \rightarrow K_1$, the Euler characteristic and the signature of the 4-dimensional cobordism $\Sigma(S)$ are given by the following identities:

$$\chi(\Sigma(S)) = -\chi(S) \quad \sigma(\Sigma(S)) = -\frac{1}{2}S \cdot S + \sigma(K_1) - \sigma(K_0)$$

The identity for the Euler characteristic is obvious. To obtain the other identity, let S_i be a Seifert surface for K_i and \tilde{S}_i is given by pushing S_i into a 4-dimensional ball that fills the sphere containing K_i . Then $\sigma(K)$ is equal to the signature of $\Sigma(\tilde{S}_i)$, the branched double cover of the



(a) A crossing of a link is an embedded ball in S^3 that its intersection with the link is equal to the union of the two black arcs.

(b) 0-resolution (1-resolution) of the crossing in Figure 3a is the link given by replacing the crossing with the left (right) picture.

Figure 3: A crossing and its resolutions

4-ball, branched along \tilde{S}_i . Gluing \tilde{S}_0 , S , and \tilde{S}_1 (with the reversed orientation) determines a closed surface \bar{S} in S^4 . The G -signature theorem implies that:

$$\begin{aligned}\sigma(\Sigma(\bar{S})) &= \sigma(S^4) - \frac{1}{2}S \cdot S \\ &= -\frac{1}{2}S \cdot S\end{aligned}$$

This identity and the additivity of the signature of 4-manifolds gives the desired identity for $\sigma(\Sigma(S))$. Using these topological formulas, the equality (41) asserts that:

$$\deg_p(\tilde{J}_{\mathbb{W},3}^{\eta}) = 2 \dim(G) - \sigma(\Sigma(S)) - \chi(\Sigma(S)) = \chi(S) + \frac{1}{2}S \cdot S - (\sigma(K_1) - \sigma(K_0)) + 2 \dim(G)$$

□

Following [10], a crossing c for a link $K \subset S^3$ is an orientation preserving embedding $c : D^3 \rightarrow S^3$ such that $c(T) = c(D^3) \cap K$ where the 1-manifold $T \subset D^3$ is demonstrated in Figure 3a. Let T be oriented as in Figure 3a. A crossing c is *positive*, if the restriction of c to T on both components is either orientation-preserving or orientation-reversing. If c is orientation preserving on exactly one connected component, then c is a *negative* crossing. Figure 3b shows how to *resolve* a crossing in order to produce the *0-resolution* and the *1-resolution*. The orientation of K on the complement of the crossing c can be extended to exactly one of these resolutions. This resolution is called the *oriented* resolution of c . If c is a positive crossing, then the 0-resolution is the oriented resolution. Otherwise, the 1-resolution is the oriented one.

For a crossing c , the branched double cover of $c(D^3)$ branched along $c(T)$ is a solid torus which is denoted by $\Sigma(c(T))$. Let l_p be the lift of the path p in Figure 3a. Then l_p is the core of $\Sigma(K)$.

The lift of the dashed path provides a parallel copy of l_p which in turn determines a framing for l_p . In fact, 0- and 1-surgery on $\Sigma(K)$ along the framed knot l_p produce 3-manifolds that are respectively branched double covers of 0- and 1-resolutions of K . The 2-resolution of a crossing is given by leaving the crossing unchanged. For a general $i \in \mathbb{Z}$, the i -resolution, denoted by $K_{(i)}$, is the same as the j -resolution where $1 \leq j \leq 2$ and $i \equiv j \pmod{3}$.

In the previous section, we define the cobordism $W_{(i)}^{(j)} : \Sigma(K)_{(i)} \rightarrow \Sigma(K)_{(j)}$ when $i, j \in \mathbb{Z}$ and $i \geq j$. This cobordism is equal to $\Sigma(S_{(i)}^{(j)})$ where $S_{(i)}^{(j)} : K_{(i)} \rightarrow K_{(j)}$ can be described as follows. The cobordism $S_{(i)}^{(i-1)}$ is the result of gluing a 1-handle along the endpoints of the path p in $K_{(i)}$ with respect to the framing given by the endpoints of the dashed red path. In particular, the Euler characteristic of $S_{(i)}^{(i-1)}$ is equal to -1. The non-trivial connected component of $S_{(i)}^{(i-1)}$, as a topological surface, is either a pair of pants or a twice punctured $\mathbb{R}P^2$. The latter case happens if and only if $K_{(i)}$ and $K_{(i-1)}$ have the same number of connected components. In general, the cobordism $S_{(i)}^{(j)}$ is equal to the composition $S_{(i)}^{(i-1)} \circ S_{(i-1)}^{(i-2)} \circ \cdots \circ S_{(j+1)}^{(j)}$. Let also $\bar{S}_{(i)}^{(j)} : K_{(i)} \rightarrow K_{(j)}$ be the reverse of the cobordism $S_{(i)}^{(j)}$. Then $S_{(i)}^{(i-2)}$, as a cobordism embedded in $I \times S^3$ is diffeomorphic to the connected sum of $\bar{S}_{(i+1)}^{(i)}$ and S_0 , the standard embedding of $\mathbb{R}P^2$ in S^4 with self intersection 2 (cf. [11, Lemma 7.2]). The cobordism $S_{(i)}^{(i-3)}$ has a similar description. Suppose $\check{S}_{(i)}^{(i-3)} : K_{(i)} \rightarrow K_{(i-3)}$ is the product cobordism. Then there is a 4-dimensional ball in $I \times S^3$ (respectively, S^4) that intersects $\check{S}_{(i)}^{(i-3)}$ (respectively, S_0) in two 2-dimensional discs. Removing these 4-balls from $I \times S^3$ and S^4 produces two pairs of 4-manifolds and embedded surfaces such that each of them has a boundary component consists of the pair of S^3 and an embedded link with two connected components which are unlinked and unknotted. Gluing these two pairs along the common boundary determines a 2-dimensional cobordism in $I \times S^3$ which is equal to $S_{(i)}^{(i-3)}$ ([11, Lemma 7.4]).

Starting with a crossing for an oriented link K , the oriented resolution inherits an orientation from K . However, there is not a canonical way to fix an orientation for the other resolution. Therefore, in general there might be an ambiguity in the definition of the self-intersection numbers of the cobordism $S_{(i)}^{(j)}$. There is an important case for our purposes that we can avoid this ambiguity. A link L is an *unlink* with n connected components if there is an embedding of a union of n 2-discs in S^3 that fills the link L . Different orientations of an unlink produce the same Seifert framing. Therefore, if both of the resolutions $K_{(i)}$ and $K_{(j)}$ are unlink, then the self-intersection of the cobordism $S_{(i)}^{(j)}$ is well-defined. For example, if $K_{(1)}$ and $K_{(0)}$ are unlink, then possible self-intersections of the cobordism $S_{(1)}^{(0)}$ for this case can be described as follows (cf. [10, Lemma 4.7]). If the non-trivial component of this cobordism is a pair of pants, then the self-intersection is zero. Otherwise, this self-intersection is equal to 2 or -2. Note that if the crossing is positive (respectively, negative), then the cobordism $S_{(0)}^{(-1)}$ (resp $S_{(2)}^{(1)}$) has zero self-intersection. Another constraint is that the sum of the self-intersections of the cobordisms $S_{(2)}^{(1)}$, $S_{(1)}^{(0)}$, and $S_{(0)}^{(-1)}$ is equal to 2 because of the special form of the cobordism $S_{(2)}^{(-1)}$, described in the previous paragraph.

As a consequence, if K , $K_{(0)}$ and $K_{(1)}$ are all unlink, then out of these three cobordisms, the self-intersection of one of them is 2 and the remaining two cobordisms have zero self-intersection.

Next, let N be a set of n crossings for K . Given $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, the link $K_{\mathbf{m}}$ is the result of replacing the i^{th} crossing with the m_i -resolution. As before, consider the path p_i connecting the two strands in the i^{th} crossing. Inverse image of this path in $\Sigma(K)$ is a framed knot and the union of these knots for the crossings in N determines a framed link in $\Sigma(K)$. It is still true that $\Sigma(K_{\mathbf{m}})$ is the same as the 3-manifold $\Sigma(K)_{\mathbf{m}}$, given by surgery on $\Sigma(K)$ along this framed link and corresponding to $\mathbf{m} \in \mathbb{Z}^n$. For a pair (\mathbf{m}, \mathbf{n}) with $\mathbf{m} \geq \mathbf{n}$, the cobordism $W_{\mathbf{m}}^{\mathbf{n}} : \Sigma(K)_{\mathbf{m}} \rightarrow \Sigma(K)_{\mathbf{n}}$ is the branched double cover of a cobordism $S_{\mathbf{m}}^{\mathbf{n}} : K_{\mathbf{m}} \rightarrow K_{\mathbf{n}}$. In the case $|\mathbf{m} - \mathbf{n}|_1 = 1$, the topological properties of this cobordism is explained above. In general, $S_{\mathbf{m}}^{\mathbf{n}}$ can be decomposed as the composition of $|\mathbf{m} - \mathbf{n}|_1$ such simpler cobordisms. In particular, the Euler characteristic of $S_{\mathbf{m}}^{\mathbf{n}}$ is equal to $-|\mathbf{m} - \mathbf{n}|_1$. If (\mathbf{m}, \mathbf{n}) is a triple-cube, then $\widetilde{W}_{\mathbf{m}}^{\mathbf{n}}$ is also branched double cover of a cobordism that is denoted by $\widetilde{S}_{\mathbf{m}}^{\mathbf{n}}$. For example, if $\mathbf{m} = \mathbf{m}_0(2)$ and $\mathbf{n} = \mathbf{n}_0(-1)$, then $\widetilde{S}_{\mathbf{m}}^{\mathbf{n}}$ is the composition of the product cobordism from $K_{\mathbf{m}_0(2)}$ to $K_{\mathbf{m}_0(-1)}$ and the cobordism $S_{\mathbf{m}_0(-1)}^{\mathbf{n}_0(-1)}$.

In the case that $K_{\mathbf{m}}$ and $K_{\mathbf{n}}$ are both unlink, the self-intersection of the cobordism $S_{\mathbf{m}}^{\mathbf{n}}$ is a well-defined even integer number. As in [10], we will write $\sigma(\mathbf{m}, \mathbf{n})$ for this self-intersection number and relax the condition $\mathbf{m} \geq \mathbf{n}$ by requiring that:

$$\sigma(\mathbf{m}, \mathbf{n}) = \sigma(\mathbf{m}, \mathbf{k}) + \sigma(\mathbf{k}, \mathbf{n}) \quad (62)$$

when $\mathbf{m}, \mathbf{k}, \mathbf{n} \in \mathbb{Z}^n$ and $K_{\mathbf{m}}, K_{\mathbf{k}}$, and $K_{\mathbf{n}}$ are unlink. As a consequence of (62), we have:

$$\begin{aligned} \sigma(\mathbf{m} + 3\mathbf{k}, \mathbf{n}) &= \sigma(\mathbf{m} + 3\mathbf{k}, \mathbf{m}) + \sigma(\mathbf{m}, \mathbf{n}) \\ &= \sigma(\mathbf{m}, \mathbf{n}) + 2|\mathbf{k}|_1 \end{aligned} \quad (63)$$

Since the set of crossings N determines a framed link with n connected components, we have the following corollary of Theorem 2.9:

Theorem 3.2. *For the set of crossings N , there exists the object $(\bigoplus_{\mathbf{m} \in \{0,1\}^n} \widehat{\text{PFC}}(\Sigma(K_{\mathbf{m}}), \zeta_{\mathbf{m}}), \widetilde{d}_p)$ of the category $\mathfrak{C}(\widetilde{\Lambda})$ that has the same chain homotopy type as $\widehat{\text{PFC}}(\Sigma(K))$. The differential \widetilde{d}_p can be decomposed as $\bigoplus_{(\mathbf{m}, \mathbf{n}) \in \{0,1\}^n} d_{\mathbf{m}}^{\mathbf{n}}$ where $d_{\mathbf{m}}^{\mathbf{n}} : \widehat{\text{PFC}}(\Sigma(K_{\mathbf{m}}), \zeta_{\mathbf{m}}) \rightarrow \widehat{\text{PFC}}(\Sigma(K_{\mathbf{n}}), \zeta_{\mathbf{n}})$ is a map induced by a family of metrics of dimension $|\mathbf{m} - \mathbf{n}|_1 - 1$ on $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$. This map is nonzero only when $\mathbf{m} \geq \mathbf{n}$.*

Recall that to define the maps $d_{\mathbf{m}}^{\mathbf{n}}$, we need to fix an appropriate metric on $\Sigma(K)$, Morse-Smale functions on $\mathcal{R}(\Sigma(K_{\mathbf{m}}))$, perturbations of the ASD equation and homology orientations of the cobordism $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$. The set of all these data together is called *auxiliary choices* to form the chain complex $(\bigoplus_{\mathbf{m} \in \{0,1\}^n} \widehat{\text{PFC}}(\Sigma(K_{\mathbf{m}}), \zeta_{\mathbf{m}}), \widetilde{d}_p)$.

3.2 Plane Knot Homology

A *filtered chain complex* (C, d) is a chain complex with a filtration of the module C :

$$C \supseteq \cdots \supseteq \mathcal{F}_{i-1}C \supseteq \mathcal{F}_iC \supseteq \mathcal{F}_{i+1}C \supseteq \cdots \supseteq \{0\}$$

such that d maps each *filtration level* \mathcal{F}_iC to itself. Suppose there exists $r \in \mathbb{R}$ such that each of the chain complexes (\mathcal{F}_iC, d) is graded with a set of the form $\{r\} + \mathbb{Z}$ and the differential d increases this degree by 1. If these gradings are compatible with respect to the filtration, then we say that (C, d) is a *filtered \mathbb{Z} -graded chain complex*. For our purposes here, we can assume that r is equal to 0 or $\frac{1}{2}$. If (C, d) and (C', d') are two filtered chain complexes, then a map $f : (C, d) \rightarrow (C', d')$ of degree $(a, b) \in \mathbb{Z} \times \mathbb{R}$ is a module homomorphism of degree b that sends \mathcal{F}_iC to $\mathcal{F}_{i+a}C'$. Note that a map of degree (a, b) has degree (a', b) for any $a' \leq a$.

Let $\mathfrak{C}_{\text{GF}}(R)$ be the category whose objects are filtered \mathbb{Z} -graded chain complexes. A morphism in this category is represented by either a chain map or an anti-chain map of degree $(0, 0)$. Two such maps represent the same morphism if there is a chain homotopy between them that is of degree $(-1, -1)$. The isomorphism class of an object of $\mathfrak{C}_{\text{GF}}(R)$ is called its *filtered chain homotopy type*. Clearly, there is also a forgetful functor from $\mathfrak{C}_{\text{GF}}(R)$ to $\mathfrak{C}(R)$.

The goal of this subsection is to assign an object of the category $\mathfrak{C}_{\text{GF}}(\Lambda)$ to a link K , equipped with appropriate additional data. We shall show that the filtered chain homotopy type of this chain complex is independent of the additional data and hence is a link invariant. In order to define this filtered chain complex, we need to work with a variant of the notion of pseudo-diagrams introduced in [10]:

Definition 3.3. A *pseudo-diagram* for a link K is a set of crossings N such that $K_{\mathbf{m}}$ is an unlink for each $\mathbf{m} \in \{0, 1\}^{|N|}$.

A planar diagram D for a link K determines a set of crossings N_D which is clearly a pseudo-diagram. This justifies the terminology of [10] for pseudo-diagrams.

Definition 3.4. A *planar pseudo-diagram* N is a pseudo-diagram that is given by a planar diagram D in one of the following ways:

- N is equal to N_D , the set of crossings of D ;
- N is given by removing an element of N_D ;
- N is given by removing two consecutive under-crossings or two consecutive over-crossings in N_D .

If adding a crossing to the pseudo-diagram N produces another pseudo-diagram N' , then we say N' is related to N by adding a crossing. Alternatively, N is related to N' by dropping a crossing. We also say that the pseudo-diagrams N and N' of the links K and K' are related by a pseudo-diagram isotopy, if there is an isotopy $\varphi_t : S^3 \rightarrow S^3$ such that $\varphi_0 = id$ and φ_1 maps K (respectively, N) to K' (respectively, N'). Note that K and K' represents the same classical link and hence N and N' can be regarded as two pseudo-diagrams for a classical link. The following observation is stated in [10]:

Lemma 3.5. *Any two planar pseudo-diagrams can be related to each other by a series of pseudo-diagram isotopies, adding and dropping crossings.*

Remark 3.6. In the upcoming definition of planar knot homology, part of the additional data is the choice of a planar pseudo-diagram. However, the construction can be applied for an arbitrary pseudo-diagram. It is not clear to the author whether the choice of planar pseudo-diagrams is essential to the definition of plane knot homology or an arbitrary pseudo-diagram determines a filtered chain complex with the same filtered chain homotopy type. A purely topological question that probably should be addressed first is the following. Is there any pseudo-diagram that cannot be related to a planar diagrams by adding or dropping crossings?

Fix a pseudo-diagram N with n crossings for a link K . The set of crossings of N can be used to form the resolutions $K_{\mathbf{m}}$ and the cobordisms $S_{\mathbf{m}}^{\mathbf{n}}$. The following topological lemma studies the self-intersection of the cobordisms $S_{\mathbf{m}}^{\mathbf{n}}$ in the case of pseudo-diagrams:

Lemma 3.7. *Suppose (\mathbf{m}, \mathbf{n}) is a cube in $\{0, 1\}^n$ and $S_{\mathbf{m}}^{\mathbf{n}}$ is the cobordism associated with a pseudo-diagram N . Then the self-intersection $S_{\mathbf{m}}^{\mathbf{n}} \cdot S_{\mathbf{m}}^{\mathbf{n}}$ is not greater than $2|\mathbf{m} - \mathbf{n}|_1$. If the equality holds, then for $\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}$ the number of connected components of $K_{\mathbf{k}}$ is independent of \mathbf{k} .*

Proof. Firstly let $|\mathbf{m} - \mathbf{n}|_1 = 1$. As it is shown in [10, Lemma 4.7], the set of possible values for the self-intersection of $S_{\mathbf{m}}^{\mathbf{n}}$ is equal to $\{-2, 0, 2\}$. If this value is equal to 2, then $S_{\mathbf{m}}^{\mathbf{n}}$ is a union of a product cobordism and a twice punctured $\mathbb{R}P^2$ (as a cobordism from the unknot to itself). In particular, $K_{\mathbf{m}}$ and $K_{\mathbf{n}}$ have the same number of the connected components. This special case can be used to verify the general case. \square

The pseudo-diagram N can be used to define an integer-valued function on the cube $\{0, 1\}^n$:

$$h : \{0, 1\}^n \rightarrow \mathbb{Z}$$

$$h(\mathbf{m}) := -|\mathbf{m}|_1 + \frac{1}{2}\sigma(\mathbf{m}, \mathbf{o})$$

where \mathbf{o} is the vertex of the cube $\{0, 1\}^n$ such that $K_{\mathbf{o}}$ is the oriented resolution of K . Extend this function in the obvious way to the elements $\mathbf{m} \in \mathbb{Z}^n$ such that $K_{\mathbf{m}}$ is an unlink. The following corollary is an immediate consequence of Lemma 3.7:

Corollary 3.8. *Let $\mathbf{m}, \mathbf{n} \in \{0, 1\}^n$ and $\mathbf{m} \geq \mathbf{n}$. Then $h(\mathbf{n}) \geq h(\mathbf{m})$.*

Suppose that auxiliary choices for N is fixed such that we can appeal to Theorem 3.2 to construct the chain complex:

$$\text{PKC}(N) := \bigoplus_{\mathbf{m} \in \{0, 1\}^n} \text{PFC}(\Sigma(K_{\mathbf{m}}))$$

with the differential d_p . Here we use Λ as the coefficient ring. In particular, there is no need to work with local coefficient systems. Furthermore, we require that the Morse-Smale function on $f_{\mathbf{m}}$ is chosen such that the differential on the complex $\text{PFC}(\Sigma(K_{\mathbf{m}}))$ vanishes. This is obviously possible because the representation variety $\mathcal{R}(\Sigma(K_{\mathbf{m}}))$ is a torus. The chain complex $(\text{PKC}(N), d_p)$

has the same chain homotopy type as $\text{PFC}(\Sigma(K))$. The map h can be used to define a grading on the chain complex $\text{PKC}(N)$ in the following way:

$$\text{deg}_h|_{\text{PFC}(\Sigma(K_{\mathbf{m}}))} = h(\mathbf{m}) + n_-(N).$$

where $n_-(N)$ denotes the number of negative crossings. We will also write $n_+(N)$ for the number of positive crossings of N . The grading deg_h is called the *homological grading*. Corollary 3.8 states that deg_h defines a filtration on $\text{PKC}(N)$ in which the i^{th} level of the filtration is generated by those elements that $\text{deg}_h \geq i$. We also define another grading on $(\text{PKC}(N), d_p)$ in the following way:

$$\delta|_{\text{PFC}(\Sigma(K_{\mathbf{m}}))} := -\frac{1}{2}\text{deg}_p - \frac{1}{2}|\mathbf{m}|_1 - \frac{1}{4}\sigma(\mathbf{m}, \mathbf{o}) + \frac{1}{2}n_+(N)$$

This grading is called the δ -grading. Since $\sigma(\mathbf{m}, \mathbf{o})$ is always an even integer, deg_δ is either an integer or a half-integer. In fact, the same argument as in [10, Lemma 4.4] shows that parity of 2deg_δ is equal to $b_0(K) - 1$. In particular, if K is a knot, then the δ -grading is always an integer number.

The definition of the homological grading and the δ -grading can be extended to $\text{PFC}(\Sigma(K_{\mathbf{m}}))$ when $K_{\mathbf{m}}$ is an unlink. Recall that if $\mathbf{m}' \equiv \mathbf{m} \pmod{3}$, then $\text{PFC}(\Sigma(K_{\mathbf{m}'})) = \text{PFC}(\Sigma(K_{\mathbf{m}}))$. The following lemma shows how the gradings on these two vector spaces compare to each other:

Lemma 3.9. *Suppose $\mathbf{m}' = \mathbf{m} + 3\mathbf{k}$ for $\mathbf{m}, \mathbf{k} \in \mathbb{Z}^n$. Then:*

$$\text{deg}_h|_{\text{PFC}(\Sigma(K_{\mathbf{m}'}))} = \text{deg}_h|_{\text{PFC}(\Sigma(K_{\mathbf{m}}))} - 2\mathbf{k}$$

$$\text{deg}_\delta|_{\text{PFC}(\Sigma(K_{\mathbf{m}'}))} = \text{deg}_\delta|_{\text{PFC}(\Sigma(K_{\mathbf{m}}))} - 2\mathbf{k}$$

Proof. These claims can be easily verified using (63). □

Lemma 3.10. *For $\mathbf{m} \geq \mathbf{n}$, suppose that $K_{\mathbf{m}}$ and $K_{\mathbf{n}}$ are unlink. Let $f : \text{PFC}(\Sigma(K_{\mathbf{m}})) \rightarrow \text{PFC}(\Sigma(K_{\mathbf{n}}))$ be a map induced by a family of metrics on $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$ (respectively, $\Sigma(\check{S}_{\mathbf{m}}^{\mathbf{n}})$). If d denotes the dimension of the parametrizing set for the family of metrics, then the degree of f with respect to the δ -grading is equal to $|\mathbf{m} - \mathbf{n}|_1 - d$ (respectively, $|\mathbf{m} - \mathbf{n}|_1 - d - 1$).*

Proof. Let f be induced by a family of metrics on $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$. According to Lemma 3.1:

$$\text{deg}_p(f) = -|\mathbf{m} - \mathbf{n}|_1 + \frac{1}{2}S_{\mathbf{m}}^{\mathbf{n}} \cdot S_{\mathbf{m}}^{\mathbf{n}} + 2d$$

As a result:

$$\text{deg}_\delta(f) = -\frac{1}{2}(-|\mathbf{m} - \mathbf{n}|_1 + \frac{1}{2}S_{\mathbf{m}}^{\mathbf{n}} \cdot S_{\mathbf{m}}^{\mathbf{n}} + 2d) - \frac{1}{2}(|\mathbf{n}|_1 - |\mathbf{m}|_1) - \frac{1}{4}(\sigma(\mathbf{n}, \mathbf{o}) - \sigma(\mathbf{m}, \mathbf{o})) = |\mathbf{m} - \mathbf{n}|_1 - d$$

A similar computation verifies the case that f is induced by a family of metrics on $\Sigma(\check{S}_{\mathbf{m}}^{\mathbf{n}})$. □

The differential of the chain complex $(\text{PKC}(N), d_p)$ is constructed via the cobordism maps associated with the family of metrics $G_{\mathbf{m}}^{\mathbf{n}}$ of dimension $|\mathbf{m} - \mathbf{n}|_1 - 1$ on the cobordisms $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$. Therefore, Lemma 3.10 asserts that d_p has degree 1 with respect to the δ -grading. In summary, the homological and δ -gradings let us regard $(\text{PKC}(N), d_p)$ as an element of $\mathfrak{C}_{\text{GF}}(\Lambda)$.

Proposition 3.11. *Suppose that a planar pseudo-diagram N of a link K is fixed. Then the filtered chain homotopy type of $(\text{PKC}(N), d_p)$ does not depend on the auxiliary choices.*

Proof. Suppose that (C_0, d_0) and (C_1, d_1) are two filtered chain complexes that are constructed with the aid of the pseudo-diagram N and two different auxiliary choices. We will write g_0 , $(f_{\mathbf{m}})_0$ and η_0 respectively for the metric on $\Sigma(K)$, the Morse-Smale function on $\mathcal{R}(\Sigma(K_{\mathbf{m}}))$, and the perturbation term. Note that we do not need to fix a homology orientation for $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$ because we are working with the coefficient ring Λ . The relevant auxiliary choices for (C_1, d_1) are also denoted by g_1 , $(f_{\mathbf{m}})_1$, and η_1 . Because the metrics g_0 and g_1 are compatible with the pseudo-diagram N , they agree in the branched double covers of the crossings of N . Fix a smooth family of metrics $\{g_t\}_{t \in I}$, connecting the metrics g_0 and g_1 on $\Sigma(K)$ such that each metric g_t is the standard metric in the branched double covers of crossings of N . From now on, in order to distinguish the 3-manifolds and the cobordisms that are involved in the construction of the two chain complexes, we use the notations $(K_{\mathbf{m}})_0$, $(K_{\mathbf{m}})_1$, and so on.

Let $(S_{\mathbf{m}}^{\mathbf{n}})_0^1$ be another copy of $S_{\mathbf{m}}^{\mathbf{n}}$ that is regarded as a cobordism from $(K_{\mathbf{m}})_0$ to $(K_{\mathbf{m}})_1$. As it was explained in subsection 2.1, g_i , for $i \in \{0, 1\}$, determines a family of metrics of dimension $|\mathbf{m} - \mathbf{n}|_1 - 1$ on $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})_i$ when (\mathbf{m}, \mathbf{n}) is a cube, double-cube, or a triple-cube. In the case that (\mathbf{m}, \mathbf{n}) is a cube, the parametrizing sets are denoted by $(G_{\mathbf{m}}^{\mathbf{n}})_0$ and $(G_{\mathbf{m}}^{\mathbf{n}})_1$. These families are constructed by decomposing $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$ to a union of a product manifold $\mathbb{R} \times Y^L$ and $|\mathbf{m} - \mathbf{n}|_1$ handles. Here Y^L is the branched double cover of the complement of the set of crossings. On each of these pieces we fixed a metric, and different ways to glue them produce the families of metrics. For $(G_{\mathbf{m}}^{\mathbf{n}})_i$, we fixed the product metric on $\mathbb{R} \times Y^L$ that is given by the restriction of g_i to the complement of the set of crossings. Alternatively, we can use the metric on $\mathbb{R} \times Y^L$ that is determined by $\{g_t\}$ to construct another family of metrics on $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$. This family can be compactified to a family of metrics parametrized with a polyhedron $J_{\mathbf{m}}^{\mathbf{n}}$ which has dimension $|\mathbf{m} - \mathbf{n}|_1$. Note that $\dim(J_{\mathbf{m}}^{\mathbf{n}}) = \dim(G_{\mathbf{m}}^{\mathbf{n}}) + 1$ because the new metric on $\mathbb{R} \times Y^L$ lacks the translational symmetry. The codimension 1 faces of $J_{\mathbf{m}}^{\mathbf{n}}$ are equal to:

$$J_{\mathbf{m}}^{\mathbf{k}} \times (G_{\mathbf{k}}^{\mathbf{n}})_1 \quad (G_{\mathbf{m}}^{\mathbf{k}})_0 \times J_{\mathbf{k}}^{\mathbf{n}}$$

where $\mathbf{m} > \mathbf{k} > \mathbf{n}$.

We can also arrange for a perturbation η_0^1 for the families of metrics $J_{\mathbf{m}}^{\mathbf{n}}$ such that the cobordism maps are well-defined and the perturbation term on each codimension 1 face is determined by η_0 , η_1 , and η_0^1 on a smaller polyhedron. With a similar argument as in the proof of Theorem 2.9, the cobordism maps associated with the family of metrics $J_{\mathbf{m}}^{\mathbf{n}}$ can be used to construct a chain map $j_{\mathbf{m}}^{\mathbf{n}} : (C_0, d_0) \rightarrow (C_1, d_1)$. Corollary 3.8 and Lemma 3.10 show that this chain map has degree $(0, 0)$.

The chain map $j_{\mathbf{m}}^{\mathbf{n}}$ is analogous to the continuation maps in Floer theories and a standard argument shows that its chain homotopy type does not depend on the choice of the connecting family of the metrics $\{g_t\}_{t \in I}$ and the perturbation η_0^1 . This argument involves defining a chain homotopy with the aid of cobordism maps associated with the product family of metrics $I \times J_{\mathbf{m}}^{\mathbf{n}}$. Another application of Corollary 3.8 and Lemma 3.10 shows that the chain homotopy has degree $(0, -1)$ (and hence degree $(-1, -1)$). Since the filtered homotopy type of the chain map $j_{\mathbf{m}}^{\mathbf{n}}$

is independent of the choices of metrics and perturbation, it defines a filtered chain homotopy equivalence. \square

Next, we want to show that if N and N' are two planar pseudo-diagrams for K , then the filtered chain homotopy types of the corresponding complexes are the same. This is clear if N and N' are related to each other by a pseudo-diagram isotopy. Suppose that N' is related to N by adding a crossing. Order the set of crossings N' such that the crossing in $N' \setminus N$ is the first one. The fact that N and N' are both pseudo-diagram implies that the resolution $Y_{\mathbf{m}(i)}$ is an unlink for $\mathbf{m} \in \{0, 1\}^n$ and $i \in \mathbb{Z}$. With appropriate auxiliary data, the chain complexes $(\text{PKC}(N), d_p)$ and $(\text{PKC}(N'), d_p)$ are respectively equal to $(C_{\mathbf{m}(2)}^{\mathbf{n}(2)}, d_{\mathbf{m}(2)}^{\mathbf{n}(2)})$ and $(C_{\mathbf{m}(1)}^{\mathbf{n}(0)}, d_{\mathbf{m}(1)}^{\mathbf{n}(0)})$ where $\mathbf{m} = (1, \dots, 1)$ and $\mathbf{n} = (0, \dots, 0)$. In particular, $(\text{PKC}(N), d_p)$ inherits two other gradings from the homological and the δ -gradings associated with the pseudo-diagram N' . These two gradings are different from the original homological and δ -gradings on $(\text{PKC}(N), d_p)$ and we denote them by $\text{deg}_{h'}$ and $\text{deg}_{\delta'}$.

Proposition 3.12. *The above gradings on $(\text{PKC}(N), d_p)$ are related to each other in the following way:*

$$\text{deg}_{\delta'} = \text{deg}_{\delta} - 1 \qquad \text{deg}_{h'} = \text{deg}_h - 1 \qquad (64)$$

Given an object (C, d) of $\mathfrak{C}_{\text{GF}}(R)$, the filtered chain complex $(C[a, b], d)$ denotes the same chain complex where the homological and the δ -grading are shifted down by a and b respectively. For example, this means that $\mathcal{F}^i C[a, b] = \mathcal{F}^{i+a} C$. Therefore, this proposition states that $C_{\mathbf{m}(2)}^{\mathbf{n}(2)}$ is equal to $\text{PKC}(N)[1, 1]$.

Proof of Proposition 3.12. Let \mathbf{o}' be the oriented resolution of the set of crossings N . If the first crossing of N' is positive, then $\mathbf{o}' = \mathbf{o}(0)$ and:

$$n_+(N') = n_+(N) + 1 \qquad n_-(N') = n_-(N) \qquad \sigma(\mathbf{k}(2), \mathbf{o}') = \sigma(\mathbf{k}, \mathbf{o}) + 2 \qquad (65)$$

for any $\mathbf{k} \in \{0, 1\}^n$. A straightforward computation using (65) proves the claim in this case. If the first crossing is negative, then the fact that $\mathbf{o}' = \mathbf{o}(1)$ and the following identities verify (64):

$$n_+(N') = n_+(N) \qquad n_-(N') = n_-(N) + 1 \qquad \sigma(\mathbf{k}(2), \mathbf{o}') = \sigma(\mathbf{k}, \mathbf{o}) \qquad (66)$$

\square

Theorem 2.9 states that the chain complexes $(C_{\mathbf{m}(1)}^{\mathbf{n}(0)}, d_{\mathbf{m}(1)}^{\mathbf{n}(0)})$ and $(C_{\mathbf{m}(2)}^{\mathbf{n}(2)}, d_{\mathbf{m}(2)}^{\mathbf{n}(2)})$ are chain homotopy equivalent. As it can be seen from the proof of Lemma 2.12, this chain homotopy equivalence is given by the following map:

$$\begin{aligned} \Phi : (C_{\mathbf{m}(2)}^{\mathbf{n}(2)}, d_{\mathbf{m}(2)}^{\mathbf{n}(2)}) &\rightarrow (C_{\mathbf{m}(1)}^{\mathbf{n}(0)}, d_{\mathbf{m}(1)}^{\mathbf{n}(0)}) = \text{Cone}(g_{\mathbf{m}(1)}^{\mathbf{n}(1)}) \\ \Phi &:= (g_{\mathbf{m}(2)}^{\mathbf{n}(2)}, n_{\mathbf{m}(2)}^{\mathbf{n}(2)}) \end{aligned}$$

In fact, if we define:

$$\Psi_i : \text{Cone}(g_{\mathbf{m}(i-1)}^{\mathbf{n}(i-1)}) \rightarrow (C_{\mathbf{m}(i-3)}^{\mathbf{n}(i-3)}, d_{\mathbf{m}(i-3)}^{\mathbf{n}(i-3)})$$

$$\Psi_i := \begin{pmatrix} n_{\mathbf{m}(i-1)}^{\mathbf{n}(i-1)} \\ n_{\mathbf{m}(i-2)}^{\mathbf{n}(i-2)} \\ g_{\mathbf{m}(i-2)}^{\mathbf{n}(i-2)} \end{pmatrix}$$

then $\Psi_2 \circ \Phi$ is chain homotopic to the isomorphism $q_{\mathbf{m}(2)}^{\mathbf{n}(2)}$ with the chain homotopy being $k_{\mathbf{m}(2)}^{\mathbf{n}(2)}$. Moreover, $\Phi \circ \Psi_5$ is chain homotopic to the isomorphism:

$$Q := \begin{pmatrix} q_{\mathbf{m}(4)}^{\mathbf{n}(4)} & 0 \\ n_{\mathbf{m}(2)}^{\mathbf{n}(2)} \circ n_{\mathbf{m}(4)}^{\mathbf{n}(4)} + k_{\mathbf{m}(3)}^{\mathbf{n}(3)} \circ g_{\mathbf{m}(4)}^{\mathbf{n}(4)} + g_{\mathbf{m}(1)}^{\mathbf{n}(1)} \circ k_{\mathbf{m}(4)}^{\mathbf{n}(4)} & q_{\mathbf{m}(3)}^{\mathbf{n}(3)} \end{pmatrix} \quad (67)$$

with the chain homotopy:

$$K := \begin{pmatrix} k_{\mathbf{m}(4)}^{\mathbf{n}(4)} & n_{\mathbf{m}(3)}^{\mathbf{n}(3)} \\ 0 & k_{\mathbf{m}(3)}^{\mathbf{n}(3)} \end{pmatrix}. \quad (68)$$

According to Lemmas 3.9 and 3.12, the above chain maps and chain homotopies induce the following diagram:

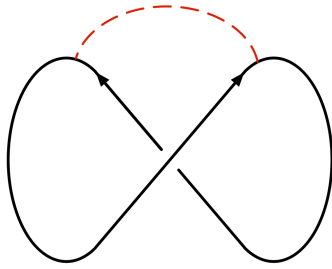
$$\begin{array}{ccc} \text{PKC}(N')[2, 2] & \xrightarrow{Q} & \text{PKC}(N') \\ & \searrow \Psi_5 & \uparrow K \\ & & \text{PKC}(N)[1, 1] \\ & & \xrightarrow{q_{\mathbf{m}(2)}^{\mathbf{n}(2)}} & \text{PKC}(N)[-1, -1] \\ & & \downarrow k_{\mathbf{m}(2)}^{\mathbf{n}(2)} & \swarrow \Psi_2 \end{array} \quad (69)$$

Next, we need to study how the maps in diagram (69) behave with respect to the homological and the δ -gradings. Firstly we start with the easier case of the δ -grading. By Lemma 3.10, the degrees of the maps Φ , Ψ_i , $k_{\mathbf{m}(i)}^{\mathbf{n}(i)}$, and K with respect to the δ -grading are equal to 1. The same lemma can be used to check that the δ -degrees of the maps $q_{\mathbf{m}(i)}^{\mathbf{n}(i)}$ and Q_i are equal to 2. In order to treat the case of the homological grading, we need to investigate further the behavior of the cobordism map $\tilde{f}_{G_{\mathbf{m}}}^{\mathbf{n}}$ when $h(\mathbf{m}) = h(\mathbf{n})$:

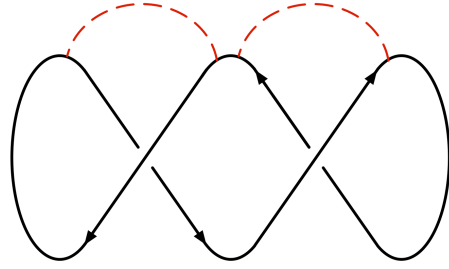
Lemma 3.13. *If (\mathbf{m}, \mathbf{n}) defines a cube in $\{0, 1\}^n$, with $S_{\mathbf{m}}^{\mathbf{n}} \cdot S_{\mathbf{m}}^{\mathbf{n}} = 2|\mathbf{m} - \mathbf{n}|_1$, then the map $\tilde{f}_{G_{\mathbf{m}}}^{\mathbf{n}}$ vanishes.*

Proof. Let D be a diagram such that N is related to N_D as in Definition 3.4. For $\mathbf{k} \in \{0, 1\}^{|\mathbf{N}|}$, the resolution $K_{\mathbf{k}}$ of the set of crossings of N can be also regarded as a resolution $K_{\mathbf{k}'}$ of the set of crossings N_D with $\mathbf{k}' \in \{0, 1, 2\}^{|\mathbf{N}_D|}$. In particular, we can consider the projection of $K_{\mathbf{k}'}$ to $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Note that this planar projection has exactly $|\mathbf{N}_D \setminus \mathbf{N}|$ crossings.

According to Lemma 3.7, for $\mathbf{m} \geq \mathbf{k} \geq \mathbf{n}$, the resolution $K_{\mathbf{k}}$ has the same number of connected components as $K_{\mathbf{m}}$ (or equivalently $K_{\mathbf{n}}$). Therefore, if N is N_D , then $\mathbf{m} = \mathbf{n}$. The map $\tilde{f}_{G_{\mathbf{m}}}^{\mathbf{n}}$ in this case vanishes because the differential of the complex $\text{PFC}(K_{\mathbf{m}})$ is zero. Now let N be given by removing a crossing from N_D . An examination of different possible cases shows that $|\mathbf{m} - \mathbf{n}|_1 \leq 1$. Furthermore, the cobordism $S_{\mathbf{m}}^{\mathbf{n}}$ is the union of a product cobordism between unlinks and a twice punched $\mathbb{R}P^2$ with self-intersection 2. Therefore, $W_{\mathbf{m}}^{\mathbf{n}}$, the branched double



(a) This diagram and a surgery along the dotted path define the 1- and 0-resolution of a pseudo-diagram with only one crossing. The cobordism between these two resolutions has self-intersection 2.



(b) This diagram and surgery along either/both of the dotted paths define vertices of the only face of a pseudo-diagram with two crossings. There are four cobordisms associated with the edges of this face and all of these cobordisms have self-intersection 2.

Figure 4

cover of $S_{\mathbf{m}}^{\mathbf{n}}$ is diffeomorphic to $(I \times \#^l S^2 \times S^1) \# \overline{\mathbb{C}P}^2$. Proposition 1.22 shows that $\text{PFH}(W_{\mathbf{m}}^{\mathbf{n}})$ (and hence $\tilde{f}_{G_{\mathbf{m}}}^{\mathbf{n}}$) is zero.

Next let $N_D \setminus N$ consists of two consecutive under-crossings or over-crossings. An examination of different possibilities shows that $|\mathbf{m} - \mathbf{n}|_1 \leq 2$. If $|\mathbf{m} - \mathbf{n}|_1 \leq 1$, then the same argument as in the above cases verifies that $\tilde{f}_{G_{\mathbf{m}}}^{\mathbf{n}} = 0$. If $|\mathbf{m} - \mathbf{n}|_1 = 2$, then there is an integer number l such that $W_{\mathbf{m}}^{\mathbf{n}}$ is diffeomorphic to $(\mathbb{R} \times \#^l S^1 \times S^2) \# \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2$. Recall that $W_{\mathbf{m}}^{\mathbf{n}}$ can be decomposed to two handles \tilde{H}_1 and \tilde{H}_2 and $\mathbb{R} \times Z$ where Z is a 3-manifold whose boundary is the union of two tori T_1 and T_2 . The family of metrics $\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}$ parametrized by the closed interval $G_{\mathbf{m}}^{\mathbf{n}}$ is given by gluing translations of H_1 and H_2 to $\mathbb{R} \times Z$. It is easy to check that in all possible cases there are two disjoint discs D_1 and D_2 in Z that fill two longitudes in T_1 and T_2 , respectively. Furthermore, gluing $\mathbb{R} \times N(D_i)$ to H_i produces \tilde{H}_i whose boundary is diffeomorphic to $\mathbb{R} \times S^2$. Here $N(D_i)$ is a regular neighborhood of the disc D_i . Removing $N(D_1)$ and $N(D_2)$ from Z produces \tilde{Z} , a three manifold whose boundary consists of two copies of the 2-dimensional sphere.

The family of metrics $\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}$ can be alternatively constructed by gluing translations of \tilde{H}_1 and \tilde{H}_2 to $\mathbb{R} \times \tilde{Z}$. The space \tilde{H}_1 includes a 2-sphere that its self-intersection is equal to -1. Therefore, we can construct a family of metrics on \tilde{H}_1 parametrized by $[0, \infty]$ given by stretching this 2-sphere. Gluing translation of this family of metrics on \tilde{H}_1 to $\mathbb{R} \times \tilde{Z}$ and \tilde{H}_2 produces a 2-dimensional family of metrics that is parametrized by $[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}$. The four faces of $[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}$ parametrizes the following families:

- The face $\{0\} \times G_{\mathbf{m}}^{\mathbf{n}}$ parametrizes the family of metrics $\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}$.
- The face $\{\infty\} \times G_{\mathbf{m}}^{\mathbf{n}}$ parametrizes a family of broken metrics. One of the factors of this family is given by a fixed metric on a copy of $\overline{\mathbb{C}P}^2 \setminus D^4$. Therefore, Lemma 1.28 implies that the corresponding cobordism map is zero.

- There are $\mathbf{k}, \mathbf{k}' \in \{0, 1\}^n$ such that $\mathbf{k} \neq \mathbf{k}'$ and $\mathbf{m} > \mathbf{k}, \mathbf{k}' > \mathbf{n}$. The two remaining faces of $[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}$ parametrizes two families of broken metrics of the form $\mathfrak{W}_{\mathbf{k}}^{\mathbf{n}} \times \mathfrak{Y}_{\mathbf{k}}$ and $\mathfrak{Y}_{\mathbf{k}'} \times \mathfrak{W}_{\mathbf{m}}^{\mathbf{k}'}$ where $\mathfrak{Y}_{\mathbf{k}}$ and $\mathfrak{Y}_{\mathbf{k}'}$ are families of metrics on the cobordisms $W_{\mathbf{m}}^{\mathbf{k}}$ and $W_{\mathbf{k}'}^{\mathbf{n}}$, respectively. Therefore, the cobordism maps for these two faces of $[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}$ are equal to $\tilde{f}_{G_{\mathbf{k}}^{\mathbf{n}}} \circ \tilde{f}_{\mathfrak{Y}_{\mathbf{k}}}^{\eta}$ and $\tilde{f}_{\mathfrak{Y}_{\mathbf{k}'}}^{\eta} \circ \tilde{f}_{G_{\mathbf{m}}^{\mathbf{k}'}}$. Since the cobordism maps $\tilde{f}_{G_{\mathbf{k}}^{\mathbf{n}}}$ and $\tilde{f}_{G_{\mathbf{m}}^{\mathbf{k}'}}$ vanish, the cobordism maps for these faces are also zero.

Extend the perturbation term of the family of metrics $G_{\mathbf{m}}^{\mathbf{n}}$ to a perturbation for $[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}$. The identity (42) and the above realization of the faces of the family of metrics parametrized by $[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}$ show that the induced cobordism map satisfies the following relation:

$$\tilde{f}_{[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}} \circ d_{\mathbf{m}} + d_{\mathbf{n}} \circ \tilde{f}_{[0, \infty] \times G_{\mathbf{m}}^{\mathbf{n}}} = \tilde{f}_{G_{\mathbf{m}}^{\mathbf{n}}}$$

Vanishing of the differentials $d_{\mathbf{m}}$ and $d_{\mathbf{n}}$ proves the claim that $\tilde{f}_{G_{\mathbf{m}}^{\mathbf{n}}} = 0$. \square

Lemma 3.13 and the proof of Corollary 3.8 imply that the cobordism maps $\tilde{f}_{G_{\mathbf{m}}^{\mathbf{n}}}$ increases the h -grading by at least 1. Given $\mathbf{m}, \mathbf{n} \in \{0, 1\}^n$, the self-intersection of the cobordism $S_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}$ can be bounded in the following way:

$$\begin{aligned} S_{\mathbf{m}(i)}^{\mathbf{n}(i-2)} \cdot S_{\mathbf{m}(i)}^{\mathbf{n}(i-2)} &= S_{\mathbf{m}(i)}^{\mathbf{n}(i)} \cdot S_{\mathbf{m}(i)}^{\mathbf{n}(i)} + S_{\mathbf{n}(i)}^{\mathbf{n}(i-2)} \cdot S_{\mathbf{n}(i)}^{\mathbf{n}(i-2)} \\ &\leq 2|\mathbf{m} - \mathbf{n}|_1 + 2 \\ &\leq 2|\mathbf{m}(i) - \mathbf{n}(i-2)|_1 - 2 \end{aligned}$$

This shows that the cobordism map $\tilde{f}_{N_{\mathbf{m}(i)}^{\mathbf{n}(i-2)}}$ increases the homological grading by at least 1. Therefore, the maps Φ and Ψ_1 have degree 1 with respect with the homological grading associated with the pseudo-diagram N . A similar argument can be used to show that $\tilde{f}_{K_{\mathbf{m}(i)}^{\mathbf{n}(i-3)}}$ increases the homological grading by at least 2, and hence all the maps involved in the definition of $k_{\mathbf{m}(i)}^{\mathbf{n}(i)}$, K , (respectively, $q_{\mathbf{m}(i)}^{\mathbf{n}(i)}$ and Q) increase the homological grading by at least 1 (respectively, 2). In summary, all the maps in Diagram (69) have the right degree to show that Φ determines a filtered chain homotopy equivalence from $\text{PKC}(N)$ to $\text{PKC}(N')$. The following theorem summarizes our analyze of the homological and δ -gradings of the maps in Diagram (69):

Theorem 3.14. *Suppose N is a planar pseudo-diagram for a link K . Then the filtered chain homotopy type of $(\text{PKC}(N), d_p)$ depends only on K .*

A filtered \mathbb{Z} -graded chain complex (C, d) determines a spectral sequence $\{(E_r, d_r)\}$ (cf., for example, [14, Chapter XI]). Moreover, $f : (C, d) \rightarrow (C', d')$, a morphism of the category $\mathfrak{C}_{\text{GF}}(R)$, induces a morphism $f_i : E_r \rightarrow E'_r$ of the corresponding spectral sequences. If $g : (C, d) \rightarrow (C', d')$ is another morphism in $\mathfrak{C}_{\text{GF}}(R)$ that represents the same element as f , then for $i \geq 2$, $f_i = g_i$ (cf. [14, Chapter XI, Proposition 3.5]). Therefore, for $i \geq 2$, there is a functor \mathcal{E}_r from $\mathfrak{C}_{\text{GF}}(R)$ to $\mathfrak{C}_g(R)$, the category of bi-graded chain complexes over the ring R , that maps a filtered \mathbb{Z} -graded chain complex to the i^{th} page of the associated spectral sequence. Note that a morphism in $\mathfrak{C}_g(R)$ is a chain map rather than a chain map up to chain homotopy.

Motivated by Theorem 3.14, we will write $(\text{PKC}(K), d_p)$ for the isomorphism class of $(\text{PKC}(N), d_p)$ for any choice of a planar pseudo-diagram. The bi-graded chain complex $\mathcal{E}_r((\text{PKC}(K), d_p))$ is also denoted by $(\text{PKE}_r(K), d_r)$:

Corollary 3.15. *For each $r \geq 2$, the isomorphism class of $(\text{PKE}_r(K), d_r)$, as a bi-graded chain complex, is an invariant of K .*

In the next section we shall explore the relationship between $\text{PKE}_2(K)$ and Khovanov cohomology of the mirror image, \bar{K} .

4 Branched Double Covers and Odd Khovanov Homology

Suppose D is a planar diagram for a link K and $(\text{PKC}(N_D), d_p)$ is the corresponding representative of $\text{PKC}(K)$. The first page of the associated spectral sequence is equal to:

$$\left(\bigoplus_{\mathbf{m} \in \{0,1\}^{|N_D|}} \text{PFH}(\Sigma(K_{\mathbf{m}})), \bigoplus_{(\mathbf{m}, \mathbf{n}) \in \mathcal{E}_c} \text{PFH}(\Sigma(S_{\mathbf{m}}^{\mathbf{n}})) \right) \quad (70)$$

where \mathcal{E}_c denotes the set of the edges in the cube $\{0,1\}^{|N_D|}$. This chain complex inherits the following gradings:

$$\text{deg}_h = -|\mathbf{m}|_1 + n_-(D) \quad (71)$$

$$\text{deg}_\delta = -\frac{1}{2} \text{deg}_p - \frac{1}{2} |\mathbf{m}|_1 + \frac{1}{2} n_+(D) \quad (72)$$

Note that the term $\sigma(\mathbf{m}, \mathbf{n})$ does not contribute in the above formulas, because D is a planar diagram and the self-intersection of the cobordism $S_{\mathbf{m}}^{\mathbf{n}}$, for $(\mathbf{m}, \mathbf{n}) \in \{0,1\}^{|N_D|}$, is zero. Corollary 3.15 asserts that the homology of this bi-graded complex is a link invariant. Our main goal in this section is to show that this link invariant, as a bi-graded module, is isomorphic to Khovanov homology of \bar{K} with coefficients in Λ . In fact, we shall prove a generalization of this claim when the coefficient ring is replaced with $\tilde{\Lambda}_0$.

Choose compatible homology orientations for the cobordisms $W_{\mathbf{m}}^{\mathbf{n}} = \Sigma(S_{\mathbf{m}}^{\mathbf{n}})$ as in Lemma 2.3. The complex (70) can be lifted to the following chain complex with coefficients in $\tilde{\Lambda}_0$:

$$(C_o(D), d_o) := \left(\bigoplus_{\mathbf{m} \in \{0,1\}^n} \widetilde{\text{PFH}}(\Sigma(K_{\mathbf{m}})), \bigoplus_{(\mathbf{m}, \mathbf{n}) \in \mathcal{E}_c} \text{sgn}(\mathbf{m}, \mathbf{n}) \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{n}})) \right) \quad (73)$$

Recall that $\text{sgn}(\mathbf{m}, \mathbf{n})$, for $(\mathbf{m}, \mathbf{n}) \in \mathcal{E}_c$, is given by Formula (48) and is equal to the orientation of the 0-dimensional parametrizing set $G_{\mathbf{m}}^{\mathbf{n}}$. It is straightforward to check that d_o defines a differential on $C_o(D)$. The same formulas as in (71) and (72) define a bi-grading on $C_o(D)$ which are also called the homological and δ -gradings.

In subsection 4.2 it is shown that the bi-graded chain complex (73) is isomorphic to the odd Khovanov homology of the mirror image \bar{K} with coefficients in $\tilde{\Lambda}_0$. This interpretation of odd

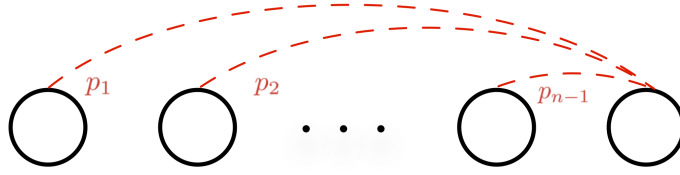


Figure 5: The unlink U_n

Khovanov homology is used to construct a spectral sequence from odd Khovanov homology that converges to $\widehat{\text{PFH}}(\Sigma(K))$. This spectral sequence is a lift of the spectral sequence in Corollary 3.15. The subsection 4.1 is devoted to make some computations related to the plane Floer homology of the unlinks. These computations provide the main input for subsection 4.2.

4.1 Plane Floer Homology of Unlinks

In this subsection, we study plane Floer homology of unlinks and maps associated with the elementary cobordisms. We take up this task firstly for a fixed link U_n . Let U_n be the n -component link in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ whose i^{th} component is a circle of radius $\frac{1}{4}$ centered around the point $(0, i)$. The link U_n is obviously an unlink. The branched double cover $\Sigma(U_n)$ is diffeomorphic to $\#^{n-1} S^2 \times S^1$. This 3-manifold has a unique torsion spin^c structure and its first Betti number is equal to $n-1$. Fix a metric on $\Sigma(U_n)$ and use the spin connection to fix an identification of $\mathcal{R}(\Sigma(U_n))$ and $J(\Sigma(U_n))$. For each $1 \leq i \leq n-1$, the path p_i in Figure 5 that connects the i^{th} connected component of U_n to the n^{th} one can be lifted to an oriented loop γ_i in the branched double cover. The set $\{\gamma_1, \dots, \gamma_{n-1}\}$ forms a basis of $H_1(\Sigma(U_n), \mathbb{R})$. Using the dual basis for $H^1(\Sigma(U_n), \mathbb{R})$, we have the following identification:

$$\mathcal{R}(\Sigma(U_n)) = S_1 \times \dots \times S_{n-1} := \{(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1} \mid |z_i| = 1\}$$

where S_i is in correspondence with the loop γ_i .

For $\tau = (\tau_1, \dots, \tau_{n-1}) \in \{-1, 1\}^{n-1}$, let $T_\tau \subset \mathcal{R}(\Sigma(U_n))$ be the subspace:

$$T_1 \times \dots \times T_{n-1} \subseteq S_1 \times \dots \times S_{n-1}$$

where $T_i = S_i$ if $\tau_i = 1$, and otherwise $T_i = \{-1\}$. The normal bundle of this sub-manifold of $\mathcal{R}(\Sigma(U_n))$ is isomorphic to $l_1 \oplus \dots \oplus l_{n-1}$ where $l_i = \{0\}$ if $T_i = S_i$ and $l_i = \mathbb{R}$ if $T_i = \{-1\}$. In the latter case, orient l_i with the standard orientation of the circle S_i . The product orientation defines a co-orientation for T_τ . The poincaré duals of these co-oriented tori determine a basis for the cohomology of $\mathcal{R}(\Sigma(U_n))$ and hence a basis for $\widehat{\text{PFH}}(U_n)$.

This basis can be used to define an isomorphism $L_n : \widehat{\text{PFH}}(\Sigma(U_n)) \rightarrow \Lambda^*(\tilde{H}^0(U_n))$, where $\tilde{H}^0(U_n)$ is the reduced cohomology of U_n . For $1 \leq i \leq n-1$, let $v_i \in \widehat{\text{PFH}}\tilde{H}^0(U_n)$ be the element that assigns 1 to the i^{th} component of U_n , -1 to the n^{th} component, and 0 to the other components. The linear map L_n , by definition, sends the generator of $\widehat{\text{PFH}}(U_n)$, represented by T_τ , to:

$$(-1)^{k(n-1)} v_{i_1} \wedge \dots \wedge v_{i_k} \in \Lambda^*(\tilde{H}^0(U_n))$$

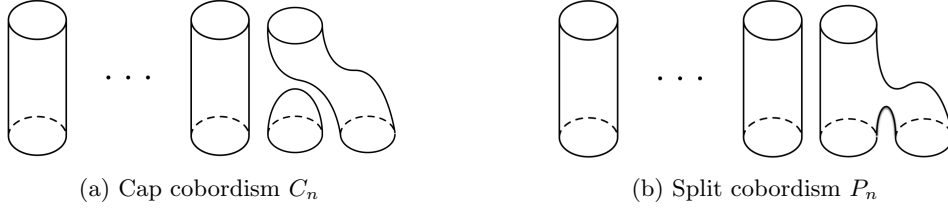


Figure 6: Elementary Cobordisms from U_n to U_{n+1}

where $i_1 < i_2 < \dots < i_k$, and $l \in \{i_1, \dots, i_n\}$ if and only if $\tau_l = -1$.

Remark 4.1. Equip $\mathcal{R}(\Sigma(U_n))$ with the standard product metric and consider the Morse-Smale function that maps (z_1, \dots, z_{n-1}) to $\sum(z_i + \bar{z}_i)$. The set of the critical points of this function is equal to $\{-1, 1\}^{n-1}$ and the closure of the unstable manifold of the critical point τ is equal to T_τ . Therefore, the basis of $\widehat{\text{PFH}}(U_n)$, mentioned above, consists of the cycles naturally associated with the generators of the Morse complex of $\mathcal{R}(\Sigma(U_n))$.

Proposition 4.2. Suppose C_n and P_n are respectively the cap and the split cobordisms from U_n to U_{n+1} (Figure 6). Then an appropriate homology orientations for these cobordisms can be fixed such that the corresponding cobordism maps are equal to:

$$\widehat{\text{PFH}}(\Sigma(C_n))(v_{i_1} \wedge \dots \wedge v_{i_k}) = v_{i_1} \wedge \dots \wedge v_{i_k} \quad \widehat{\text{PFH}}(\Sigma(P_n))(v_{i_1} \wedge \dots \wedge v_{i_k}) = v_n \wedge v_{i_1} \wedge \dots \wedge v_{i_k} \quad (74)$$

Similarly, the cup cobordism $\bar{C}_n : U_{n+1} \rightarrow U_n$ and the merge cobordism $\bar{P}_n : U_{n+1} \rightarrow U_n$ (Figure 8) can be given homology orientations such that:

$$\begin{aligned} \widehat{\text{PFH}}(\Sigma(\bar{C}_n))(v_{i_1} \wedge \dots \wedge v_{i_k}) &= 0 & \widehat{\text{PFH}}(\Sigma(\bar{P}_n))(v_{i_1} \wedge \dots \wedge v_{i_k}) &= v_{i_1} \wedge \dots \wedge v_{i_k} \\ \widehat{\text{PFH}}(\Sigma(\bar{C}_n))(v_n \wedge v_{i_1} \wedge \dots \wedge v_{i_k}) &= v_{i_1} \wedge \dots \wedge v_{i_k} & \widehat{\text{PFH}}(\Sigma(\bar{P}_n))(v_n \wedge v_{i_1} \wedge \dots \wedge v_{i_k}) &= 0 \end{aligned} \quad (75)$$

where $1 \leq i_l \leq n-1$.

Proof. The cobordism $\Sigma(C_n)$ is diffeomorphic to:

$$((D^3 \times S^1) \setminus D^4) \natural (I \times (\#^{n-1} S^2 \times S^1)) : \#^{n-1} S^2 \times S^1 \rightarrow \#^n S^2 \times S^1$$

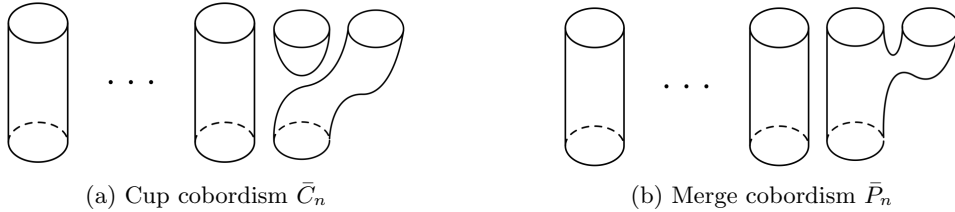


Figure 7: Elementary Cobordisms from U_{n+1} to U_n

Here $(D^3 \times S^1) \setminus D^4$ is regarded as a cobordism from S^3 to $S^1 \times S^2$. Given two cobordisms $W : Y_0 \rightarrow Y_1$ and $W' : Y'_0 \rightarrow Y'_1$, $W \natural W'$ is a cobordism from the connected sum $Y_0 \# Y'_0$ to the connected sum $Y_1 \# Y'_1$. This cobordism is constructed by removing a neighborhood of paths in W and W' which connect the incoming and the outgoing ends and then gluing the resulted manifolds along the boundary of the regular neighborhoods of the paths. Therefore, $b^+(\Sigma(C_n)) = 0$, $b^1(\Sigma(C_n)) = n$, and there is exactly one spin^c structure that restricts to the torsion spin^c structure on the boundary. This spin^c structure is the lift of the spin structure on $\Sigma(C_n)$. Because $b^+(\Sigma(C_n)) = 0$, we can use the second case of Proposition 1.22 to compute the cobordism map associated with $\Sigma(C_n)$. The corresponding pull-up-push-down diagram is given by:

$$\begin{array}{ccc}
& S_1 \times S_2 \times \cdots \times S_{n-1} \times S_n & \\
j_n \swarrow & & \searrow id \\
S_1 \times S_2 \times \cdots \times S_{n-1} & & S_1 \times S_2 \times \cdots \times S_{n-1} \times S_n
\end{array} \tag{76}$$

where the map j_n sends the point $(z_1, \dots, z_{n-1}, z_n)$ to (z_1, \dots, z_{n-1}) . The spaces $M(\Sigma(C_n))$ and $\mathcal{R}(U_{n+1})$ are the same and we orient them with the same orientations. Use these orientations of $M(\Sigma(C_n))$ and $\mathcal{R}(U_{n+1})$ to fix a homology orientation for $\Sigma(C_n)$. Our conventions from subsection 1.4 implies that:

$$\widetilde{\text{PFH}}(\Sigma(C_n))(\text{PD}[T_\tau]) = (-1)^k \text{PD}[T_\tau \times S_n]$$

where k is the codimension of T_τ . Therefore, $\widetilde{\text{PFH}}(\Sigma(C_n))$ has the form in (74).

In the case of the split cobordism $P_n : U_n \rightarrow U_{n+1}$, the branched double cover $\Sigma(P_n)$ is diffeomorphic to:

$$((D^2 \times S^2) \setminus D^4) \natural (I \times (\#^{n-1} S^2 \times S^1)) : \#^{n-1} S^2 \times S^1 \rightarrow \#^n S^2 \times S^1$$

Here $(D^2 \times S^2) \setminus D^4$ is regarded as a cobordism from S^3 to $S^1 \times S^2$. The diagram of the moduli space and the restriction maps is isomorphic to:

$$\begin{array}{ccc}
& S_1 \times S_2 \times \cdots \times S_{n-1} & \\
id \swarrow & & \searrow i_n \\
S_1 \times S_2 \times \cdots \times S_{n-1} & & S_1 \times S_2 \times \cdots \times S_{n-1} \times S_n
\end{array} \tag{77}$$

where the map i_n maps (z_1, \dots, z_{n-1}) to $(z_1, \dots, z_{n-1}, 1)$. Fix an arbitrary orientation of $M(\Sigma(P_n))$ and let $\mathcal{R}(U_{n+1}) = M(\Sigma(P_n)) \times S_n$ have the product orientation. Again, define a homology orientation of $\Sigma(P_n)$ using these orientations. Then we have:

$$\widetilde{\text{PFH}}(\Sigma(P_n))(\text{PD}[T_\tau]) = (-1)^n \text{PD}[T_\tau]$$

which finishes the proof of (74).

Note that $C_n \circ \bar{P}_n$ and $P_n \circ \bar{C}_n$ are product cobordisms. Fix homology orientations for \bar{P}_n and \bar{C}_n such that the composed orientations on the product cobordisms $C_n \circ \bar{P}_n$ and $P_n \circ \bar{C}_n$ are trivial. Therefore, the following compositions are identity:

$$\widetilde{\text{PFH}}(\Sigma(\bar{P}_n)) \circ \widetilde{\text{PFH}}(\Sigma(C_n)) \qquad \widetilde{\text{PFH}}(\Sigma(\bar{C}_n)) \circ \widetilde{\text{PFH}}(\Sigma(P_n))$$

On the other hand, an application of Proposition 1.22 shows that the following cobordism maps vanish:

$$\widetilde{\text{PFH}}(\Sigma(\bar{P}_n)) \circ \widetilde{\text{PFH}}(\Sigma(P_n)) \qquad \widetilde{\text{PFH}}(\Sigma(\bar{C}_n)) \circ \widetilde{\text{PFH}}(\Sigma(C_n))$$

This is true because $b^+(\Sigma(P_n \circ \bar{P}_n)) > 0$ and $\Sigma(C_n \circ \bar{C}_n) \cong (I \times \#^{n-1}S^2 \times S^1) \# S^3 \times S^1$. Identity (75) can be verified with the characterization of the above compositions. \square

From now on we use the term unlink in the following more restrictive sense:

Definition 4.3. An unlink is a pair $(\mathcal{U}, \mathcal{D})$ of a link \mathcal{U} with n connected components and a Seifert surface \mathcal{D} that consists of n discs. An *unlink isotopy* \mathcal{J} from $(\mathcal{U}, \mathcal{D})$ to $(\mathcal{U}', \mathcal{D}')$ is an isotopy of surfaces (with boundary) that starts from \mathcal{D} and ends at \mathcal{D}' .

The link U_n comes with an obvious disc filling D_n that is embedded in the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, and (U_n, D_n) is our standard example of unlink with n components. Note that restriction to the boundary of an unlink isotopy $\mathcal{J} : (\mathcal{U}, \mathcal{D}) \rightarrow (\mathcal{U}', \mathcal{D}')$ induces an isotopy $J : \mathcal{U} \rightarrow \mathcal{U}'$. If \mathcal{J} and \mathcal{J}' are two unlink isotopies, then a homotopy between \mathcal{J} and \mathcal{J}' is a smooth family of unlink isotopies $\mathcal{J}_t : (\mathcal{U}, \mathcal{D}) \rightarrow (\mathcal{U}', \mathcal{D}')$ such that $\mathcal{J}_0 = \mathcal{J}$ and $\mathcal{J}_1 = \mathcal{J}'$. The unlink isotopy \mathcal{J} induces a bijection from the set of connected components of \mathcal{U} to that of \mathcal{U}' which does not change by homotopies of unlink isotopies. In fact, \mathcal{J} , up to homotopy, is determined by this bijection. This is a consequence of the fact that the fundamental group of configuration space of n (labeled) points in S^3 is trivial. The cobordism $\Sigma(J)$ is diffeomorphic to $I \times \Sigma(\mathcal{U}')$ and hence it comes with the product homology orientation. In the following, we use this homology orientation for the branched double cover of an unlink isotopy. Also, we will drop the disc filling \mathcal{D} from our notation for an unlink unless there is a chance of confusion.

Proposition 4.4. Let $\mathcal{J} : U_n \rightarrow U_n$ be an unlink isotopy that induces the permutation σ of the connected components of U_n . That is to say, \mathcal{J} connects the i^{th} connected component of the incoming unlink to the $\sigma(i)^{\text{th}}$ connected component of the outgoing end. The permutation σ induces an automorphism of $H^0(U_n)$ and hence an automorphism of $\Lambda^*(\tilde{H}^0(U_n))$. Once $\widetilde{\text{PFH}}(\Sigma(U_n))$ is identified with $\Lambda^*(\tilde{H}^0(U_n))$ as above, this automorphism is equal to $\widetilde{\text{PFH}}(\Sigma(J))$.

Proof. The restriction maps from $M(\Sigma(J))$ to the representation varieties of the ends are diffeomorphisms of tori of dimension $n - 1$. However, these maps are not equal to each other. An appropriate parametrization of $M(\Sigma(J))$ gives rise to the following restriction maps:

$$\begin{array}{ccc} & S_1 \times S_2 \times \cdots \times S_{n-1} & \\ r_\sigma \swarrow & & \searrow id \\ S_1 \times S_2 \times \cdots \times S_{n-1} & & S_1 \times S_2 \times \cdots \times S_{n-1} \times S_{n-1} \end{array} \quad (78)$$

where $r_\sigma(z_1, \dots, z_{n-1})$ is equal to:

$$\begin{cases} (z_{\sigma(1)}, \dots, z_{\sigma(n-1)}) & \text{if } \sigma(n) = n \\ (z_{\sigma(1)}z_i^{-1}, \dots, z_{\sigma(j-1)}z_i^{-1}, z_i^{-1}, z_{\sigma(j+1)}z_i^{-1}, \dots, z_{\sigma(n-1)}z_i^{-1}) & \text{if } i := \sigma(n) \leq n-1, j := \sigma^{-1}(n) \end{cases}$$

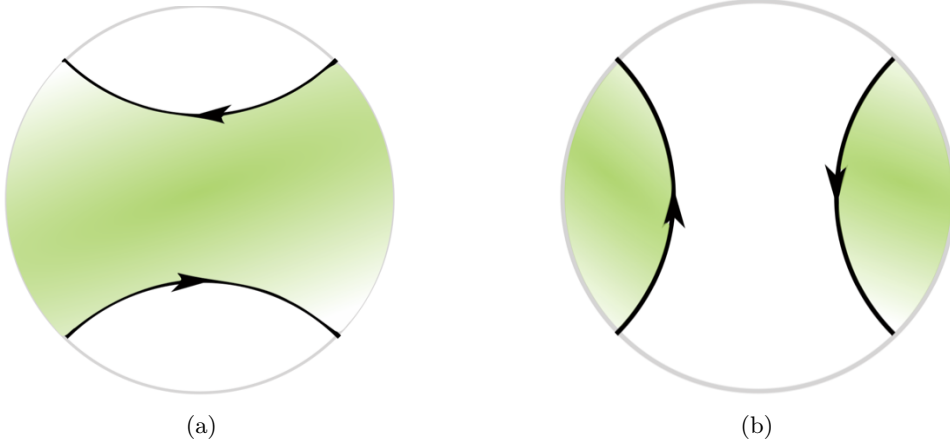


Figure 8

Therefore, the map $\widetilde{\text{PFH}}(\Sigma(J))$ is equal to $(r_\sigma)^*$. It is also true that the induced map by σ on $\Lambda^*(\tilde{H}^0(U_n))$ is the same as $(r_\sigma)^*$. \square

Corollary 4.5. *If $(\mathcal{U}, \mathcal{D})$ is an arbitrary unlink, then $\widetilde{\text{PFH}}(\Sigma(\mathcal{U}))$ is canonically isomorphic to $\Lambda^*(\tilde{H}^0(\mathcal{U}))$.*

Proof. Let \mathcal{U} have n components and fix an unlink isotopy $\mathcal{J} : (\mathcal{U}, \mathcal{D}) \rightarrow (U_n, D_n)$. Since J is the product cobordism, $\widetilde{\text{PFH}}(\Sigma(J))$ is an isomorphism. Furthermore, \mathcal{J} defines a bijection between the connected components of \mathcal{U} and U_n . Therefore, it induces an isomorphism $i_{\mathcal{J}} : \Lambda^*(\tilde{H}^0(\mathcal{U})) \rightarrow \Lambda^*(\tilde{H}^0(U_n))$. The map $i_{\mathcal{J}}^{-1} \circ L_n \circ \text{PFH}(\Sigma(J)) : \widetilde{\text{PFH}}(\Sigma(\mathcal{U})) \rightarrow \Lambda^*(\tilde{H}^0(\mathcal{U}))$ is an isomorphism. Proposition 4.4 shows that this isomorphism is independent of the choice of \mathcal{J} . \square

Now that the plane Floer homology of an unlink is characterized more canonically, we can improve the statement of Proposition 4.2. Let c be a crossing of an unlink \mathcal{U} with n components such that the two strands belong to the same component of \mathcal{U} and the intersection of the disc filling \mathcal{D} with c has the form in Figure 8a. Then replacing the crossing with Figure 8b introduces another unlink \mathcal{U}' with $n + 1$ components and a disc filling \mathcal{D}' . The connected components of \mathcal{U}' that contains the strands in Figure 8b are denoted by K_1 and K_2 .

As in subsection 3.1, we can define a cobordism $\mathcal{P} : \mathcal{U} \rightarrow \mathcal{U}'$ which is a split cobordism in this case. Fix unlink isotopies $\mathcal{J} : \mathcal{U} \rightarrow U_n$ and $\mathcal{J}' : \mathcal{U}' \rightarrow U_{n+1}$ such that $P_n \circ J$ and $J' \circ \mathcal{P}$ are isotopic to each other with an isotopy that fixes the ends. Define the homology orientation of $\Sigma(\mathcal{P})$ such that the composed orientations of $\Sigma(\mathcal{P})$ and $\Sigma(\mathcal{J}')$ is equal to that of $\Sigma(\mathcal{J})$ and $\Sigma(P_n)$. Proposition 4.2 shows that with the chosen homology orientation, the map $\widetilde{\text{PFH}}(\Sigma(\mathcal{P}))$ acts in the following way:

$$F_{\mathcal{P}} : w \in \Lambda^*(\tilde{H}^0(\mathcal{U})) \rightarrow v \wedge w \in \Lambda^*(\tilde{H}^0(\mathcal{U}')) \quad (79)$$

where v , as an element of $\tilde{H}^0(\mathcal{U})$, is non-zero only on the connected components K_1, K_2 . It evaluates to 1 on one of these components and -1 on the other one. There are two possibilities for v and both of them are feasible by different choices of \mathcal{J} and \mathcal{J}' . In fact, if we change \mathcal{J}' by composing it with an unlink isotopy that permutes the n^{th} and the $(n+1)^{\text{st}}$ components, then v turns into $-v$. However, if we fix one of the two possibilities for v , then all different choices of \mathcal{J} and \mathcal{J}' define the same cobordism map $F_{\mathcal{P}}$. Consequently, they determine the same homology orientation on \mathcal{P} .

Next, let \mathcal{U} be an unlink with $n+1$ connected components. Suppose c is a crossing such that the two strands belong to the different connected components and the intersection of the disc filling \mathcal{D} with the crossing is as in Figure 8b. As in the previous paragraph, we can construct an unlink \mathcal{U}' and a merge cobordism $\bar{\mathcal{P}} : \mathcal{U} \rightarrow \mathcal{U}'$. Choose unlink isotopies $\mathcal{J} : \mathcal{U} \rightarrow U_{n+1}$, $\mathcal{J}' : \mathcal{U}' \rightarrow U_n$, with $\bar{\mathcal{P}} \circ \mathcal{J}'$ and $\mathcal{J} \circ \bar{\mathcal{P}}_n$ being isotopic to each other, and use these to define a homology orientation on $\Sigma(\bar{\mathcal{P}})$ as in the previous case. Each connected component of \mathcal{U} is connected to one component of \mathcal{U}' via $\bar{\mathcal{P}}$ and $j : H^0(\mathcal{U}) \rightarrow H^0(\mathcal{U}')$ is the map that sends the Poincaré dual of the fundamental class of a component of \mathcal{U} to the Poincaré dual of the fundamental class of the corresponding component of \mathcal{U}' . This map induces:

$$F_{\bar{\mathcal{P}}} : w \in \Lambda^*(\tilde{H}^0(\mathcal{U})) \rightarrow j_*(w) \in \Lambda^*(\tilde{H}^0(\mathcal{U}')) \quad (80)$$

and by Proposition 4.2, $\widetilde{\text{PFH}}(\Sigma(\bar{\mathcal{P}}))$ is equal to this map.

Let $(\mathcal{U}, \mathcal{D})$, $(\mathcal{U}', \mathcal{D}')$ be unlinks such that $(\mathcal{U}, \mathcal{D})$ is given by dropping one of the connected components of $(\mathcal{U}', \mathcal{D}')$. We can construct a cap cobordism $\mathcal{C} : \mathcal{U} \rightarrow \mathcal{U}'$ that is the union of two parts. The first part is $I \times \mathcal{U}$ and the second part is the pushing of the extra component of \mathcal{D}' from $\{1\} \times S^3$ into $I \times S^3$. We can proceed as above to define a homology orientation for $\Sigma(\mathcal{C})$. Let $i : H^0(\mathcal{U}) \rightarrow H^0(\mathcal{U}')$ be the map that extends an element of $H^0(\mathcal{U})$ by zero on the extra component of \mathcal{U}' . Then the cobordism map associated with \mathcal{C} is equal to:

$$F_{\mathcal{C}} : w \in \Lambda^*(\tilde{H}^0(\mathcal{U})) \rightarrow i_*(w) \in \Lambda^*(\tilde{H}^0(\mathcal{U}')) \quad (81)$$

By reversing the role of the incoming and the outgoing ends in the previous paragraph, we can also construct a cup cobordism $\bar{\mathcal{C}}$ from an unlink \mathcal{U} with $n+1$ components to an unlink \mathcal{U}' with n components. By following the same strategy to define a homology orientation for $\Sigma(\bar{\mathcal{C}})$, we will have the following cobordism map:

$$F_{\bar{\mathcal{C}}} : w \in \Lambda^*(\tilde{H}^0(\mathcal{U})) \rightarrow \iota_K(w) \in \Lambda^*(\tilde{H}^0(\mathcal{U}')) \quad (82)$$

where ι_K is contraction with respect to the element of $H_0(\mathcal{U})$ that is represented by the extra component. Note that in the case of merge, cup, and cap cobordisms the chosen homology orientations are independent of the unlink isotopes. Because in these cases the cobordism maps are independent of these choices.

4.2 Odd Khovanov Homology Revisited

In this subsection, we firstly review the definition of reduced odd Khovanov homology. The reduced odd Khovanov homology of the mirror image with coefficients in $\hat{\Lambda}_0$ interacts better with plane

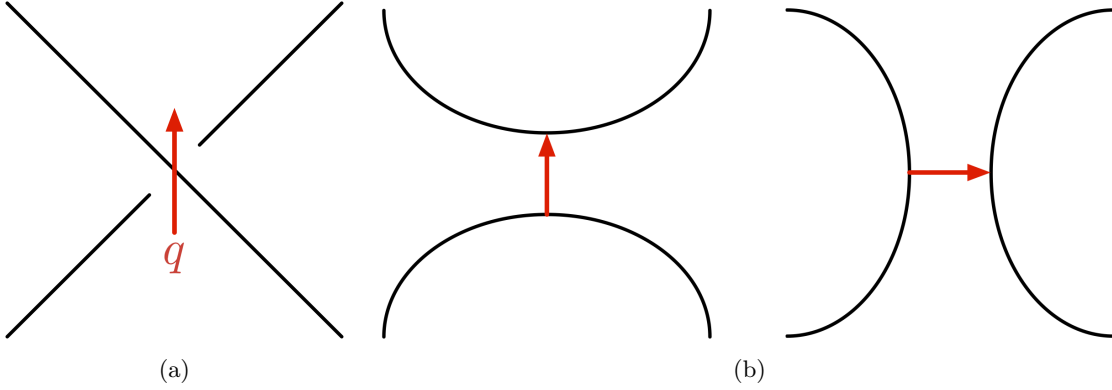


Figure 9: A decorated crossing and its decorated resolutions

Floer homology. Therefore, we focus on this version of odd Khovanov homology. The main input in the original definition of odd Khovanov homology is the notion of *decorated link diagrams*. In the context of this paper, oriented link diagrams can be used as a means to fix homology orientations for the cobordisms in the cube of resolutions:

Definition 4.6. A *decorated link diagram* D for a link K is a planar diagram such that at each crossing of D , one of the two possible orientations for the line segment q in Figure 9a is fixed. The resolutions of a decorated planar diagram are also equipped with extra decorations: if the line segment q is oriented as Figure 9a, then orient the line segment connecting the two strands as in Figure 9b. Otherwise these orientations for the resolutions are reversed.

Fix a decorated diagram D for a link K . Any resolution $K_{\mathbf{m}}$ of K , associated with D , is an unlink embedded in $\mathbb{R}^2 \times \{0\}$ of \mathbb{R}^3 . A disc filling of $K_{\mathbf{m}}$ can be fixed in the following way. The most inner circles in $K_{\mathbf{m}}$ bound planar discs, i.e., discs that live in the plane $\mathbb{R}^2 \times \{0\}$. Fix these discs as the filling of the corresponding components. Next, remove the filled circles and consider the discs that fill the most inner circles among the remaining ones. We can push the interior of such discs into $\mathbb{R}^2 \times \mathbb{R}^+$ such that they lie above the discs constructed in the first stage and hence they avoid intersecting with each other. Iteration of this construction determines a disc filling for $K_{\mathbf{m}}$.

For an edge (\mathbf{m}, \mathbf{n}) of the cube $\{0, 1\}^{|N_D|}$, consider $S_{\mathbf{m}}^{\mathbf{n}} : K_{\mathbf{m}} \rightarrow K_{\mathbf{n}}$ which is a pair of pants cobordism. A straightforward examination of the different possible cases shows that $S_{\mathbf{m}}^{\mathbf{n}}$, with respect to the isotopy class of the fixed disc fillings, is either \mathcal{P} or $\bar{\mathcal{P}}$ constructed in subsection 4.1. Homology orientations for the merge cobordisms are already fixed in the previous subsection. If $S_{\mathbf{m}}^{\mathbf{n}}$ is a split cobordism, the ambiguity of the homology orientation of $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$ can be fixed by ordering the two components of the outgoing end that intersects the crossing. Denote these two components by A_1 and A_2 . The decorated resolution of the crossing provides us with such an ordering. Suppose that the decoration of the line segment between $\widetilde{A_1}$ and A_2 points from A_1 to A_2 . We fix the homology orientation such that the cobordism map $\text{PFH}(\Sigma(S_{\mathbf{m}}^{\mathbf{n}}))$ is equal to:

$$F_S : w \in \Lambda^*(\tilde{H}^0(K_{\mathbf{m}})) \rightarrow v \wedge w \in \Lambda^*(\tilde{H}^0(K_{\mathbf{n}}))$$

where $v \in \tilde{H}^0(K_{\mathbf{n}})$ evaluates to 1 (respectively, -1) on the representative of the component A_1 (respectively, A_2).

In [15], the first step to construct the reduced odd Khovanov chain complex of \bar{K} is to assign a module to each resolution $K_{\mathbf{m}}$. It can be easily checked that this module is the same as $\widetilde{\text{PFH}}(\Sigma(K_{\mathbf{m}})) = \Lambda^*(\tilde{H}^0(K_{\mathbf{m}}))$. The next step in the definition of odd Khovanov homology is to define a map from $\Lambda^*(\tilde{H}^0(K_{\mathbf{m}}))$ to $\Lambda^*(\tilde{H}^0(K_{\mathbf{n}}))$ for each edge (\mathbf{m}, \mathbf{n}) of the cube. It is also straightforward to check that this map is the same as $\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{n}}))$, using the fixed homology orientations.

Suppose (\mathbf{m}, \mathbf{n}) is a face of the cube $\{0, 1\}^{|N_D|}$ and \mathbf{k}, \mathbf{k}' are the two vertices that $\mathbf{m} > \mathbf{k}, \mathbf{k}' > \mathbf{n}$. Homology orientations of the cobordisms $S_{\mathbf{m}}^{\mathbf{k}}, S_{\mathbf{m}}^{\mathbf{k}'}, S_{\mathbf{k}}^{\mathbf{n}}$ and $S_{\mathbf{k}'}^{\mathbf{n}}$ are already fixed. This face of the cube is called *commutative* if the composed homology orientations of $S_{\mathbf{m}}^{\mathbf{k}}$ and $S_{\mathbf{k}}^{\mathbf{n}}$ is equal to the composed homology orientations of $S_{\mathbf{m}}^{\mathbf{k}'}$ and $S_{\mathbf{k}'}^{\mathbf{n}}$. Otherwise this face is called *non-commutative*. If the composition $\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}}))$ (or equivalently $\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}'}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}'})$) is non-zero then plane Floer homology can be used to detect whether (\mathbf{m}, \mathbf{n}) is commutative or not. To be more specific, if:

$$\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}})) = \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}'}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}'})$$

then (\mathbf{m}, \mathbf{n}) is commutative and it is non-commutative if:

$$\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}})) = -\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}'}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}'})$$

Any face (\mathbf{m}, \mathbf{n}) of the cube $\{0, 1\}^{|N_D|}$ is determined by the resolution of the vertex $K_{\mathbf{m}}$ and the corresponding decorations of the two crossings associated with the face. A type X face (respectively, type Y face) is a face that its decorated resolution $K_{\mathbf{m}}$ is equal to the union of the arrangement in Figure 10a (respectively, Figure 10b) and several (possibly zero) other components that do not intersect the crossings of the face. Type X and type Y crossings are the only types of the faces such that $\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}}))$ vanishes.

Lemma 4.7. *Out of type X and type Y faces, faces of one type are commutative and faces of the other type are non-commutative.*

Proof. Suppose that L is the decorated planar diagram of a link consisting of two crossings such that its $(1, 1)$ resolution has only one component and it has the form of Figure 10a. Therefore, the only face of this planar diagram has type X . Label these crossings by requiring that the first crossing is represented by the vertical red line in Figure 10a and the second crossing is the other one. Let L' be another decorated diagram for the same link that is given by Figure 10b. The diagram L' has only one face that is of type Y . We can use these two decorated diagrams to define two homology orientations for each of the cobordisms $S_{11}^{10}, S_{11}^{01}, S_{10}^{00}$ and S_{01}^{00} . Out of these cobordisms, only the two homology orientations of the cobordism S_{11}^{10} differ from each other. Therefore, one of L and L' has a commutative face and the other one has a non-commutative face. In the following, we assume that L is the decorated diagram with a commutative face. The proof in the other case is similar.

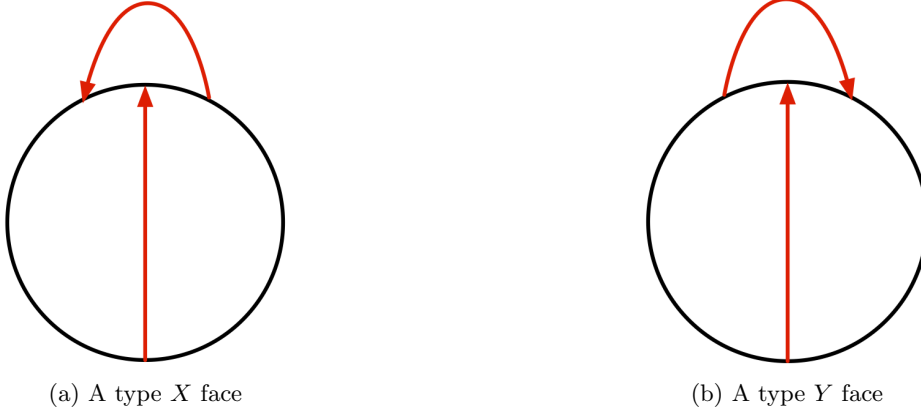


Figure 10

Let (\mathbf{m}, \mathbf{n}) be a face of a decorated diagram K with type X . Therefore $K_{\mathbf{m}}$ has a component that intersects the two crossings determined by this face. This component has the form in Figure 10a and we fix the labeling such that:

$$\mathbf{m} = (1, 1, 0 \dots, 0) \quad \mathbf{n} = (0, 0, 0 \dots, 0)$$

and if:

$$\mathbf{k} = (1, 0, 0 \dots, 0) \quad \mathbf{k}' = (0, 1, 0 \dots, 0)$$

then $K_{\mathbf{k}}$ is given by the 1-resolution of the first crossing and the 0-resolution of the second crossing in Figure 10a. Suppose that $S : K_{\mathbf{m}} \rightarrow L_{11}$, $S' : K_{\mathbf{n}} \rightarrow L_{00}$, $S''_1 : K_{\mathbf{k}} \rightarrow L_{10}$ and $S''_2 : K_{\mathbf{k}'} \rightarrow L_{01}$ are composition of cup cobordisms that fill the components which do not intersect the crossings. Furthermore, we have:

$$S_{\mathbf{m}}^{\mathbf{k}'} \circ S_{\mathbf{k}'}^{\mathbf{n}} \circ S' = S_{\mathbf{m}}^{\mathbf{k}'} \circ S''_2 \circ S''_1 \circ S_{01}^{00} = S \circ S_{11}^{01} \circ S_{01}^{00} = S \circ S_{11}^{10} \circ S_{10}^{00} = S_{\mathbf{m}}^{\mathbf{k}} \circ S''_1 \circ S_{10}^{00} = S_{\mathbf{m}}^{\mathbf{k}} \circ S_{\mathbf{k}}^{\mathbf{n}} \circ S' \quad (83)$$

We use the homology orientation of the cup cobordisms to define homology orientations for $\Sigma(S)$, $\Sigma(S')$, $\Sigma(S''_1)$ and $\Sigma(S''_2)$. The cobordism compositions in (83) define 6 homology orientations on the same cobordism. By assumption the homology orientation determined by the compositions $S \circ S_{11}^{10} \circ S_{10}^{00}$ and $S \circ S_{11}^{01} \circ S_{01}^{00}$ are the same. One can also easily check that the non-zero maps $\widetilde{\text{PFH}}(\Sigma(S')) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}}^{\mathbf{n}}))$ and $\widetilde{\text{PFH}}(\Sigma(S_{01}^{00})) \circ \widetilde{\text{PFH}}(\Sigma(S''_2))$ are equal to each other. Thus $S_{\mathbf{m}}^{\mathbf{k}'} \circ S_{\mathbf{k}'}^{\mathbf{n}} \circ S'$ and $S_{\mathbf{m}}^{\mathbf{k}'} \circ S''_2 \circ S''_1 \circ S_{01}^{00}$ determine the same homology orientations. A similar argument shows that the remaining pair of compositions that appear as consecutive terms in (83) determine the same homology orientations. In particular, the homology orientations of the compositions $S_{\mathbf{m}}^{\mathbf{k}'} \circ S_{\mathbf{k}}^{\mathbf{n}} \circ S'$ and $S_{\mathbf{m}}^{\mathbf{k}} \circ S_{\mathbf{k}}^{\mathbf{n}} \circ S'$ are the same. This proves the claim that the face (\mathbf{m}, \mathbf{n}) is even. \square

The following definition is a reformulation of Definition 1.1. in [15]:

Definition 4.8. A type X edge assignment for D is a map σ from the set of edge of the cube $\{0, 1\}^n$ to $\{1, -1\}$ such that for any face (\mathbf{m}, \mathbf{n}) :

$$\sigma(\mathbf{m}, \mathbf{k})\sigma(\mathbf{k}, \mathbf{n})\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}})) + \sigma(\mathbf{m}, \mathbf{k}')\sigma(\mathbf{k}', \mathbf{n})\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{k}'}^{\mathbf{n}})) \circ \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{k}'})) = 0. \quad (84)$$

Furthermore, we require that if (\mathbf{m}, \mathbf{n}) is a face of type X , then:

$$\sigma(\mathbf{m}, \mathbf{k}) \cdot \sigma(\mathbf{k}, \mathbf{n}) = \sigma(\mathbf{m}, \mathbf{k}') \cdot \sigma(\mathbf{k}', \mathbf{n})$$

and if it is a face of type Y , then:

$$\sigma(\mathbf{m}, \mathbf{k}) \cdot \sigma(\mathbf{k}, \mathbf{n}) = -\sigma(\mathbf{m}, \mathbf{k}') \cdot \sigma(\mathbf{k}', \mathbf{n}).$$

In the definition, if we reverse the role of type X and type Y faces, then the resulting edge assignment has type Y .

Given a type X or a type Y edge assignment σ , the reduced odd Khovanov chain complex of \bar{K} associated with the decorated link diagram D is equal to:

$$\left(\bigoplus_{\mathbf{m} \in \{0,1\}^n} \widetilde{\text{PFH}}(\Sigma(K_{\mathbf{m}})), \bigoplus_{(\mathbf{m}, \mathbf{n}) \in \mathcal{E}_c} \sigma(\mathbf{m}, \mathbf{n}) \widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{n}})) \right) \quad (85)$$

Relation (84) implies that (85) is in fact a chain complex. The chain complex (85) is equipped with two gradings in [15] which are called the homological and quantum gradings. If we will write h_o and q_o for these gradings, then these gradings are equal to:

$$\deg_{h_o} = (|\bar{D}| - |\mathbf{m}|_1) - n_-(\bar{D}) \quad (86)$$

$$\deg_{q_o} = (1 + \deg_p) + (|\bar{D}| - |\mathbf{m}|_1) + n_+(\bar{D}) - 2n_-(\bar{D}) \quad (87)$$

Here \bar{D} is the digram for \bar{K} that is induced by D . This term appears in our formulas because the chain complex (85) computes the odd Khovanov homology of \bar{K} using the diagram \bar{D} .

Theorem 4.9. *There is an isomorphism between the chain complexes (73) and (85). Furthermore, this isomorphism relates the bi-gradings in the following way:*

$$\deg_{h_o} = \deg_h, \quad \deg_{q_o} = 2(\deg_h - \deg_\delta) + 1$$

Proof. Note that the definition of the chain complex (73) depends on D and a choice of homology orientations on the cobordisms $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$ that satisfies the assumption of Lemma 2.3. On the other hand, the chain complex (85) depends on D and a type X or type Y edge assignment. In particular, the cobordism maps $\widetilde{\text{PFH}}(\Sigma(S_{\mathbf{m}}^{\mathbf{n}}))$, involved in the definition, are defined by homology orientation $o_{N_D}(\mathbf{m}, \mathbf{n})$ determined by D . It is shown in [15] that for any decorated diagram, type X and type Y edge assignments exist and the corresponding chain complexes for a fixed diagram are isomorphic.

Now suppose type X faces are non-commutative and a type X edge assignment σ of the diagram is fixed. For an edge (\mathbf{m}, \mathbf{n}) of the cube $\{0, 1\}^n$ define:

$$o(\mathbf{m}, \mathbf{n}) := \sigma(\mathbf{m}, \mathbf{n}) \text{sgn}(\mathbf{m}, \mathbf{n}) o_{N_D}(\mathbf{m}, \mathbf{n}) \in o(\Sigma(S_{\mathbf{m}}^{\mathbf{n}}))$$

Suppose (\mathbf{m}, \mathbf{n}) is a face of the cube $\{0, 1\}^n$ and \mathbf{k} and \mathbf{k}' are the vertices that $\mathbf{m} > \mathbf{k}, \mathbf{k}' > \mathbf{n}$. Then we have:

$$o(\mathbf{k}, \mathbf{n}) \circ o(\mathbf{m}, \mathbf{k}) = o(\mathbf{k}', \mathbf{n}) \circ o(\mathbf{m}, \mathbf{k}'). \quad (88)$$

In the case of a type X face it holds by definition. The case of type Y face is a consequence of Lemma 4.7. For the remaining faces it can be derived from (84). Relation (88) implies that o can be extended to a homology orientation for the cobordisms $\Sigma(S_{\mathbf{m}}^{\mathbf{n}})$ for all cubes (\mathbf{m}, \mathbf{n}) in \mathbb{Z}^n such that:

$$o(\mathbf{k}, \mathbf{n}) \circ o(\mathbf{m}, \mathbf{k}) = o(\mathbf{m}, \mathbf{n}).$$

Therefore, we can use this homology orientation to define the chain complex (73). With this choice of homology orientation the identity map defines an isomorphism of the chain complexes. In the case that type X faces are commutative, the same argument can be used with the aid of a type Y edge assignment. The claim about the relations between gradings is also an immediate consequence of the definition. \square

Corollary 4.10. *Given a link K , there is a spectral sequence that its first page is equal to $Kh_o(\bar{K}) \otimes \tilde{\Lambda}$ and it converges to $\widetilde{\text{PFH}}(\Sigma(K))$.*

Proof. In section 2, starting from a pair (Y, ζ) and a framed link L with n components, we constructed a pair $(Y_{\mathbf{m}}, \zeta_{\mathbf{m}})$ for each $\mathbf{m} \in \mathbb{Z}^n$. The 1-manifold $\zeta_{\mathbf{m}}$ is the union $\cup_i \zeta_{\mathbf{m}, i}$ and $\zeta_{\mathbf{m}, i}$ is non-empty if and only if $\mathbf{m}_i \equiv 0 \pmod{3}$, and in this case $\zeta_{\mathbf{m}, i}$ is the core of the tours $D_{\mathbf{m}, i}$. In the definition of $\zeta_{\mathbf{m}}$, we could replace the condition $\mathbf{m}_i \equiv 0 \pmod{3}$ with $\mathbf{m}_i \equiv 2 \pmod{3}$. Then the definition of $Z_{\mathbf{m}}^{\mathbf{n}}$ can be modified accordingly, and another version of Theorem 2.9 can be proved. This version can be used to prove a variation of Theorem 3.2. In particular, this variation states that a planar diagram D determines a chain complex $(\oplus_{\mathbf{m} \in \{0, 1\}^n} \widetilde{\text{PFC}}(\Sigma(K_{\mathbf{m}})), d_p)$ which has the same chain homotopy type as $\widetilde{\text{PFC}}(\Sigma(K), \zeta)$. The differential d_p is given by the maps $d_{\mathbf{m}}^{\mathbf{n}} : \widetilde{\text{PFC}}(\Sigma(K_{\mathbf{m}})) \rightarrow \widetilde{\text{PFC}}(\Sigma(K_{\mathbf{n}}))$ where $d_{\mathbf{m}}^{\mathbf{n}}$ is the cobordism map for the family of metrics $\mathfrak{W}_{\mathbf{m}}^{\mathbf{n}}$ parametrized by $G_{\mathbf{m}}^{\mathbf{n}}$ (constructed in subsection 2.1). Furthermore, the 1-manifold ζ is the branched double cover of $\gamma = \cup_i p_i$ where p_i is the path that connects the two strands in the i^{th} crossing of D (Fig. 3a).

Let $[\zeta]$ denote the $\mathbb{Z}/2\mathbb{Z}$ -homology class of the 1-manifold ζ . The $\mathbb{Z}/2\mathbb{Z}$ action on $\Sigma(K)$ induces a $\mathbb{Z}/2\mathbb{Z}$ action τ on the set of simplicial complexes $C_*^{\text{simp}}(\Sigma(K))$. Therefore, we have a map:

$$i : H_*((1 + \tau)C_*^{\text{simp}}(\Sigma(K)); \mathbb{Z}/2\mathbb{Z}) \rightarrow H_*(\Sigma(K); \mathbb{Z}/2\mathbb{Z})$$

and $[\zeta]$ lies in the image of i . The homology group $H_*((1 + \tau)C_*^{\text{simp}}(\Sigma(K)); \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $H_*(S^3, K)$, and under this isomorphism $[\zeta]$ corresponds to $[\gamma] \in H_1(S^3, K)$. The boundary of $[\gamma]$ represents the trivial class in $H_0(K)$, because the set of crossings is induced by a diagram of the link K . Therefore, $[\zeta]$ vanishes and by Lemma 1.24, $\widetilde{\text{PFH}}(\Sigma(K), \zeta) = \widetilde{\text{PFH}}(\Sigma(K))$. Consequently, the chain complex $(\oplus_{\mathbf{m} \in \{0, 1\}^n} \widetilde{\text{PFC}}(\Sigma(K_{\mathbf{m}})), d_p)$ gives rise to a spectral sequence whose first page is the complex (73) and it abuts to $\widetilde{\text{PFH}}(\Sigma(K))$. \square

References

- [1] M. F. Atiyah. Algebraic topology and operators in Hilbert space. In *Lectures in Modern Analysis and Applications. I*, pages 101–121. Springer, Berlin, 1969.
- [2] Michael Atiyah, Vijay Kumar Patodi, and Isadore Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.
- [3] Jonathan Bloom. A link surgery spectral sequence in monopole Floer homology. *Adv. Math.*, 226(4):3216–3281, 2011.
- [4] S. K. Donaldson. Polynomial invariants for smooth four-manifolds. *Topology*, 29(3):257–315, 1990.
- [5] S. K. Donaldson. *Floer homology groups in Yang-Mills theory*, volume 147 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick.
- [6] Simon Donaldson and Peter Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990. Oxford Science Publications.
- [7] Andreas Floer. An instanton-invariant for 3-manifolds. *Comm. Math. Phys.*, 118(2):215–240, 1988.
- [8] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.
- [9] Peter Kronheimer and Tomasz Mrowka. *Monopoles and three-manifolds*, volume 10 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2007.
- [10] Peter Kronheimer and Tomasz Mrowka. Filtrations on instanton homology. 2011, arXiv:1110.1290.
- [11] Peter Kronheimer and Tomasz Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.
- [12] Peter Kronheimer, Tomasz Mrowka, Peter Ozsváth, and Zoltán Szabó. Monopoles and lens space surgeries. *Ann. of Math. (2)*, 165(2):457–546, 2007.
- [13] Robert Lipshitz and Sucharit Sarkar. A Khovanov stable homotopy type. *J. Amer. Math. Soc.*, 27(4):983–1042, 2014.
- [14] Saunders Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.
- [15] Peter Ozsváth, Jacob Rasmussen, and Zoltán Szabó. Odd Khovanov homology. *Algebr. Geom. Topol.*, 13(3):1465–1488, 2013.

- [16] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.
- [17] C. W. Scaduto. Instantons and odd Khovanov homology. *ArXiv e-prints*, January 2014, 1401.2093.
- [18] Katrin Wehrheim. Smooth structures on morse trajectory spaces, featuring finite ends and associative gluing. 2012, arXiv:1205.0713.