

Lagrangians, $SO(3)$ -instantons and the Atiyah-Floer Conjecture

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Abstract

A version of the Atiyah-Floer conjecture, adapted to admissible $SO(3)$ -bundles, is established.

Contents

1	Introduction	3
2	Floer homology groups	6
2.1	3-manifolds and $SO(3)$ -bundles	6
2.2	The moduli space $\mathcal{M}(\Sigma, F)$	7
2.3	3-manifold Lagrangians	8
2.4	Floer homology groups	12
3	Proof of the Main Theorem	23
3.1	Special quintuples	23
3.2	Moduli spaces associated to special quintuples and the isomorphism \mathbf{N}	26
4	Linear analysis	33
4.1	The configuration space of mixed pairs	33
4.2	Fredholm property of the mixed operator	38
4.2.1	Mixed cylinders and mixed operators	39
4.2.2	Proof of Theorem 4.12	44
4.3	Proof of Proposition 3.19	47

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4.4	Mixed shifting	52
4.5	Proof of Proposition 3.20	56
4.6	Proof of Propositions 2.27 and 2.54	57
4.7	Orientability of the mixed determinant lines	59
5	Non-linear analysis	60
5.1	Review of the results of [DFL] on regularity and compactness	60
5.2	Mixed Chern-Simons functional	63
5.3	Exponential decay	68
5.4	Compactness	71
6	Perturbations	74
6.1	Cylinder functions and holonomy perturbations	74
6.2	Secondary perturbations	80
7	Extensions of the main theorem	82
7.1	Filtered framed Floer homology groups	83
7.2	Quantum cohomology and the μ -operator	87

1 Introduction

Gauge theoretic methods in low dimensional topology and holomorphic curve methods in symplectic geometry are responsible for many revolutionary advancements in respective fields. These two approaches have many formal similarities, despite the fact that low dimensional topology and symplectic geometry have different origins. The Atiyah–Floer conjecture is a manifestation of these similarities. According to this conjecture, instanton Floer homology, a 3-manifold invariant constructed in the context of Yang–Mills gauge theory can be recovered using holomorphic curve methods. The main goal of the present paper is to prove a version of the Atiyah–Floer conjecture.

Main Results

Suppose $Y_\#$ is an orientable connected closed 3-manifold and $E_\#$ is an $\mathrm{SO}(3)$ -bundle on $Y_\#$. The pair $(Y_\#, E_\#)$ is *admissible* if the Stiefel–Whitney class $w_2(E_\#)$ lifts to a non-torsion element of $H^2(Y_\#; \mathbf{Z})$. This condition is equivalent to the existence of an element $R \in H_2(Y_\#; \mathbf{Z})$ such that the pairing of $w_2(E_\#)$ and R is non-trivial. Any such R is called a *nice* homology class for the pair $(Y_\#, E_\#)$. Associated to any such admissible pair $(Y_\#, E_\#)$, we have the *instanton Floer homology* of $(Y_\#, E_\#)$, which is a relatively $\mathbf{Z}/8\mathbf{Z}$ -graded group [Flo95, BD95]. A nice homology class R induces a degree 4 involution on the instanton Floer homology of $(Y_\#, E_\#)$, and the invariant subspace with respect to this involution determines a relatively $\mathbf{Z}/4\mathbf{Z}$ -graded group $I_*(Y_\#, E_\#)$.

Any admissible pair with a choice of a nice homology class admits an *admissible splitting*. An admissible splitting

$$(Y, E) \cup_{(\Sigma, F)} (Y', E') \tag{1.1}$$

consists of connected 3-manifolds Y, Y' with boundary Σ and $\mathrm{SO}(3)$ -bundles E, E' on Y, Y' whose restrictions to Σ are identified with an $\mathrm{SO}(3)$ -bundle F . The restriction of $w_2(F)$ to each connected component of Σ is required to be non-trivial. This assumption implies that Σ has an even number of connected components and as an additional assumption we require that Σ has exactly two connected components Σ_0 and Σ_1 . We also assume that Y and Y' are oriented such that the induced orientations on their boundaries are the orientation on Σ . Thus, after reversing the orientation of Y' , we may glue Y and Y' to form an oriented closed 3-manifold $Y_\#$. The bundles E and E' can be also glued to form an $\mathrm{SO}(3)$ -bundle $E_\#$ on $Y_\#$. Connected components of Σ determine homologous homology classes in $Y_\#$ which are denoted by R . Since $w_2(E_\#)$ has a non-trivial pairing with R , the pair $(Y_\#, E_\#)$ is admissible and R is a nice homology class for this pair. We say (1.1) is an admissible splitting of $(Y_\#, E_\#)$ compatible with R .

Suppose $(Y_\#, E_\#)$ is a pair with an admissible splitting as in (1.1), and $\mathcal{M}(\Sigma, F)$ denotes the moduli space of flat connections on F modulo determinant one automorphisms of F [AB83]. The manifold $\mathcal{M}(\Sigma, F)$ admits a canonical symplectic structure Ω . (See Subsection 2.2 for our conventions.) Flat connections on E after a small perturbation gives rise to an immersed Lagrangian submanifold $L(Y, E)$, which is called the *3-manifold Lagrangian* associated to (Y, E) [Her94]. (See Subsection 2.3 for more details.) Similarly, we can associate a 3-manifold Lagrangian $L(Y', E')$ to (Y', E') . We say that the 3-manifold Lagrangians $L(Y, E)$ and $L(Y', E')$ are embedded if there are arbitrarily small perturbations

such that the associated Lagrangians are embedded. A more precise version of this assumption is stated as Hypothesis 1. For instance, if the fundamental group of one of the connected components of Σ surjects into the fundamental group of Y (resp. Y'), then $L(Y, E)$ (resp. $L(Y', E')$) is embedded. (See Proposition 2.21.)

Theorem 1. *Suppose $L(Y, E)$ and $L(Y', E')$ are embedded submanifolds of $\mathcal{M}(\Sigma, F)$. Then the pair of Lagrangians $(L(Y, E), L(Y', E'))$ is monotone with minimal Maslov number 4, and configuration space of strips associated to these Lagrangians can be coherently oriented.*

Let (L, L') be a pair of embedded Lagrangians with minimal Maslov number N . Building on Floer's work [Flo88b], Oh defines a $\mathbf{Z}/N\mathbf{Z}$ -graded Lagrangian Floer homology group $\mathrm{HF}(L, L')$ in [Oh93], which is a vector space over $\mathbf{Z}/2$. Coherent orientations for configuration spaces of strips allow us to work with integer coefficients (see Definition 2.37). Let $L(Y, E)$ and $L(Y', E')$ be embedded. Then Oh's Lagrangian Floer homology of these two Lagrangians is a $\mathbf{Z}/4\mathbf{Z}$ -graded abelian group which is called *symplectic instanton Floer homology* of $(Y_{\#}, E_{\#})$ and is denoted by $\mathrm{SI}_*(Y_{\#}, E_{\#})$. The following theorem is our main result.

Theorem 2. *Suppose an admissible splitting for a pair $(Y_{\#}, E_{\#})$ is given such that the associated 3-manifold Lagrangians are embedded. Then there is an isomorphism of relatively $\mathbf{Z}/4\mathbf{Z}$ -graded abelian groups $\mathbf{N} : \mathrm{I}_*(Y_{\#}, E_{\#}) \rightarrow \mathrm{SI}_*(Y_{\#}, E_{\#})$.*

The proof of Theorem 2 modulo some analytical results is given in Section 3. Sections 4, 5 and 6 of the paper are devoted to verifying the analytical results which are used in Section 3.

Instanton Floer homology group $\mathrm{I}_*(Y_{\#}, E_{\#})$ is an invariant of the topological type of $(Y_{\#}, E_{\#})$. However, symplectic instanton Floer homology $\mathrm{SI}_*(Y_{\#}, E_{\#})$, a priori, depends on the choice of an admissible splitting. As a consequence of Theorem 2, we have the following corollary.

Corollary 1. *Symplectic instanton Floer homology group $\mathrm{SI}_*(Y_{\#}, E_{\#})$ depends only on the topological type of $(Y_{\#}, E_{\#})$. In particular, it is independent of the admissible splitting of $(Y_{\#}, E_{\#})$.*

Corollary 2. *Let $\phi : \Sigma_g \rightarrow \Sigma_g$ be a diffeomorphism and Y_{ϕ} be the mapping cylinder $[0, 1] \times Y / \{(x, 1) \sim (\phi(x), 0)\}$. Let also E_{ϕ} be the $\mathrm{SO}(3)$ -bundle on Y_{ϕ} which is induced by the non-trivial $\mathrm{SO}(3)$ -bundle on Σ_g . The map ϕ induces a symplectomorphism $\phi_* : \mathcal{M}(\Sigma_g, F_g) \rightarrow \mathcal{M}(\Sigma_g, F_g)$. Then instanton Floer homology group $\mathrm{I}_*(Y_{\phi}, E_{\phi})$ is isomorphic to the Lagrangian Floer homology of the diagonal and the graph Γ_{ϕ_*} of ϕ_* , which are Lagrangians in $(\mathcal{M}(\Sigma_g, F_g) \times \mathcal{M}(\Sigma_g, F_g), -\Omega \times \Omega)$.*

This corollary of Theorem 2 is essentially the same as Dostoglou and Salamon's celebrated result in [DS94b]. It is shown in [DS94b] that $\mathrm{I}_*(Y_{\phi}, E_{\phi})$ is isomorphic to the fixed point Floer homology of ϕ_* . It is a folklore theorem that fixed Floer homology of a symplectomorphism is isomorphic to the Lagrangian Floer homology of the diagonal and the graph of the symplectomorphism.

Proof. The pair (Y_{ϕ}, E_{ϕ}) has an obvious admissible splitting as the union of (Y, E) and (Y', E') where Y, Y' are diffeomorphic to $[0, 1] \times \Sigma_g$ and E, E' are pull-backs of the non-trivial $\mathrm{SO}(3)$ -bundle F_g on Σ_g . The Lagrangian submanifolds associated to Y, Y' are the diagonal Δ and Γ_{ϕ_*} . \square

In [KM11], Kronheimer and Mrowka use instanton Floer homology to define an invariant of 3-manifolds which is called *framed Floer homology*. Let $T^3 = S^1 \times T^2$ be the 3-dimensional torus and E_1 be the $\mathrm{SO}(3)$ -bundle on T^3 which is the pullback of the non-trivial bundle F_1 on T^2 . The trivial $\mathrm{SO}(3)$ -bundle on a 3-manifold M and E_1 induces a bundle $E_\#$ on $Y_\# = M \# T^3$. The pair $(Y_\#, E_\#)$ is admissible and the factor T^2 of T^3 determines a nice homology class for this pair. The framed Floer homology $I_*^\sharp(M)$ of M is defined to be the associated instanton Floer homology. Given a Heegaard splitting $H \cup_{\Sigma_g} H'$ of Y , we can obtain an admissible splitting of $(Y_\#, E_\#)$. Let Y (resp. Y') be the boundary sum of H (resp. H') and $[0, 1] \times T^2$. Then Y and Y' are 3-manifolds whose boundary components are $\Sigma = \Sigma_{g+1} \sqcup T^2$. The non-trivial $\mathrm{SO}(3)$ -bundle on $[0, 1] \times T^2$ induces $\mathrm{SO}(3)$ -bundles E and E' on Y and Y' . The restriction F of E (or equivalently E') to Σ is given by the non-trivial $\mathrm{SO}(3)$ -bundle F_{g+1} on Σ_{g+1} and the bundle F_1 on T^2 . In particular, $\mathcal{M}(\Sigma, F) = \mathcal{M}(\Sigma_{g+1}, F_{g+1})$. The subspaces of elements of $\mathcal{M}(\Sigma_{g+1}, F_{g+1})$ which extend as flat connections to E and E' determine embedded Lagrangian submanifolds L and L' of $\mathcal{M}(\Sigma_{g+1}, F_{g+1})$. In [WW16, Definition 4.4.1], Wehrheim and Woodward, define a 3-manifold invariant as the Lagrangian Floer homology of L and L' . We call this invariant *symplectic framed Floer homology* of M , and denote it by $\mathrm{SI}_*^\sharp(M)$.

Theorem 3. *The 3-manifold invariants $I_*^\sharp(M)$ and $\mathrm{SI}_*^\sharp(M)$ together with their Chern-Simons filtrations, are isomorphic to each other.*

The Chern-Simons filtrations on $I_*^\sharp(M)$ and $\mathrm{SI}_*^\sharp(M)$ are defined in Section 7.1, where the proof of Theorem 3 is given. Forgetting this additional structure, Theorem 3 is a special case of Theorem 2.

For a pair of monotone Lagrangians (L, L') in a symplectic manifold (M, ω) , $\mathrm{HF}(L, L')$ is a module over the quantum cohomology ring $QH^*(M)$. See, for example, [Flo89, Sei01, Alb08, FOOO09a, BC09] for this structure on Lagrangian Floer homology and related constructions in symplectic geometry. For our purposes in the present paper, this structure determines an action of $QH^*(\mathcal{M}(\Sigma, F))$ on $\mathrm{SI}_*(Y_\#, E_\#)$. To simplify the discussion we work with the coefficients in \mathbf{Q} for the rest of the introduction. An explicit set of generators for the ring $QH^*(\mathcal{M}(\Sigma, F))$ (or equivalently $H^*(\mathcal{M}(\Sigma, F))$) is given in [AB83]. There is a *universal* $\mathrm{SO}(3)$ -bundle \mathbb{F} over the product $\mathcal{M}(\Sigma, F) \times \Sigma$, and the slant products

$$\frac{1}{4}p_1(\mathbb{F}) \setminus \sigma, \quad \sigma \in H_*(\Sigma) \tag{1.2}$$

defines an element of $H^*(\mathcal{M}(\Sigma, F))$. These cohomology classes as σ varies over a generating set for $H_*(\Sigma)$ determine multiplicative generating for the cohomology ring of $\mathcal{M}(\Sigma, F)$. We write $m_\sigma^S : \mathrm{SI}_*(Y_\#, E_\#) \rightarrow \mathrm{SI}_*(Y_\#, E_\#)$ for the induced action of (1.2), as an element of $QH^*(\mathcal{M}(\Sigma, F))$, on $\mathrm{SI}_*(Y_\#, E_\#)$.

On the gauge theoretical side, there is an action of $H_*(Y_\#)$ on $I_*(Y_\#, E_\#)$ for any admissible pair $(Y_\#, E_\#)$. For $\sigma \in H_*(Y_\#)$, we denote the corresponding action by $m_\sigma^G : I_*(Y_\#, E_\#) \rightarrow I_*(Y_\#, E_\#)$. This action plays a crucial role in certain topological applications of instanton Floer homology of admissible pairs (see [KM10] for some instances of such topological applications), and it is related to μ -classes and polynomial invariants in Donaldson theory of smooth closed 4-manifolds.

Theorem 4. *Suppose $\sigma \in H_1(\Sigma) \oplus H_2(\Sigma)$, and the endomorphism m_σ^G of $I_*(Y_\#, E_\#)$ is defined using the inclusion of Σ in $Y_\#$. Then the isomorphism \mathbf{N} of Theorem 2 is compatible with m_σ^G and the homomorphism m_σ^S on the symplectic side. That is to say, $\mathbf{N} \circ m_\sigma^G = m_\sigma^S \circ \mathbf{N}$.*

This theorem is meant to feature an instance of a more general result. In particular, we believe that this theorem should generalize to the case that one uses arbitrary homology classes in $H_*(\Sigma)$ and the homology classes are defined with arbitrary coefficient ring.

The original version of the Atiyah-Floer conjecture was stated in [Ati88]. This version of the Atiyah-Floer conjecture concerns the invariants of 3-manifolds with the same integral homology as S^3 [Flo88a]. One can generalize this conjecture so that it has the conjecture of [Ati88] and Theorem 3 as two special cases. A strategy to approach the Atiyah-Floer conjecture for admissible bundles, similar to the method of this paper, was proposed in [Fuk]. The key geometrical tool to prove the results of this paper is the mixed equation [Lip14, DFL] whereas the proposal of [Fuk] is based on a version of the ASD equation defined using degenerate metrics. Another major approach to the Atiyah-Floer conjecture makes use of adiabatic limits. The adiabatic limits method was already used in [DS94b] and it forms a crucial part of the programs of [Weh05, Dun15]. Other attempts to the Atiyah-Floer conjecture can be found in [Yos, LL95].

Notations. In this paper we shall be concerned with connections on manifolds of dimensions 2, 3 and 4. To avoid confusion, we denote a typical connection on a 4-manifold by A (possibly with an index), on a 3-manifold by B (possibly with an index) and on a surface by a greek letter.

The Euclidean space \mathbf{R}^3 with the standard cross product defines a Lie algebra with an action of $\mathrm{SO}(3)$. This Lie algebra with the action of $\mathrm{SO}(3)$ is isomorphic to $\mathfrak{so}(3)$, linear space of skew-adjoint endomorphisms of \mathbf{R}^3 , and $\mathfrak{su}(2)$, the linear space of trace free skew-Hermitian endomorphisms of \mathbf{C}^2 . The Lie algebra structure and the action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are respectively given by the commutator map and the adjoint action. Throughout this paper, we use this isomorphism to identify an $\mathrm{SO}(3)$ vector bundle V with the bundle $\mathfrak{so}(V)$ of skew-adjoint endomorphisms of V . We also define a bi-linear form $\mathrm{tr} : \mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}$ given by $-\frac{1}{2}$ of the inner product. Using the identification with $\mathfrak{su}(2)$, this bi-linear form can be identified with $\mathrm{tr} : \mathfrak{su}(2) \times \mathfrak{su}(2) \rightarrow \mathbf{R}$ which maps a pair of a skew-Hermitian matrices A and B to $\mathrm{tr}(AB)$. The bi-linear form tr induces a bi-linear form on sections of any $\mathrm{SO}(3)$ -vector bundle, which is denoted by the same notation.

2 Floer homology groups

In this section, we recall the definitions of Floer homology groups $I_*(Y_\#, E_\#)$ and $\mathrm{SI}_*(Y_\#, E_\#)$ for an admissible pair $(Y_\#, E_\#)$. The definition of the latter Floer homology group requires some preparation. First we recall the definition of the symplectic manifold $\mathcal{M}(\Sigma, F)$. Then we define the 3-manifold Lagrangian $L(Y, E)$, which is an immersed Lagrangian of $\mathcal{M}(\Sigma, F)$. In the case that F is replaced with the trivial bundle, analogues of the 3-manifold Lagrangian $L(Y, E)$ are the main subject of study in [Her94]. Our case of interest is less complicated because there is no singular point in $\mathcal{M}(\Sigma, F)$.

2.1 3-manifolds and $\mathrm{SO}(3)$ -bundles

Suppose Σ is a Riemann surface with two connected components. We assume that Y, Y' are oriented connected 3-manifolds such that an identification of collar neighborhoods of their boundary components

with $[-1, 2) \times \Sigma$, $(-2, 1] \times \Sigma$ are fixed, which are respectively orientation preserving and orientation reversing. Throughout this paper, we use outward-normal-first convention to orient the boundary of an oriented 3-manifold and the first-factor-first convention to orient the product of two oriented manifolds. We reverse the orientation of Y' and glue it to Y using the rule

$$(t, x) \in [-1, 1] \times \Sigma \subset Y \sim (t, x) \in [-1, 1] \times \Sigma \subset Y'$$

to form a closed oriented 3-manifold $Y_{\#}$. It will be useful to fix a notation for the following subspaces of Y, Y'

$$Y_0 := Y \setminus [-1, 1] \times \Sigma, \quad Y'_0 := Y' \setminus (-1, 1] \times \Sigma \quad (2.1)$$

which are clearly diffeomorphic to Y, Y' . Let g, g' be Riemannian metrics on Y, Y' that restrict to the product metric on the collar neighborhoods of the boundaries of Y and Y' corresponding to a fixed metric on Σ . Gluing these metrics produces a metric $g_{\#}$ on $Y_{\#}$, and in the following we use g, g' or $g_{\#}$ when we need a metric on Y, Y' or $Y_{\#}$. We will impose further constraints on the metrics g and g' in Section 6.

Suppose F is an $\text{SO}(3)$ -bundle on Σ with non-trivial restrictions to the connected components of Σ . Suppose $\text{SO}(3)$ -bundles E, E' on Y, Y' are also given such that their restrictions to $[-1, 2) \times \Sigma \subset Y, (-2, 1] \times \Sigma \subset Y'$ are identified with $[-1, 2) \times F, (-2, 1] \times F$. Then we can glue these two $\text{SO}(3)$ -bundles to form the bundle $E_{\#}$ on $Y_{\#}$. In particular, $(Y_{\#}, E_{\#})$ is an admissible pair with an admissible splitting as in (1.1) determined by (Y, E) and (Y', E') . This admissible pair with the given splitting shall be fixed for the rest of the paper.

2.2 The moduli space $\mathcal{M}(\Sigma, F)$

Throughout this section, we need to consider the space of connections on $\text{SO}(3)$ -bundles over manifolds of dimensions 2, 3 and 4. As the first instance, let $\mathcal{A}(\Sigma, F)$ be the space of connections on the bundle F . For analytical purposes, it is convenient to allow for Sobolev connections. To that end, we fix an integer $l \geq 2$, and assume that $\mathcal{A}(\Sigma, F)$ is defined using L^2_{l-1} connections.¹ Let $F \times_{\text{ad}} \text{SU}(2)$ be the fiber bundle on Σ associated to the framed bundle of F via the adjoint action of $\text{SO}(3)$ on $\text{SU}(2)$. Any section of this fiber bundle is called a *determinant one gauge transformation* of F . We use this standard terminology for $\text{SO}(3)$ bundles over manifolds of any dimension. The space of L^2_l sections of $F \times_{\text{ad}} \text{SU}(2)$ forms the Banach Lie group $\mathcal{G}(F)$. Taking pullbacks with respect to elements of $\mathcal{G}(F)$ determines an action of this group on $\mathcal{A}(\Sigma, F)$, and the quotient space, the *configuration space* of connections on Σ , is denoted by $\mathcal{B}(\Sigma, F)$. Embedded in this infinite dimensional Banach manifold, there is the moduli space $\mathcal{M}(\Sigma, F)$, which consists of the elements of $\mathcal{B}(\Sigma, F)$ that are represented by flat connections. The dimension of $\mathcal{M}(\Sigma, F)$ is equal to $-3\chi(\Sigma)$.

Remark 2.2. An alternative gauge group $\mathcal{G}_{\text{ex}}(F)$ can be defined by considering the sections of the fiber bundle $F \times_{\text{ad}} \text{SO}(3)$ induced by the adjoint action of $\text{SO}(3)$ on itself. There is an obvious map from $\mathcal{G}(F)$ to $\mathcal{G}_{\text{ex}}(F)$ induced by the quotient map $\text{SU}(2)$ to $\text{SO}(3)$. This map fits into an exact sequence:

$$\mathbf{Z}/2 \oplus \mathbf{Z}/2 \hookrightarrow \mathcal{G}(F) \rightarrow \mathcal{G}_{\text{ex}}(F) \twoheadrightarrow H^1(\Sigma; \mathbf{Z}/2).$$

¹The same integer l is used throughout the paper for Sobolev spaces associated to manifolds of various dimensions. The exact choice of the Sobolev parameter l does not play any role in this paper.

The first map is the inclusion of the elements of $\mathcal{G}(F)$ which are locally equal to ± 1 , and the last map for $g \in \mathcal{G}_{\text{ex}}(F)$ is given as the obstruction of lifting g to $\mathcal{G}(F)$ over the 1-skeleton of Σ .

For a flat connection σ on F , the tangent space of the smooth manifold $\mathcal{M}(\Sigma, F)$ at $[\sigma]$ is given by

$$\mathcal{H}^1(\Sigma; \sigma) = \{c \in \Omega^1(\Sigma, \Lambda^1 \otimes F) \mid d_\sigma c = 0, d_\sigma^* c = 0\}. \quad (2.3)$$

We consider the L^2 inner product

$$\langle c, c' \rangle := - \int_{\Sigma} \text{tr}(c \wedge *_2 c') \quad (2.4)$$

on (2.3), where tr is defined by applying the inner product of F to the vector factor of c and c' . In this paper, we use a similar convention to define inner products of differential forms of any degree with values in an $\text{SO}(3)$ -bundle over a Riemannian manifold. We similarly define a symplectic form Ω on $\mathcal{H}^1(\Sigma; \sigma)$:

$$\Omega(c, c') := - \int_{\Sigma} \text{tr}(c \wedge c'). \quad (2.5)$$

The complex structure $J_* : \mathcal{H}^1(\Sigma; \sigma) \rightarrow \mathcal{H}^1(\Sigma; \sigma)$, defined as $J_*(c) = *_2 c$, can be used to relate the metric and the symplectic form:

$$\langle c, c' \rangle = \Omega(c, J_* c').$$

The basic topological invariants of $\mathcal{M}(\Sigma, F)$ are well understood. This manifold is simply connected and $\pi_2(\mathcal{M}(\Sigma, F)) = \pi_2(\mathcal{M}(\Sigma_0, F_0)) \oplus \pi_2(\mathcal{M}(\Sigma_1, F_1))$ where $\pi_2(\mathcal{M}(\Sigma_i, F_i)) = 0$ if the genus of the connected component Σ_i of Σ is 1, and $\pi_2(\mathcal{M}(\Sigma_i, F_i)) = \mathbf{Z}$ otherwise [AB83, DS94a]. In fact, for a surface Σ_g of genus $g \geq 2$, a generator of $\pi_2(\mathcal{M}(\Sigma_g, F_g))$ can be constructed in the following way. There is a connection A on the pullback of F_g to $D^2 \times \Sigma_g$ such that for any point $z \in D^2$ the restriction $A|_{\{z\} \times \Sigma_g}$ is flat, the restriction of A to the boundary $S^1 \times \Sigma_g$ is flat and

$$\frac{1}{8\pi^2} \int_{D^2 \times \Sigma_g} \text{tr}(F_A \wedge F_A) = \frac{1}{2}.$$

In particular, for any $z \in S^1$, the flat connection $A|_{\{z\} \times \Sigma_g}$ represents a fixed element α of $\mathcal{M}(\Sigma_g, F_g)$. Therefore, A induces a map $s : (D^2, S^1) \rightarrow (\mathcal{M}(\Sigma_g, F_g), \alpha)$ determining a generator of $\pi_2(\mathcal{M}(\Sigma_g, F_g))$. Since the connections $A|_{\{z\} \times \Sigma_g}$ for $z \in S^1$ are gauge equivalent to a fixed (irreducible) flat connection, we also obtain a loop in $\mathcal{G}(F_g)/\{\pm 1\}$ from the restriction of A to $S^1 \times \Sigma_g$ which gives a generator of $\pi_1(\mathcal{G}(F_g)/\{\pm 1\})$.

2.3 3-manifold Lagrangians

Suppose (Y, E) is as in Subsection 2.1. Fix a smooth connection B_0 on E , and define

$$\mathcal{A}(Y, E) := \{B_0 + b \mid b \in L^2_l(Y, \Lambda^1 \otimes E)\}$$

The group $\mathcal{G}(E)$ of determinant one gauge transformations of E with finite L^2_{l+1} norm acts smoothly on $\mathcal{A}(Y, E)$ by taking pull back. We denote the quotient space by $\mathcal{B}(Y, E)$. Let $B \in \mathcal{A}(Y, E)$ be an

irreducible connection, that is to say, the stabilizer of B with respect to the action of $\mathcal{G}(E)$ consists of only ± 1 . Then $\mathcal{B}(Y, E)$ is a Banach smooth manifold at the class of B . The tangent space of $\mathcal{B}(Y, E)$ at this point can be identified with the Banach space

$$X_B := \{b \in L_l^2(Y, \Lambda^1 \otimes E) \mid d_B^* b = 0, *_3 b|_\Sigma = 0\}. \quad (2.6)$$

In particular, for any b in $L_l^2(Y, \Lambda^1 \otimes E)$, the tangent space of $\mathcal{A}(Y, E)$ at B , there is $\zeta \in L_{l+1}^2(Y, E)$ such that $b - d_B \zeta$ belongs to X_B . There is a variation of this fact which shall be useful. Let $b \in L_l^2(Y, \Lambda^1 \otimes E)$ such that $b|_\Sigma = 0$. Then there is $\zeta \in L_{l+1}^2(Y, E)$ with $\zeta|_\Sigma = 0$ such that $d_B^*(b - d_B \zeta) = 0$.

Define a $\mathcal{G}(E)$ -equivariant map $\phi : \mathcal{A}(Y, E) \rightarrow L_{l-1}^2(Y, \Lambda^1 \otimes E)$ by

$$\phi(B) = *_3 F_B. \quad (2.7)$$

In general the space $L(Y, E) := \phi^{-1}(0)/\mathcal{G}(E)$ might not be smooth because ϕ might have zeros which are not cut down transversely. To achieve transversality, we perturb ϕ following a standard scheme used in various places including [Don87, Flo88a, Tau90, Her94, KM11].

In Section 6.1, we review the definition of a family of functions defined on $\mathcal{A}(Y, E)$, which are known as *cylinder functions*. Given a cylinder function $h : \mathcal{A}(Y, E) \rightarrow \mathbf{R}$, we may define its formal gradient $\nabla h : \mathcal{A}(Y, E) \rightarrow L_l^2(Y, \Lambda^1 \otimes E)$ with respect to the L^2 metric on $\mathcal{A}(Y, E)$. This determines a gauge invariant perturbation of (2.7):

$$\phi_h(B) = *_3 F_B + \nabla_B h. \quad (2.8)$$

The function h depends only on the restriction of the connection B to the interior of Y_0 . In particular, $\nabla_B h$ vanishes on a neighborhood of the boundary of Y . Moreover, invariance of h with respect to the action of $\mathcal{G}(E)$ implies that $d_B^* \nabla_B h$ vanishes. This together with Bianchi identity implies that $\phi_h(B)$ belongs to the kernel of d_B^* . We write $L_h(Y, E)$ for the quotient space $\phi_h^{-1}(0)/\mathcal{G}(E)$. Any element of $L_h(Y, E)$ restricts to a flat connection on the boundary Riemann surface Σ . In particular, this defines a map $r : L_h(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$.

The linearization of ϕ_h at any connection B modulo the action of the gauge group defines a map from X_B to $L_{l-1}^2(Y, \Lambda^1 \otimes E)$ as follows:

$$b \rightarrow *_3 d_B b + \text{Hess}_B h(b). \quad (2.9)$$

This map has an infinite dimensional co-kernel and is not Fredholm. To resolve this issue, let Π_B be the projection to the kernel of d_B^* acting on $L_{l-1}^2(Y, \Lambda^1 \otimes E)$. Since $d_{B'}^* \phi_h(B') = 0$ for any $B' \in \mathcal{A}(Y, E)$, the zeros of ϕ_h and $\Pi_B \circ \phi_h$, in a neighborhood of B , agree with each other. Therefore, we may consider linearization of the operator $\Pi_B \circ \phi_h$ to study the deformation theory of the space $L_h(Y, E)$.

Proposition 2.10 ([Her94]). *Suppose B represents an element of $L_h(Y, E)$. Then*

$$d_B^*(*_3 d_B + \text{Hess}_B h) = 0.$$

In particular, the linearization of $\Pi_B \circ \phi_h$, denoted by $L_B : X_B \rightarrow \ker(d_B^)$, is given by (2.9). The operator L_B is Fredholm with index $-\frac{3}{2}\chi(\Sigma)$. The kernel of this operator can be identified with*

$$\mathcal{H}_h^1(Y; B) := \{b \in L_l^2(Y, \Lambda^1 \otimes E) \mid *_3 b|_\Sigma = 0, d_B^* b = 0, *_3 d_B(b) + \text{Hess}_B h(b) = 0\}, \quad (2.11)$$

and its cokernel is given by

$$\mathcal{H}_h^1(Y, \Sigma; B) := \{b \in L_t^2(Y, \Lambda^1 \otimes E) \mid b|_\Sigma = 0, d_B^* b = 0, *_3 d_B(b) + \text{Hess}_B h(b) = 0\}. \quad (2.12)$$

Proof. Let Ψ be a smooth 0-form on Y with values in E which is supported in the interior of Y , and $b \in L_t^2(Y, \Lambda^1 \otimes E)$. If B_t is the connection $B + tb$, then the inner product of $*_3 F_{B_t} + \nabla_{B_t} h$ and $d_{B_t} \Psi$ vanishes. Taking derivative with respect to t implies that:

$$\langle *_3 d_B b + \text{Hess}_B h(b), d_B \Psi \rangle_{L^2(Y)} + \langle *_3 F_B + \nabla_B h, [b, \Psi] \rangle_{L^2(Y)} = 0$$

Now the claim follows from the assumption that $*_3 F_B + \nabla_B h$ vanishes. The remaining claims can be treated as in [Her94]. \square

As another useful property of cylinder functions, we record the following lemma. It is a consequence of the symmetric property of Hessians.

Lemma 2.13. *Suppose $B \in \mathcal{A}(Y, E)$ and $b, b' \in L_t^2(Y, \Lambda^1 \otimes E)$. Then we have*

$$\int_Y \text{tr}(b \wedge *_3 \text{Hess}_B h(b')) = \int_Y \text{tr}(*_3 \text{Hess}_B h(b) \wedge b'). \quad (2.14)$$

To study regularity properties of $L_h(Y, E)$ for a general choice of h , it is useful to consider the family version of this construction. Let \mathcal{P} denote the *parameter space* of cylinder functions (to be defined in Section 6.1). Define a gauge invariant map $\Phi : \mathcal{A}(Y, E) \times \mathcal{P} \rightarrow L_{t-1}^2(Y, \Lambda^1 \otimes E)$ as follows:

$$\Phi(B, \rho) = *_3 F_B + \nabla_B h_\rho. \quad (2.15)$$

Then $\mathbf{L}(Y, E) = \Phi^{-1}(0)/\mathcal{G}(E)$ determines the family version of $L_h(Y, E)$. There is an obvious map $\pi : \mathbf{L}(Y, E) \rightarrow \mathcal{P}$ such that the fibers of this map are the spaces $L_h(Y, E)$. Any element of $\mathbf{L}(Y, E)$ restricts to a flat connection on the boundary Riemann surface Σ . In particular, we have a map $\mathbf{r} : \mathbf{L}(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$. Obviously the restriction of this map to each subspace $L_h(Y, E)$ is r . The proof of the following proposition will be given in Section 6.1.

Proposition 2.16 ([Her94]). *The map Φ is a submersion, and hence the space $\mathbf{L}(Y, E)$ is a smooth Banach manifold. The projection map $\pi : \mathbf{L}(Y, E) \rightarrow \mathcal{P}$ is a proper Fredholm map of index $-\frac{3}{2}\chi(\Sigma)$. Moreover, the restriction map $\mathbf{r} : \mathbf{L}(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ is a submersion.*

Proposition 2.16 and Sard-Smale theorem imply that the space $L_h(Y, E)$ is a smooth compact manifold of dimension $-\frac{3}{2}\chi(\Sigma)$ for a generic choice of h . Thus, Proposition 2.10 implies that for any such h and any $[B] \in L_h(Y, E)$, $\mathcal{H}_h^1(Y; B)$ has dimension $-\frac{3}{2}\chi(\Sigma)$ and $\mathcal{H}_h^1(Y, \Sigma; B)$ is trivial. For any such B and any $b \in \mathcal{H}_h^1(Y; B)$, the restriction $c := b|_\Sigma$ is d_σ closed where $\sigma = B|_\Sigma$. Thus, there is ξ such that $c - d_\sigma \xi$ belongs to $\mathcal{H}^1(\Sigma; \sigma)$, and the derivative of the restriction map $r : L_h(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ at B is given by $b \rightarrow c - d_\sigma \xi$. Thus, if b belongs to the kernel of this map, then there is $\zeta \in L_{t+1}^2(Y, E)$ such that $b - d_B \zeta$ restricts to the trivial 1-form on Σ . In particular, by applying the fact mentioned at the beginning of this subsection, we may assume that $b - d_B \zeta \in \mathcal{H}_h^1(Y, \Sigma; B)$, which implies that $b = d_B \zeta$ because of our assumption on h . Since $b \in X_b$, this implies that $b = 0$. Thus, the restriction map r is an immersion.

Proposition 2.17. *Suppose h is a regular value of the projection map $\pi : \mathbf{L}(Y, E) \rightarrow \mathcal{P}$. Then the immersion $r : L_h(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ is Lagrangian. Moreover, given a finite dimensional smooth manifold N and a smooth map $s : N \rightarrow \mathcal{M}(\Sigma, F)$, for a generic h , the map r is transversal to s .*

In the following, if h satisfies the assumption of the first part of this proposition, we say $L_h(Y, E)$ is *regular*.

Proof. Suppose B represents an element of $L_h(Y, E)$ and $b, b' \in \mathcal{H}_h^1(Y; B)$. Suppose also c and c' denote the restrictions of these elements to Σ :

$$\begin{aligned} \int_{\Sigma} \text{tr}(c \wedge c') &= \int_Y \text{tr}(d_B b \wedge b') - \text{tr}(b \wedge d_B b') \\ &= - \int_Y \text{tr}(*_3 \text{Hess}_B h(b) \wedge b') - \text{tr}(b \wedge *_3 \text{Hess}_B h(b')) \\ &= 0 \end{aligned}$$

where the last identity is a consequence of Lemma 2.13. To verify the second part, notice that $\mathbf{r} : \mathbf{L}(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ is transversal to $s : N \rightarrow \mathcal{M}(\Sigma, F)$ because the former map is a submersion. In particular, the following space is a smooth Banach manifold:

$$\mathbf{L}(Y, E)_{\mathbf{r}} \times_s N := \{([A], x) \mid [A] \in \mathbf{L}(Y, E), x \in N, \mathbf{r}([A]) = s(x)\}.$$

The map p induces a map from $\mathbf{L}(Y, E)_{\mathbf{r}} \times_s N$ to \mathcal{P} which is Fredholm (with index $-\frac{3}{2}\chi(\Sigma) + \dim(N) - \dim(\mathcal{M}(\Sigma, F))$). If h is a regular value of this map, then $r : L_h(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ is transversal to $s : N \rightarrow \mathcal{M}(\Sigma, F)$. Therefore, the second part is a consequence of the Sard-Smale theorem. \square

Proposition 2.18. *Suppose $h, h' \in \mathcal{P}$ are two cylinder functions such that $L_h(Y, E)$ and $L_{h'}(Y, E)$ are immersed Lagrangians of $\mathcal{M}(\Sigma, F)$. Then $L_h(Y, E)$ and $L_{h'}(Y, E)$ are Lagrangian cobordant. That is to say, there is a Lagrangian immersion $R : V \rightarrow \mathbf{R}^2 \times \mathcal{M}(\Sigma, F)$ such that there are subspaces $V_-, V_+ \subset V$ with the property that*

$$R^{-1}((-\infty, -1] \times \mathbf{R} \times \mathcal{M}(\Sigma, F)) = V_-, \quad R^{-1}([1, \infty) \times \mathbf{R} \times \mathcal{M}(\Sigma, F)) = V_+,$$

V_-, V_+ can be identified with $(-\infty, -1] \times L_h(Y, E)$, $[1, \infty) \times L_{h'}(Y, E)$, and the restrictions of R to V_{\pm} is given by $(\text{id}, 0, r)$. Here the symplectic structure on the product space $\mathbf{R}^2 \times \mathcal{M}(\Sigma, F)$ is induced by the standard symplectic structure $dx \wedge dy$ on \mathbf{R}^2 and the symplectic form Ω on $\mathcal{M}(\Sigma, F)$.

A closely related result is proved in [Her94]. In the case that the bundles F and E over Σ and Y are trivial, Herlad shows that $L_h(Y, E)$, which is a singular Lagrangian in the singular symplectic manifold $\mathcal{M}(\Sigma, F)$, has a Legendrian lift to a certain S^1 -bundle over $\mathcal{M}(\Sigma, F)$, and there is a Legendrian cobordism from the Legendrian lift of $L_h(Y, E)$ to the Legendrian lift of $L_{h'}(Y, E)$ for any two choices of perturbation functions h and h' . Herlad's result has a counterpart in the admissible setup of this paper. However, we content ourselves with Proposition 2.18.

Proof. For $t \in \mathbf{R}$, suppose $h_t \in \mathcal{P}$ is a 1-parameter family of cylinder functions depending smoothly on t such that $h_t = h$ for $t \leq -1$ and $h_t = h'$. Let also g_t be the function $\frac{dh_\tau}{d\tau}|_{\tau=t}$. Define

$$V = \{(t, [B]) \in \mathbf{R} \times \mathcal{B}(Y, E) \mid *_3 F_B + \nabla_B h_t = 0\}.$$

As a consequence of Proposition 2.16, this family of cylinder functions can be chosen such that V is a smooth manifold, and its tangent space at $(t, [B])$ is given as

$$T_{(t, [B])}V = \{(s, b) \mid *_3 b|_\Sigma = 0, d_B^* b = 0, *_3 d_B(b) + \text{Hess}_B h(b) + s \nabla_B g_t = 0\}. \quad (2.19)$$

Since V is cut down transversely, the vector space

$$\{b \in L_t^2(Y, \Lambda^1 \otimes E) \mid b|_\Sigma = 0, d_B^* b = 0, *_3 d_B(b) + \text{Hess}_B h(b) = 0, \langle \nabla_B g_t, b \rangle = 0\} \quad (2.20)$$

is trivial. Consider the map $R : V \rightarrow \mathbf{R}^2 \times \mathcal{M}(\Sigma, F)$ defined as

$$R(t, [B]) = (t, g_t([B]), r([B])).$$

Analogous to the map r and using the triviality of (2.20), one can see that R is an immersion. Let (s, b) and (s', b') be two vectors in (2.19), and c, c' denote the restrictions of b, b' to ∂Y . Then we have

$$\begin{aligned} \int_\Sigma \text{tr}(c \wedge c') &= \int_Y \text{tr}(d_B b \wedge b') - \text{tr}(b \wedge d_B b') \\ &= - \int_Y \text{tr}((*_3 \text{Hess}_B h(b) + s *_3 \nabla_B g_t) \wedge b') - \text{tr}(b \wedge (*_3 \text{Hess}_B h(b') + s' *_3 \nabla_B g_t)) \\ &= s \langle \nabla_B g_t, b' \rangle - s' \langle \nabla_B g_t, b \rangle. \end{aligned}$$

It is easy to see from this identity that R induces a Lagrangian immersion. □

2.4 Floer homology groups

In this subsection, the pairs (Y, E) , (Y', E') and the glued up pair $(Y_\#, E_\#)$ is given as in Subsection 2.1. The construction of Subsection 2.3 can be used to form two immersed Lagrangians $L_h(Y, E)$ and $L_{h'}(Y', E')$ of $\mathcal{M}(\Sigma, F)$. The essential assumption that we make throughout the paper is:

Hypothesis 1. *For any positive real number ε , there is h such that the associated parameter in \mathcal{P} is smaller than ε , $L_h(Y, E)$ is regular, and $r : L_h(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ defines a submanifold. The pair (Y', E') satisfies a similar property.*

The following proposition provides a special case where the assumptions in Hypothesis 1 are always satisfied.

Proposition 2.21. *If the inclusion map induces a surjection of the fundamental group of a connected component Σ_0 of Σ into $\pi_1(Y)$, then $L_h(Y, E)$ with $h = 0$, denoted by $L_0(Y, E)$, is regular and $r : L_0(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ defines an embedded submanifold.*

Proof. It is clear that $r : L_0(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ is injective. For $B \in L_0(Y, E)$, it suffices to show that $\mathcal{H}^1(Y, \Sigma; B)$ is trivial. Let b be a 1-form with values in E such that

$$b|_{\Sigma} = 0, \quad d_B^* b = 0, \quad d_B b = 0.$$

Let ζ be the section of E defined as

$$\zeta(p) := \int_{\gamma} \gamma^* b \quad \text{for } p \in Y.$$

Here γ is a path from a base point $p_0 \in \Sigma_0$ to p and to define the integral, we trivialize E along γ using parallel transport with respect to B . This integral does not depend on the choice of γ , because the integral of b over any closed path based at p_0 vanishes. To see the latter claim, note that any closed path based at p_0 can be homotoped into a closed path in Σ_0 and the Stokes theorem and the assumption $d_B b = 0$ show that the integral does not change throughout the homotopy. Since $b|_{\Sigma_0} = 0$, the integral over a path in Σ clearly vanishes. The definition of ζ implies that $b = d_B \zeta$. Using the Stokes theorem we have

$$\begin{aligned} \|b\|_{L^2} &= - \int_Y \text{tr}(d_B \zeta \wedge *_3 d_B \zeta) \\ &= - \int_{\Sigma} \text{tr}(\zeta \wedge *_3 d_B \zeta) + \int_Y \text{tr}(\zeta \wedge d_B *_3 d_B \zeta). \end{aligned} \quad (2.22)$$

The second term in the last expression vanishes because $d_B^* b = 0$. Since $d_B \zeta|_{\Sigma} = 0$ and the restriction of B to each boundary component of Σ is irreducible, $\zeta|_{\Sigma} = 0$, which implies the vanishing of the first term in (2.22). \square

Suppose h and h' are chosen such that $L_h(Y, E)$ and $L_{h'}(Y', E')$ are embedded Lagrangian submanifolds of $\mathcal{M}(\Sigma, F)$. Proposition 2.17 implies that by a small perturbation of the cylinder function h (or h'), we may assume that the submanifolds $L_h(Y, E)$ and $L_{h'}(Y', E')$ are transversal to each other. To define symplectic instanton Floer homology $\text{SI}_*(Y_{\#}, E_{\#})$ as the Lagrangian Floer homology of these two Lagrangians, we need to guarantee that $L_h(Y, E)$ and $L_{h'}(Y', E')$ satisfy further restrictive assumptions.

Definition 2.23. Suppose L is a Lagrangian submanifold of a symplectic manifold (M, ω) . Suppose $\mu : \pi_2(X, L) \rightarrow \mathbf{Z}$ is the Maslov index function. Integrating the symplectic form ω on discs with boundary values in L defines another map $[\omega] : \pi_2(X, L) \rightarrow \mathbf{R}$. The Lagrangian L is *monotone* if there is a positive constant c such that

$$[\omega] = c\mu. \quad (2.24)$$

The positive generator of the image μ is called the *minimal Maslov number* of L .

Definition 2.25. Suppose L and L' are Lagrangian submanifolds of a symplectic manifold (M, ω) . Suppose $\Omega(L, L')$ denotes the space of all paths from L' to L . Any $\alpha \in L \cap L'$ determines a constant path $o_{\alpha} \in \Omega(L, L')$. Any element of $\pi_1(\Omega(L, L'), o_{\alpha})$ determines a continuous map from $S^1 \times [-1, 1]$ to M with boundary components $S^1 \times \{1\}, S^1 \times \{-1\}$ mapped to L and L' . In particular, the Maslov index and the symplectic form induces maps $\mu : \pi_1(\Omega(L, L'), o_{\alpha}) \rightarrow \mathbf{Z}$ and $[\omega] : \pi_1(\Omega(L, L'), o_{\alpha}) \rightarrow \mathbf{R}$. We say the pair (L, L') is *monotone*, if there is a positive constant c such that:

$$[\omega] = c\mu. \quad (2.26)$$

for any choice of α . The positive generator of the image μ is called the *minimal Maslov number* of (L, L') .

The following proposition will be proved in Subsection 4.6. The proof of this proposition utilizes the linear theory of the mixed equation, discussed in the next subsection.

Proposition 2.27. *The Lagrangians $L_h(Y, E)$ and $L_{h'}(Y', E')$ are orientable and monotone with minimal Maslov numbers 4. In fact, the pair $(L_h(Y, E), L_{h'}(Y', E'))$ is monotone.*

Remark 2.28. If L and L' are monotone Lagrangians in a simply connected symplectic manifold (M, ω) , then (L, L') is also a monotone pair. In particular, the second part of Proposition 2.27 is an immediate consequence of the first part.

Now we review how Proposition 2.27 allows us to define Lagrangian Floer homology of the monotone pair $(L_h(Y, E), L_{h'}(Y', E'))$ [Oh93]. Suppose α, β belong to the finite set $\mathfrak{C}_S := L_h(Y, E) \cap L_{h'}(Y', E')$. Let $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ be a smooth map such that

$$u(-s, \theta) = \alpha, \quad u(s, \theta) = \beta, \quad \forall (s, \theta) \in [1, \infty) \times [-1, 1] \quad (2.29)$$

and it satisfies the boundary conditions

$$u|_{\mathbf{R} \times \{1\}} \subset L_h(Y, E), \quad u|_{\mathbf{R} \times \{-1\}} \subset L_{h'}(Y', E'). \quad (2.30)$$

Given another map $u' : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ with similar properties, we say u and u' are homotopic, if there is a smooth map $U : \mathbf{R} \times [-1, 1] \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ such that

- (i) for any $t \in [-1, 1]$, $U|_{\mathbf{R} \times [-1, 1] \times \{t\}}$ satisfies (4.74) and (2.30);
- (ii) $U|_{\mathbf{R} \times [-1, 1] \times \{-1\}} = u$ and $U|_{\mathbf{R} \times [0, 1] \times \{1\}} = u'$.

Equivalence classes of this relation can be regarded as homotopy classes of paths from α to β in $\Omega((L_h(Y, E), L_{h'}(Y', E')))$. The set of all such homotopy classes is denoted by $\pi_2(\alpha, \beta)$. Maslov class of any $p \in \pi_2(\alpha, \beta)$, denoted by $\mu(p)$, is defined to be the Maslov class of any representative of p . For a path p , it is helpful to form a space $\mathcal{B}_S(\alpha, \beta)_p$ consisting of more general representatives of p .

To define $\mathcal{B}_S(\alpha, \beta)_p$, let \exp denote the exponential map with respect to an arbitrary metric on $\mathcal{M}(\Sigma, F)$. A continuous map $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ satisfying (2.30) is an element of $\mathcal{B}_S(\alpha, \beta)_p$ if the following conditions hold. The covariant derivatives $\nabla^k(du)$ have finite L^2 norms for $0 \leq k \leq l-1$. Moreover, there is t_0 such that the restriction of u to $(-\infty, -t_0]$ and $[t_0, \infty)$ is given by $\exp_\alpha v_-$ and $\exp_\beta v_+$ for $v_- \in L_l^2((-\infty, -t_0] \times [-1, 1], T_\alpha \mathcal{M}(\Sigma, F))$ and $v_+ \in L_l^2([t_0, \infty) \times [-1, 1], T_\beta \mathcal{M}(\Sigma, F))$. In particular, u determines an element of $\pi_2(\alpha, \beta)$, which we denote by p . We call $\mathcal{B}_S(\alpha, \beta)_p$ the *configuration space of strips* corresponding to the path p .

The set \mathfrak{C}_S can be decomposed as

$$\mathfrak{C}_S := \bigcup_o \mathfrak{C}_{S,o},$$

where o runs over the set of the connected components of $\Omega(L_h(Y, E), L_{h'}(Y', E'))$ and $\mathfrak{C}_{S,o}$ consists of $\alpha \in \mathfrak{C}_S$ such that $o_\alpha = o$. There is a relative $\mathbf{Z}/4\mathbf{Z}$ -grading \deg_S on each $\mathfrak{C}_{S,o}$, which is called *Floer grading*. Let $\alpha, \beta \in \mathfrak{C}_{S,o}$ and p be a path from α to β in $\Omega(L_h(Y, E), L_{h'}(Y', E'))$. Then define

$$\deg_S(\alpha) - \deg_S(\beta) \equiv \mu(p) \pmod{4} \quad (2.31)$$

Proposition 2.27 implies that the value of $\mu(p) \bmod 4$ is independent of the choice of p and hence \deg_S is well-defined. A relative $\mathbf{Z}/4\mathbf{Z}$ -grading on \mathfrak{C}_S is *compatible with the Floer grading* if its restriction to each $\mathfrak{C}_{S,o}$ agrees with the Floer grading.

Remark 2.32. Since $\mathcal{M}(\Sigma, F)$ is simply connected, the connected component of α_α is determined by the connected components of $L_h(Y, E)$ and $L_{h'}(Y', E')$ that contain α . In particular, if $L_h(Y, E)$ and $L_{h'}(Y', E')$ are connected, then $\Omega(L_h(Y, E), L_{h'}(Y', E'))$ is path connected. Consequently, there is a unique relative $\mathbf{Z}/4\mathbf{Z}$ -grading on \mathfrak{C}_S compatible with the Floer grading. This, for example, happens for the Lagrangians involved in the definition of symplectic framed Floer homology.

Fix a 1-parameter family of Ω -compatible almost complex structures $\mathcal{J} = \{J_\theta\}_{\theta \in [-1, 1]}$ on $\mathcal{M}(\Sigma, F)$, and consider the Cauchy-Riemann equation

$$\frac{\partial u}{\partial \theta} - J_\theta \frac{\partial u}{\partial s} = 0 \quad (2.33)$$

where $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ satisfies the Lagrangian boundary condition of (2.30). Any solution of (2.33), with $\|du\|_{L^2}$ being finite, belongs to $\mathcal{B}_S(\alpha, \beta)_p$ for some choice of α, β and the homotopy class of a path p from α to β . The space of all such solutions of (2.33) is denoted by $M_S(\alpha, \beta)_p$. Translation along the \mathbf{R} factor defines an \mathbf{R} -action on $M_S(\alpha, \beta)_p$, which is free unless $M_S(\alpha, \beta)_p$ contains the constant map to α . The quotient space by this action is denoted by $\check{M}_S(\alpha, \beta)_p$.

For any $u \in \mathcal{B}_S(\alpha, \beta)_p$, let \mathcal{D}_u denote the linearization of (2.33). Then \mathcal{D}_u is an operator acting on L^2_l sections of $u^*T\mathcal{M}(\Sigma, F)$ with the boundary condition that the restriction of u to $\mathbf{R} \times \{1\}$ and $\mathbf{R} \times \{-1\}$ belong to $TL_h(Y, E)$ and $TL_{h'}(Y', E')$, and $L^2_{l-1}(\mathbf{R} \times [-1, 1], u^*T\mathcal{M}(\Sigma, F))$ is the target of this operator. For a section ζ of $u^*T\mathcal{M}(\Sigma, F)$ in the domain of \mathcal{D}_u , we have

$$\mathcal{D}_u \zeta = \nabla \zeta - J_\theta(u) \nabla_s \zeta - (\nabla_\zeta J_\theta) \frac{du}{ds}, \quad (2.34)$$

where the connection ∇ is defined by pulling back the Levi-Civita connection on $\mathcal{M}(\Sigma, F)$. The index of this elliptic operator is equal to $\mu(p)$. The equation (2.33) is cut down transversely at u if \mathcal{D}_u is surjective. In a neighborhood of u , the moduli space $M_S(\alpha, \beta)_p$ is a smooth manifold of dimension $\mu(p)$, which is equal to $\deg_S(\alpha) - \deg_S(\beta) \bmod 4$.

Lemma 2.35 ([Oh93]). *There is a family of almost complex structures $\mathcal{J} = \{J_\theta\}_{\theta \in [-1, 1]}$ such that the moduli space $M_S(\alpha, \beta)_p$ is cut-down transversely.*

The moduli spaces $M_S(\alpha, \beta)_p$ are orientable. Using a standard construction, we may define the *determinant line bundle* δ_p^S on $\mathcal{B}_S(\alpha, \beta)_p$, where the fiber over u is given by

$$\Lambda^{\max} \ker(\mathcal{D}_u) \otimes (\Lambda^{\max} \operatorname{coker}(\mathcal{D}_u))^*.$$

If $M_S(\alpha, \beta)_p$ is cut down transversely at u , then an orientation of the fiber of δ_p^S determines an orientation of $T_u M_S(\alpha, \beta)_p$. Thus, to orient $M_S(\alpha, \beta)_p$ it suffices to fix a trivialization of δ_p^S , which always exists (see Proposition 2.38). We denote the set of trivializations of this bundle by Λ_p^S , which is a $\mathbf{Z}/2\mathbf{Z}$ -torsor. The set Λ_p^S can be identified with the trivializations of δ_p^S over the subspace $\mathcal{B}_S^c(\alpha, \beta)_p$ of $\mathcal{B}_S(\alpha, \beta)_p$ consisting maps u which satisfy (4.74).

If p is a path from α_0 to α_1 and p' is a path α_1 to α_2 , then there is an obvious strip gluing map $\mathcal{B}_S^c(\alpha_0, \alpha_1)_p \times \mathcal{B}_S^c(\alpha_1, \alpha_2)_{p'} \rightarrow \mathcal{B}_S^c(\alpha_0, \alpha_2)_{p\sharp p'}$ which induces the map

$$\Phi_{p,p'} : \Lambda_p^S \otimes_{\mathbf{Z}/2\mathbf{Z}} \Lambda_{p'}^S \rightarrow \Lambda_{p\sharp p'}^S. \quad (2.36)$$

using additivity of the index of the Fredholm operator \mathcal{D}_u with respect to gluing strips.

Definition 2.37. A *coherent system of orientations for strips associated to the Lagrangians* $L_h(Y, E)$ and $L_{h'}(Y', E')$ is an association of an element $\lambda_p \in \Lambda_p^S$ to each homotopy class p of a path between two elements of \mathfrak{C}_S which is compatible with the map $\Phi_{p,p'}$. That is to say, for any two paths p and p' , where the terminal point of p is equal to the initial point of p' , we have

$$\Phi_{p,p'}(\lambda_p \otimes \lambda_{p'}) = \lambda_{p\sharp p'}.$$

Two systems of coherent orientations $\{\lambda_p\}$ and $\{\lambda'_p\}$ are ε -*equivalent* if there is $\varepsilon : \mathfrak{C}_S \rightarrow \mathbf{Z}/2$ such that for any path p from α to β

$$\lambda'_p = (-1)^{\varepsilon(\beta) - \varepsilon(\alpha)} \lambda_p.$$

Proposition 2.38. *The line bundles δ_p^S are orientable. Moreover, there is a coherent system of orientations for strips associated to the Lagrangians $L_h(Y, E)$ and $L_{h'}(Y', E')$.*

A proof of this proposition will be given in Subsection 4.6. In fact, we will also give a recipe in the proof of Proposition 2.38 to fix a coherent system of orientations for strips associated to the Lagrangians $L_h(Y, E)$ and $L_{h'}(Y', E')$.

Remark 2.39. Although Proposition 2.38 is sufficient for our purposes here, there is still room to improve this proposition. For instance, the Lagrangians $L_h(Y, E)$ and $L_{h'}(Y', E')$ are in fact spin, and the spin structure can be used to fix orientations for the line bundles δ_p^S following [FOOO09a, FOOO09b]. The authors expect that there is a preferred choice of spin structures for $L_h(Y, E)$ and $L_{h'}(Y', E')$ and the induced orientations by these spin structures agree with the coherent system of orientations $\{\lambda_p\}$ constructed in Subsection 4.6. Another issue related to orientations of the determinant bundles which is not completely addressed here is compatibility of the coherent system of orientations $\{\lambda_p\}$ with *gluing spheres*. For an arbitrary smooth map $s : S^2 \rightarrow \mathcal{M}(\Sigma, F)$, we may similarly define the linearized Cauchy-Riemann operator \mathcal{D}_s as a map

$$L_l^2(S^2, s^*T\mathcal{M}(\Sigma, F)) \rightarrow L_{l-1}^2(S^2, \Lambda^{0,1} \otimes s^*T\mathcal{M}(\Sigma, F)).$$

Since S^2 is a closed Riemann surface, the operator \mathcal{D}_s is complex linear up to compact terms and hence its determinant line has a canonical orientation. Gluing s to elements of $\mathcal{B}_S^c(\alpha, \beta)_p$ determines a map $\mathcal{B}_S^c(\alpha, \beta)_p \rightarrow \mathcal{B}_S^c(\alpha, \beta)_{p'}$ where $p' := p\sharp s$ is the induced path from α to β . This gluing map gives

$$\Psi_{p,s} : \Lambda_p^S \rightarrow \Lambda_{p'}^S,$$

which depends only on the homotopy class of s . Recall that $\pi_2(\mathcal{M}(\Sigma, F)) = \mathbf{Z}^i$ where i is the number of the connected components of Σ which have genus greater than 1. We can guarantee that the system of orientations $\{\lambda_p\}$ given by Proposition 2.38 is compatible with the map $\Psi_{p,s}$ in the case that $i = 1$. For $i = 2$, we can only obtain a system of orientations $\{\lambda_p\}$ compatible with $\Psi_{p,s}$ when s belongs to one of the summands of $\pi_2(\mathcal{M}(\Sigma, F))$. See Remark 4.76 for more details.

Let $C_S((Y, E), (Y', E'))$ be the abelian group freely generated by the elements of \mathfrak{C}_S . Fix a family of almost complex structures as in Lemma 2.35 and orient the smooth manifolds $\check{M}_S(\alpha, \beta)_p$ using the orientation given by Proposition 2.38. The space $\check{M}_S(\alpha, \beta)_p$ is a fiber bundle over $\check{M}_S(\alpha, \beta)_p$ with fiber \mathbf{R} , and the total space and the fiber of this bundle are oriented. We orient $\check{M}_S(\alpha, \beta)_p$ such that the orientation of $\check{M}_S(\alpha, \beta)_p$ is obtained from those of \mathbf{R} and $\check{M}_S(\alpha, \beta)_p$ using the fiber-first convention. Let $d_S : C_S((Y, E), (Y', E')) \rightarrow C_S((Y, E), (Y', E'))$ be the linear map whose value at $\alpha \in \mathfrak{C}_S$ is given by

$$d_S(\alpha) := \sum_{p:\alpha \rightarrow \beta} \# \check{M}_S(\alpha, \beta)_p \cdot \beta,$$

where the above sum is taken over all paths p such that $\check{M}_S(\alpha, \beta)_p$ is zero dimensional, and $\# \check{M}_S(\alpha, \beta)_p$ denotes the signed count of the elements of $\check{M}_S(\alpha, \beta)_p$. In particular, d_S decreases the $\mathbf{Z}/4\mathbf{Z}$ -grading by 1. This map is a differential, i.e., $d_S^2 = 0$, and the homology of the chain complex $(C_S((Y, E), (Y', E')), d_S)$ is independent of the choice of the family of almost complex structures \mathcal{J} . In fact, our main theorem shows that this homology group depends only on $(Y_\#, E_\#)$, and the symplectic instanton Floer homology $SI_*(Y_\#, E_\#)$ of the pair $(Y_\#, E_\#)$ is defined to be this relatively $\mathbf{Z}/4\mathbf{Z}$ -graded homology group.

Proposition 2.40. *The chain homotopy type of the chain complex $(C_S((Y, E), (Y', E')), d_S)$ is an invariant of the pair $(Y_\#, E_\#)$. In particular, it does not depend on the family of almost complex structures \mathcal{J} , the cylinder functions h and h' , and the coherent system of orientations provided by Proposition 2.38.*

Proof. This is a consequence of Proposition 2.57 below and Theorem 3.29, proved in the next section. \square

Remark 2.41. It is desirable to give a direct proof for the above proposition. The invariance with respect to the choice of almost complex structures is standard. Proposition 2.18 asserts that changing cylinder functions h and h' gives rise to cobordant Lagrangians. Thus, one would expect that the results of [BC13] imply that changing h and h' give chain homotopy equivalent chain complexes $(C_S((Y, E), (Y', E')), d_S)$. However, it is not clear that the Lagrangian cobordism V provided by Proposition 2.18 is embedded. If so, then it is reasonable to expect that the analogue of Proposition 2.27 holds, and V is monotone. We hope to come back to this issue in a sequel where we pursue generalization of the results of this paper to the case of immersed Lagrangians.

Remark 2.42. Suppose the Lagrangians $L_h(Y, E)$ and $L_{h'}(Y', E')$ are simply connected, and $\{\lambda_p\}$ and $\{\lambda'_p\}$ are two systems of coherent orientations, which are compatible with the maps $\Psi_{p,s}$. (For example, the Lagrangians involved in the definition of symplectic framed Floer homology together with the systems of orientations of [WW16] and Proposition 2.38 have this property.) Then there is a map $\kappa : \mathfrak{C}_S \times \mathfrak{C}_S \rightarrow \mathbf{Z}/2\mathbf{Z}$ such that for any path p from α to β we have

$$\lambda'_p = (-1)^{\kappa(\alpha, \beta)} \lambda_p.$$

This follows from the fact that any two paths from α to β are related to each other by gluing an element of $\pi_2(\mathcal{M}(\Sigma, F))$. Since $\{\lambda_p\}$ and $\{\lambda'_p\}$ are both systems of coherent orientations, we have

$$\kappa(\alpha_1, \alpha_3) = \kappa(\alpha_1, \alpha_2) + \kappa(\alpha_2, \alpha_3).$$

Therefore, there exists $\varepsilon : \mathfrak{C}_S \rightarrow \mathbf{Z}/2\mathbf{Z}$ such that

$$\kappa(\alpha, \beta) = \varepsilon(\beta) - \varepsilon(\alpha).$$

That is to say, $\{\lambda_p\}$ and $\{\lambda'_p\}$ are ε -equivalent. Thus, Lagrangian Floer homology groups of $L_h(Y, E)$ and $L_{h'}(Y', E')$ with respect to $\{\lambda_p\}$ and $\{\lambda'_p\}$ are isomorphic to each other. In particular, our definition of symplectic framed Floer homology agree with [WW16].

Next, we turn into the gauge theoretical component of our main theorem. For the pair $(Y_\#, E_\#)$, the class $w_2(E_\#) \in H^2(Y_\#, \mathbf{Z}/2\mathbf{Z})$ has a non-trivial pairing with each connected component of the copy of Σ in $Y_\#$. In particular, $E_\#$ is admissible in the sense of [BD95]. We review the definition of a version of instanton Floer homology for the admissible pair $(Y_\#, E_\#)$ which is more suitable for our purposes.

Suppose $\mathcal{A}(Y_\#, E_\#)$ is the space of all L^2_t connections on $E_\#$. This is an affine space modeled on $L^2_t(Y_\#, \Lambda^1 \otimes E_\#)$. Let $\mathcal{G}(E_\#)$ be the space of global sections of the fiber bundle $E_\# \times_{\text{ad}} \text{SU}(2)$ of class L^2_{l+1} . As in the case of 3-manifolds with boundary, the gauge group $\mathcal{G}(E_\#)$ acts on $\mathcal{A}(Y_\#, E_\#)$, and we denote the quotient space by $\mathcal{B}(Y_\#, E_\#)$. Analogous to Remark 2.2, we may form the gauge group $\mathcal{G}_{\text{ex}}(E_\#)$ using sections of $E_\# \times_{\text{ad}} \text{SO}(3)$. There is an obvious homomorphism $\mathcal{G}(E_\#) \rightarrow \mathcal{G}_{\text{ex}}(E_\#)$, whose cokernel can be identified with $H^1(Y, \mathbf{Z}/2\mathbf{Z})$. The group $\mathcal{G}_{\text{ex}}(E_\#)$ acts on $\mathcal{A}(Y_\#, E_\#)$, extending the action of $\mathcal{G}(E_\#)$. In particular, there is an action of $H^1(Y, \mathbf{Z}/2\mathbf{Z})$ on $\mathcal{B}(Y_\#, E_\#)$. We shall be interested in the action of $l_\Sigma \in H^1(Y, \mathbf{Z}/2\mathbf{Z})$ given as the Poincaré dual of any of the connected components of Σ . We write ι for the involution determined by l_Σ .

Lemma 2.43. *An element of $\mathcal{B}(Y_\#, E_\#)$ is fixed by the action of ι if it is represented by an $\text{O}(2)$ -connection such that its orientation bundle is determined by the cohomology class l_Σ . In particular, ι does not have any fixed point which restricts to a flat connection on one of the connected components of Σ .*

Proof. Suppose $\tilde{E}_\#$ denotes a $\text{U}(2)$ -bundle such that $c_1(\tilde{E}_\#)$ is a lift of $w_2(E_\#)$. Then the vector bundle associated to $\tilde{E}_\#$ by the adjoint map $\text{U}(2) \rightarrow \text{SO}(3)$ is isomorphic to $E_\#$. Moreover, the determinant map $\text{U}(2) \rightarrow \text{U}(1)$ induces a complex line bundle $\det(\tilde{E}_\#)$, and we fix a connection b_0 on this line bundle. The configuration space of connections on $\tilde{E}_\#$ with the induced connection on $\det(\tilde{E}_\#)$ being b_0 can be identified with $\mathcal{B}(Y_\#, E_\#)$. Using this identification, the involution ι on $\mathcal{B}(Y_\#, E_\#)$ is given by taking the tensor product with a real line bundle determined by l_Σ . For a $\text{U}(2)$ connection \tilde{B} representing an element of $\mathcal{B}(Y_\#, E_\#)$ and a loop γ based at a point $x \in Y_\#$, if $\text{hol}_\gamma(\tilde{B})$ denotes the holonomy of \tilde{B} along γ , then the holonomy of $\iota(\tilde{B})$ is $(-1)^{l_\Sigma(\gamma)} \text{hol}_\gamma(\tilde{B})$. Thus \tilde{B} represents a fixed point of ι , if and only if there is $g \in \text{SU}(2)$ such that for any loop γ based at x we have

$$(-1)^{l_\Sigma(\gamma)} \text{hol}_\gamma(\tilde{B}) = g \text{hol}_\gamma(\tilde{B}) g^{-1}.$$

By picking a loop γ_0 with $l_\Sigma(\gamma_0) = 1$, we conclude that $\text{tr}(g) = 0$. Now, if γ represents an element in $\ker(l_\Sigma)$, then $\text{hol}_\gamma(\tilde{B})$ commutes with g , and otherwise $\text{hol}_{\gamma_0}(\tilde{B}) \text{hol}_\gamma(\tilde{B})$ commutes with g . It is easy to see from this that the $\text{SO}(3)$ connection induced by \tilde{B} is an $\text{O}(2)$ connection with orientation bundle l_Σ . In particular, the restriction of any such connection to a connected component of Σ is an S^1 connection on F , and hence this restriction cannot be flat. \square

The cylinder functions h and h' may be used to define a perturbation of the flat equation on $\mathcal{A}(E_\#)$:

$$\phi_{h,h'}(B) = *_3 F_B + \nabla_B h + \nabla_B h'. \quad (2.44)$$

This map is equivariant with respect to the automorphisms of $E_\#$. Any solution of $\phi_{h,h'}(B) = 0$ restricts to a flat connection on a neighborhood of Σ . In particular, such connections are irreducible and ι acts freely on them by Lemma 2.43. We write $\widetilde{\mathfrak{C}}_G$ and \mathfrak{C}_G respectively for the subspaces of $\mathcal{B}(Y_\#, E_\#)$ and $\mathcal{B}(Y_\#, E_\#)/\iota$ which are represented by the solutions of (2.44). If $B_\# \in \mathcal{A}(Y_\#, E_\#)$ represents an element of $\widetilde{\mathfrak{C}}_G$, then the restrictions of $B_\#$ to Y and Y' determine an element of $\mathfrak{C}_S = L_h(Y, E) \cap L_{h'}(Y', E')$. Since the restrictions of l_Σ to Y and Y' are trivial, this element of \mathfrak{C}_S depends only on the equivalence class of $A_\#$ in \mathfrak{C}_G . Moreover, if the pair of $[B] \in L_h(Y, E)$ and $[B'] \in L_{h'}(Y', E')$ represents an element of \mathfrak{C}_S , then gluing these connections gives rise to two equivalence classes of connections in $\widetilde{\mathfrak{C}}_G$ which are related to each other by ι . Thus, any element of \mathfrak{C}_S determines a well-defined element of \mathfrak{C}_G . We summarize this discussion in the following lemma.

Lemma 2.45. *The space \mathfrak{C}_G is compact and can be identified with \mathfrak{C}_S .*

As the first step to study (2.44), we may consider the linearization of (2.44) as in the case of 3-manifolds with boundary. For any connection α representing an element of \mathfrak{C}_G , we may define X_α as in (2.6) where the condition $*_3 a|_\Sigma = 0$ is dropped. There is also a Fredholm operator $L_B : X_B \rightarrow X_B$ as in Proposition 2.10 with index 0. An element of $\mathcal{B}(Y_\#, E_\#)$ represented by B is regular if L_B is a surjective operator. This is equivalent to say that the kernel of L_B given as follows

$$\mathcal{H}_{h,h'}^1(Y_\#; B) := \{b \in L_l^2(Y_\#, \Lambda^1 \otimes E_\#) \mid d_B^* b = 0, *_3 d_B(b) + \text{Hess}_B h(b) + \text{Hess}_B h'(b) = 0\} \quad (2.46)$$

is trivial. The following lemma is a consequence of Mayer-Vietoris principle for the space $\mathcal{H}_{h,h'}^1(Y_\#, B)$.

Lemma 2.47. *An element of \mathfrak{C}_G is regular if the corresponding element of \mathfrak{C}_S is given by a transversal intersection of $L_h(Y, E)$ and $L_{h'}(Y', E')$.*

Proof. Let $B_\#$ represent an element of \mathfrak{C}_G , and B, B' denote its restrictions to Y and Y' . Since the connection $B_\#$ is flat on the overlapping region $[-1, 1] \times \Sigma$ of Y and Y' , we may assume that $B_\#$ is the pull-back of a flat connection σ on the bundle F . Suppose $b_\# \in \mathcal{H}_{h,h'}^1(Y_\#; B_\#)$ whose restrictions to Y and Y' are denoted by b and b' . There are $\zeta \in L_{l+1}^2(Y, E)$ and $\zeta' \in L_{l+1}^2(Y', E')$ such that

$$b - d_B \zeta \in \mathcal{H}_h^1(Y; B), \quad b' - d_{B'} \zeta' \in \mathcal{H}_{h'}^1(Y'; B').$$

The restrictions of $b - d_B \zeta$ and $b' - d_{B'} \zeta'$ to ∂Y and $\partial Y'$ represent the same element of $\mathcal{H}^1(\Sigma; \sigma)$ because they are equal to the cohomology classes represented by $b_\#|_{\{t\} \times \Sigma}$ for any $t \in [-1, 1]$. Transversality of the intersection of Lagrangians $L_h(Y, E)$ and $L_{h'}(Y', E')$ implies that $b - d_B \zeta = 0$ and $b' - d_{B'} \zeta' = 0$. In particular, on the overlap region $[-1, 1] \times \Sigma$, we have $d_{B_\#}(\zeta - \zeta') = 0$. Since $B_\#$ restricted to the overlap region is irreducible, $\zeta = \zeta'$, and hence ζ and ζ' determine a 0-form $\zeta_\#$ on $Y_\#$ such that $b_\# = d_{B_\#} \zeta_\#$. This identity and the condition $d_{B_\#}^* b_\# = 0$ imply that $\zeta_\# = 0$. Thus, $b_\#$ vanishes. \square

Fix $\alpha, \beta \in \mathfrak{C}_G$, and let A_0 be a smooth connection on the bundle $E_\# \times \mathbf{R}$ over the cylinder 4-manifold $\mathbf{R} \times Y_\#$, such that the restriction of A_0 to $(-\infty, -1] \times Y_\#$ (resp. $[1, \infty) \times Y_\#$) is the pull-back of a representative B of α (resp. B' of β). We say two such connections A_0 and A_1 represent the same path, if there is a smooth section g of $(\mathbf{R} \times E_\#) \times_{\text{ad}} \text{SU}(2)$ over $\mathbf{R} \times Y_\#$ such that $A_1 - g^* A_0$ or $A_1 - \iota(g^* A_0)$ is compactly supported. This defines an equivalence relation, and any equivalence class of this relation is

called a *path* along $\mathbf{R} \times Y_\#$ from α to β . For a fixed pair α and β , a set of representatives for all paths from α to β can be given as follows. Suppose A_0 is a connection as above which restricts to the pullbacks of the representative B for α and the representative B' for β . Suppose also A_1 is a connection on $\mathbf{R} \times Y_\#$ which restricts to the pullbacks of B' and $\iota(B')$ on the ends $(-\infty, -1] \times Y_\#$ and $[1, \infty) \times Y_\#$. Then any path from α to β is represented by gluing an $SU(2)$ connection on S^4 with $c_2 = k \in \mathbf{Z}$ to A_0 and then possibly gluing the resulting connection to A_1 .

Given a path p from α to β , fix a representative A_0 for p , and let $\mathcal{A}_G(\alpha, \beta)_p$ be the space of connections of the form $A_0 + a$ where $a \in L_t^2(\mathbf{R} \times Y, \Lambda^1 \otimes E_\#)$. The configuration space $\mathcal{B}_G(\alpha, \beta)_p$ is defined as the quotient of $\mathcal{A}_G(\alpha, \beta)_p$ with respect to the action of the sections g of $(\mathbf{R} \times E_\#) \times_{\text{ad}} SU(2)$ over $\mathbf{R} \times Y_\#$ such that $\nabla_{A_0} g$ is in L_t^2 . For $A \in \mathcal{A}_G(\alpha, \alpha)_p$, define the *perturbed ASD equation*

$$F_A^+ + (*_3 \nabla_{A_t} h)^+ + (*_3 \nabla_{A'_t} h')^+ = 0 \quad (2.48)$$

where F_A^+ denotes the self-dual part of the curvature of A , defined with respect to the product metric on $\mathbf{R} \times Y_\#$. For each $t \in \mathbf{R}$, A_t (resp. A'_t) denotes the restriction of A to $\{t\} \times Y$ (resp. $\{t\} \times Y'$). Thus $\nabla_{A_t} h$ (resp. $\nabla_{A'_t} h'$) can be regarded as a 1-form on $\{t\} \times Y$ (resp. $\{t\} \times Y'$) with values in E (resp. E'), and $*_3 \nabla_{A_t} h$ (resp. $*_3 \nabla_{A'_t} h'$) is the Hodge dual of $\nabla_{A_t} h$ (resp. $\nabla_{A'_t} h'$) with respect to the metric on Y (resp. Y'). This equation is gauge invariant and determines a subspace of $\mathcal{B}_G(\alpha, \beta)_p$ which is denoted by $M_G(\alpha, \beta)_p$. Translation along the first factor of $\mathbf{R} \times Y_\#$ determines an action of \mathbf{R} on $M_G(\alpha, \beta)_p$ and the quotient space with respect to this action is denoted by $\check{M}_G(\alpha, \beta)_p$.

For a connection $A \in \mathcal{A}_G(\alpha, \beta)_p$, define the ASD operator

$$\mathcal{D}_A : L_t^2(\mathbf{R} \times Y, \Lambda^1 \otimes E_\#) \rightarrow L_{t-1}^2(\mathbf{R} \times Y, (\Lambda^0 \otimes \Lambda^+) \otimes E_\#)$$

as follows:

$$\mathcal{D}_A(a) := (d_A^* a, d_A^+ a + (*_3 \text{Hess}_{A_t} h(a_t))^+ + (*_3 \text{Hess}_{A'_t} h'(a'_t))^+). \quad (2.49)$$

The first component of \mathcal{D}_A takes into account the gauge fixing condition and the second component is given by the linearization of (2.48). This operator \mathcal{D}_A is the perturbation of the standard ASD operator by a compact term induced by h and h' . Since α and β are regular, \mathcal{D}_A is a Fredholm operator. The index of this operator depends only on the path p and otherwise is independent of A . Moreover, switching p with another path from α to β changes the index by a multiple of 4. In fact, gluing a connection on S^4 with $c_2 = k$ to the path p changes the index by $8k$, and gluing p to a path from a representative B' of β to $\iota(B')$, changes the index by an integer of the form $8k + 4$. In particular, we may use the index of the path p to define a relative $\mathbf{Z}/4\mathbf{Z}$ -grading deg_G on \mathfrak{C}_G :

$$\text{deg}_G(\alpha) - \text{deg}_G(\beta) \equiv \text{index}(\mathcal{D}_A) \pmod{4}. \quad (2.50)$$

There is another useful number associated to a path p from α to β . For any connection $A \in \mathcal{A}_G(\alpha, \beta)_p$, define the topological energy of A as follows:

$$\mathcal{E}(A) := \frac{1}{8\pi^2} \int_{\mathbf{R} \times Y} \text{tr} \left((F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A'_t} h') \wedge (F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A'_t} h') \right) \quad (2.51)$$

It is straightforward to check that

$$\mathcal{E}(A) = \frac{1}{4\pi^2} (h(\alpha) + h'(\alpha)) - \frac{1}{4\pi^2} (h(\beta) + h'(\beta)) + \frac{1}{8\pi^2} \int_{\mathbf{R} \times Y} \text{tr}((F_A) \wedge (F_A)). \quad (2.52)$$

The last term in the above sum, which is the more standard definition for the topological energy of A , depends only on the path. This implies that $\mathcal{E}(A)$ also depends only on p . For a connection A that represents an element of $\mathcal{M}_G(\alpha, \beta)_p$, $\mathcal{E}(A)$ is non-negative and is zero if and only if $\alpha = \beta$, p is the constant path and A is the pullback of a representative of β . Another straightforward observation about topological energy is that $2\mathcal{E}(A) \in \mathbf{Z}$ for any connection $A \in \mathcal{A}_G(\alpha, \alpha)_p$. This is a consequence of (2.52) and the fact that the Chern-Weil integral in (2.52) satisfies a similar property.

The following proposition gives a relationship between $\mathcal{E}(A)$ and the index of $\mathcal{D}(A)$.

Proposition 2.53. *To each $\alpha \in \mathfrak{C}_G$, we can associate a real number $\varepsilon(\alpha)$ such that for any $A \in \mathcal{A}_G(\alpha, \beta)_p$, we have*

$$\text{index}(\mathcal{D}_A) := 8\mathcal{E}(A) + \varepsilon(\beta) - \varepsilon(\alpha).$$

Proof. The standard index formula for the ASD operator on a manifold with cylindrical end [Tau93, MMR94] asserts that

$$\text{index}(\mathcal{D}_A) = \frac{\rho_\beta - \rho_\alpha}{2} + \frac{1}{\pi^2} \int_{\mathbf{R} \times Y} \text{tr}(F_A \wedge F_A),$$

where for $\alpha \in \mathfrak{C}_G$, ρ_α is the ρ -invariant associated to the connection α . This identity and (2.52) give the desired result. \square

Following proposition can be regarded as a linear version of our main theorem. It is also a variation of the main result of [Tau90] for the admissible setting. Proof of this result will be given in Subsection 4.6.

Proposition 2.54. *The relative $\mathbf{Z}/4$ -grading deg_G on $\mathfrak{C}_S \cong \mathfrak{C}_G$ is compatible with the Floer grading deg_S . In particular, in the case of framed Floer homology, the two gradings deg_G and deg_S agree with each other.*

As in the symplectic case, orientations of determinant lines of \mathcal{D}_A for connections A representing elements of $\mathcal{B}_G(\alpha, \beta)_p$ determine a real line bundle δ_p^G on $\mathcal{B}_G(\alpha, \beta)_p$. This line bundle is oriented [Flo88a, Don02] and the set of the two orientations of this line bundle is denoted by Λ_p^G . For paths p from $\alpha_0 \in \mathfrak{C}_G$ to $\alpha_1 \in \mathfrak{C}_G$ and p' from α_1 to $\alpha_2 \in \mathfrak{C}_G$ along $\mathbf{R} \times Y$, we may again define an isomorphism

$$\Phi_{p,p'} : \Lambda_p^G \otimes_{\mathbf{Z}/2\mathbf{Z}} \Lambda_{p'}^G \rightarrow \Lambda_{p\#p'}^G. \quad (2.55)$$

For a path p from α to β and the path p' obtained by gluing a connection on S^4 with $kc_2 = k$ to p , there is an isomorphism obtained from gluing the standard orientations of the ASD complexes for S^4 :

$$\Psi_{p,k} : \Lambda_p^G \rightarrow \Lambda_{p'}^G.$$

We can fix a system of orientations for the line bundles δ_p^G , which is compatible with the maps $\Phi_{p,p'}$ and $\Psi_{p,k}$ [Don02, Section 5.4]. To achieve this goal, fix $\alpha_0 \in \mathfrak{C}_G$ with a representative connection B_0 on $E_\#$. We also fix a path p_0 from α_0 to α_0 represented by a connection A_0 whose restrictions to the ends $(-\infty, -1] \times Y_\#$ and $[1, \infty) \times Y_\#$ are pullback of B_0 and $\iota(B_0)$. Fix an element $\lambda_{p_0} \in \Lambda_{p_0}^G$. For any $\alpha \in \mathfrak{C}_G$, we pick an arbitrary path p from α to α_0 , and pick an element $\lambda_p \in \Lambda_p^G$. Then we extend this choice of orientations of the line bundle δ_p^G to all paths from α to α_0 using the maps Φ_{p,p_0} and $\Psi_{p,k}$ and

the orientation element λ_{p_0} . Finally for $\alpha, \beta \in \mathfrak{C}_G$ and a path p from α to β , we pick an arbitrary path p_+ from β to α_0 , and pick $\lambda_p \in \Lambda_p^G$ such that

$$\Phi_{p,p_+}(\lambda_p \otimes \lambda_{p_+}) = \lambda_{p_-},$$

where p_- is the path obtained by gluing p to p_+ .

A connection A representing an element of $M_G(\alpha, \beta)_p$ is regular if \mathcal{D}_A is surjective. The moduli space $M_G(\alpha, \beta)_p$ in a neighborhood of a regular connection $[A]$ is a smooth manifold of dimension $\text{index}(\mathcal{D}_A)$, and a trivialization of δ_p^G fixes an orientation of this manifold. The following lemma, which will be proved in Subsection 6.1, asserts that we can ensure regularity of the elements of moduli spaces which are essential for the definition of instanton Floer homology.

Lemma 2.56. *There are Riemannian metrics g, g' on Y, Y' and small enough perturbations of the cylinder functions h and h' such that the sets \mathfrak{C}_S and \mathfrak{C}_G do not change and all solutions of (2.48) with index at most seven are regular.*

From now on, we assume that h and h' are chosen such that the spaces $L_h(Y, E), L_{h'}(Y', E')$ are smooth embedded Lagrangians which intersect transversely and the claim of Lemma 2.56 holds. We also drop h and h' from our notations for the 3-manifolds Lagrangians.

Let $C_G(Y_\#, E_\#)$ be the abelian group freely generated by the elements of \mathfrak{C}_G . Fix orientations of the determinant line bundles δ_p^G as above, and use them to orient the moduli spaces $\check{M}_G(\alpha, \beta)_p$. Let $d : C_G(Y_\#, E_\#) \rightarrow C_G(Y_\#, E_\#)$ be the map defined as

$$d(\alpha) := \sum_{p:\alpha \rightarrow \beta} \# \check{M}_G(\alpha, \beta)_p \cdot \beta.$$

where the sum is over all paths p from α to another element $\beta \in \mathfrak{C}_G$ such that the associated ASD operator has index 1. Thus, the operator d decreases the relative grading by 1. Moreover, $d^2 = 0$ and the instanton Floer group $I_*(Y_\#, E_\#)$ is defined to be the homology of the relatively $\mathbf{Z}/4$ -graded complex $(C_G(Y_\#, E_\#), d)$.

Proposition 2.57. *The chain homotopy type of the chain complex $(C_G(Y_\#, E_\#), d)$ is independent of the choice of the Riemannian metrics on Y, Y' , the cylinder functions h and h' and the choices of orientation elements $\lambda_p \in \Lambda_p^G$. In particular, $I_*(Y_\#, E_\#)$ is a topological invariant of $(Y_\#, E_\#)$.*

The proof is standard and we refer the reader to [Flo88a, Don02] for more details. We only remark on the dependence of $I_*(Y_\#, E_\#)$ on the involution ι , which is determined by the $\mathbf{Z}/2$ cohomology class dual to the connected component Σ_0 of Σ . We may define a variant of $I_*(Y_\#, E_\#)$, where we do not pass to the quotient by the action of ι . The resulting invariant is a $\mathbf{Z}/8\mathbf{Z}$ -graded chain complex, which is a topological invariant of $(Y_\#, E_\#)$ and does not depend on the $\mathbf{Z}/2$ cohomology class of Σ_0 . Moreover, ι induces an involution of degree 4 on this complex and the quotient space is isomorphic to $I_*(Y_\#, E_\#)$. Thus, the isomorphism type of $I_*(Y_\#, E_\#)$ does not depend on ι .

3 Proof of the Main Theorem

In this section, we prove our main result, Theorem 2. Our key tool in the proof is the *mixed equation*, which is defined using a combination of the Cauchy-Riemann equation and the ASD equation. In the prequel to this paper [DFL] and following [Lip14], we defined mixed equation for any *quintuple*. We recall the notion of quintuples in Subsection 3.1, and introduce *special quintuples*, which are the specific type of quintuples used in our proof. In the next subsection, we use the moduli spaces of solutions to the mixed equation associated to special quintuples, and construct the desired isomorphism for Theorem 2.

3.1 Special quintuples

A quintuple $\mathfrak{q} = (X, V, S, \mathcal{M}(\Sigma, F), \mathbb{L})$ consists of a Riemannian 4-manifold X , an $\mathrm{SO}(3)$ -bundle V over X , a Riemann surface (S, j) , the symplectic manifold $\mathcal{M}(\Sigma, F)$ and a collection of Lagrangians $\mathbb{L} = \{L_1, L_2, \dots, L_k\}$ in $\mathcal{M}(\Sigma, F)$. There is a (possibly non-compact) oriented 1-manifold γ such that the boundary of the 4-manifold X is identified with $\gamma \times \Sigma$ where Σ is the disconnected Riemann surface that we fixed in the previous section. Moreover, the restriction of V to ∂X is identified with the pullback of the $\mathrm{SO}(3)$ -bundle F on Σ to $\gamma \times \Sigma$. The boundary components of the Riemann surface S are given as

$$\partial S = -\gamma \sqcup \eta_1 \sqcup \eta_2 \sqcup \dots \sqcup \eta_k. \quad (3.1)$$

In particular, we regard L_i as a Lagrangian attached to the boundary component η_i . In [DFL], we considered quintuples in the more general case that $\mathcal{M}(\Sigma, F)$ is replaced with an arbitrary symplectic manifold (M, ω) . In that case, \mathbb{L} includes some additional information in the form a certain type of Lagrangian correspondence from $\mathcal{A}(\Sigma, F)$ to M .

The mixed equation associated to the quintuple \mathfrak{q} is defined for a pair of a connection A on V and a map $u : S \rightarrow \mathcal{M}(\Sigma, F)$:

$$\begin{cases} F_A^+ = 0, \\ \bar{\partial}_{J_*} u = 0, \end{cases} \quad (3.2)$$

where $\bar{\partial}_{J_*} u = 1/2(du + J_* \circ du \circ j)$. The restriction of A to $\{x\} \times \Sigma \subset \gamma \times \Sigma$, for each $x \in \gamma$, is required to be a flat connection representing $u(x)$. Moreover, $u(x) \in L_i$ for $x \in \eta_i$. These two conditions are respectively called the *matching* and the Lagrangian boundary conditions. The Cauchy-Riemann-equation in (3.2) is defined using the standard complex structure J_* on $\mathcal{M}(\Sigma, F)$. Eventually, we shall be interested in the case that the ASD equation in (3.2) is perturbed and the Cauchy-Riemann equation is defined by a domain dependent family of almost complex structures.

To define special quintuples, let U be the domain in the complex plane which is sketched in Figure 1. This non-compact space has four boundary components, denoted by η_+ , η_- , η'_+ , η'_- , and contains the following subspaces of \mathbb{C} :

$$[-2, 2] \times [2, \infty), \quad [-2, 2] \times (-\infty, -2], \quad [3, \infty) \times [-1, 1], \quad (-\infty, -3] \times [-1, 1].$$

This space is decomposed as the union of the regions U_+ and U_- which share the imaginary line in \mathbb{C} , denoted by U_∂ , as their common boundary components. We identify a regular neighborhood of the

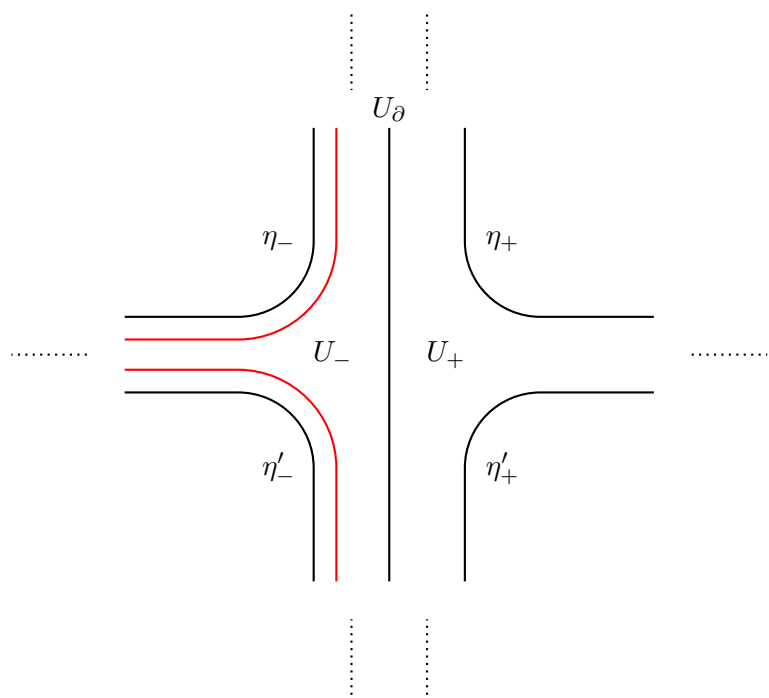


Figure 1: The space U , a subspace of the complex plane \mathbf{C} . The chosen regular neighborhoods of η_- and η'_- are determined by the red curves.

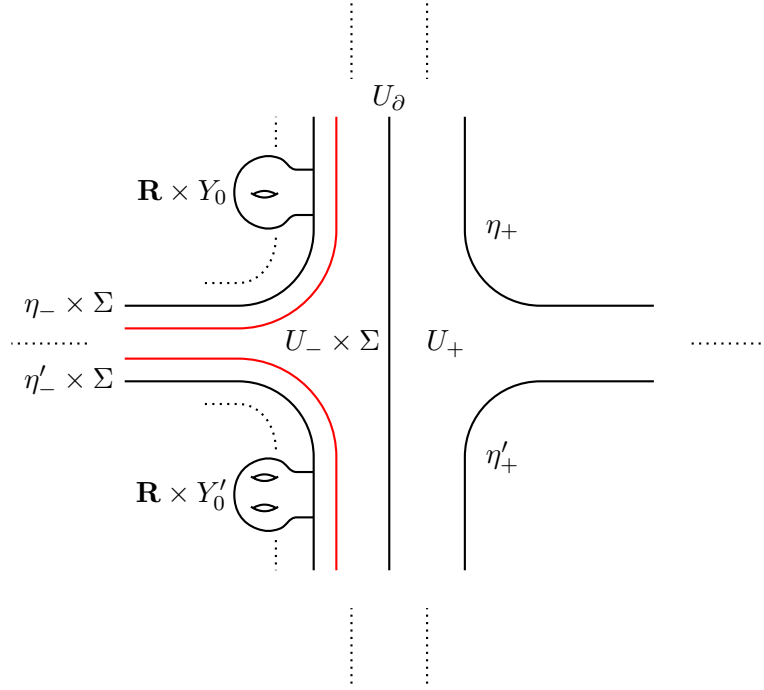


Figure 2: The Riemannian 4-manifold X and U_+

boundary components η_-, η'_- with $\mathbf{R} \times (\frac{1}{2}, 1]$, $\mathbf{R} \times [-1, -\frac{1}{2})$ and fix a Riemannian metric g_- on U_- which is equal to product metrics on these regular neighborhoods, and is equal to the standard metric of the complex plane on the subsets

$$[-2, 0] \times [2, \infty), \quad [-2, 0] \times (-\infty, -2], \quad (-\infty, -3] \times [-1, 1], \quad [-1, 0] \times [-2, 2].$$

Let X be the oriented smooth 4-manifold given by gluing the following 4-manifolds along their common boundaries

$$\mathbf{R} \times Y_0 \cup U_- \times \Sigma \cup \mathbf{R} \times -Y'_0,$$

where $Y_0 \subset Y$, $Y'_0 \subset Y'$ are given in (2.1). The subspaces $[-2, 0] \times [2, \infty)$, $[-2, 0] \times (-\infty, -2]$ and $(-\infty, -3] \times [-1, 1]$ of U_- determine subspaces Z , Z' and $Z_\#$ of X which are, respectively, diffeomorphic to $[2, \infty) \times Y$, $(-\infty, -2] \times Y'$ and $(-\infty, -3] \times Y_\#$. The projection maps from Z , Z' and $Z_\#$ to Y , Y' and $Y_\#$ are respectively denoted by π , π' and $\pi_\#$. The fixed Riemannian metrics on Σ , Y and Y' in Subsection 2.1 and the metric g_- on U_- give rise to a Riemannian metric on X , which we denote by g_X . Moreover, the $\text{SO}(3)$ bundles E , E' and F determine an $\text{SO}(3)$ -bundle on X , which we denote by V .

Remark 3.3. The 4-manifold X contains two subspaces which are naturally parametrized as $\mathbf{R} \times Y_0$ and $[2, \infty) \times Y$ and their intersection is $[2, \infty) \times Y_0$. Similarly, there are two subspaces diffeomorphic to $\mathbf{R} \times -Y'_0$ and $(-\infty, -2] \times Y'$, whose intersection is the subspace $(-\infty, -2] \times Y'_0$ of $(-\infty, -2] \times Y'$ which is identified with the subspace $[2, \infty) \times -Y'_0$ of $\mathbf{R} \times -Y'_0$ using the orientation preserving map $(t, y) \rightarrow (-t, y)$ with $(t, y) \in (-\infty, -2] \times Y'_0$. To avoid confusion in the rest of the paper, we write

$\{t\} \times -Y'_0$ for the subspace of $\mathbf{R} \times -Y'_0$ with $t \in \mathbf{R}$, and $\{t\} \times Y'$ for the subspace of $(-\infty, -2] \times Y'$ with $t \in (-\infty, -2]$. In particular, $\{-t\} \times -Y'_0$ is a subspace of $\{t\} \times Y'$ for any $t \in (-\infty, -2]$.

Let S be the surface given by the domain U_+ . Using the notation in (3.1), the boundary components U_∂ , η_+ and η'_+ of S respectively play the roles of γ , η_1 and η_2 . We associate the 3-manifold Lagrangians $L(Y, E)$, $L(Y', E')$ to the boundary components η_+ , η'_+ of U_+ . We write \mathbb{L} for these two Lagrangians together. The quintuple $q_s := (X, V, S, M, \mathbb{L})$ is called the *special quintuple associated to (Y, E) and (Y', E')* . The subspaces $(-\infty, -3] \times Y_\#$ and $[3, \infty) \times [-1, 1]$ are respectively called the gauge theoretic and symplectic ends of X and S . Moreover, the subspaces $[2, \infty) \times Y \subset X$ and $[0, 2] \times [2, \infty) \subset S$ together are called the mixed end associated to (Y, E) . The mixed end associated to (Y', E') is defined in an analogous way.

We shall need a slight modification of the mixed equation in (3.2) associated to the special quintuple q_s . First, we fix a family of compatible almost complex structures $\{J_{(s,\theta)}\}_{(s,\theta) \in U_+}$ on the symplectic manifold $\mathcal{M}(\Sigma, F)$ such that $J_{(s,\theta)}$ is equal to the standard complex structure J_* for $s \leq 1$ and is equal to J_θ , the complex structure given by Lemma 2.35 for $s \geq 1$. Moreover, $J_{s,\theta}$ is constant in the θ direction if $|\theta| > 2$. For a connection A on V and a map $u : U_+ \rightarrow \mathcal{M}(\Sigma, F)$, which satisfy matching and Lagrangian boundary conditions, we define the mixed equation as

$$\begin{cases} F_A^+ + (*_3 \nabla_{A_t} h)^+ + (*_3 \nabla_{A'_t} h')^+ = 0 \\ \frac{\partial u}{\partial s} + J_{(s,\theta)} \frac{\partial u}{\partial \theta} = 0 \end{cases} \quad (3.4)$$

Here the self-dual part of the 2-forms in the first equation are defined with respect to the Riemannian metric g_X . For each $t \in \mathbf{R}$, A_t (resp. A'_t) denotes the restriction of A to $\{t\} \times Y_0$ (resp. $\{t\} \times -Y'_0$), and the perturbation terms $(*_3 \nabla_{A_t} h)^+$ and $(*_3 \nabla_{A'_t} h')^+$ are defined as in (2.48) and are respectively supported in the interior of $\mathbf{R} \times Y_0$ and $\mathbf{R} \times Y'_0$. The Cauchy-Riemann equation in (3.4) is defined with respect to the family of complex structures $\{J_{(s,\theta)}\}_{(s,\theta) \in U_+}$.

3.2 Moduli spaces associated to special quintuples and the isomorphism \mathbf{N}

Working with the space of all solutions of (3.4) is unmanageable due to non-compactness of X and S , and we need to impose some decay conditions on the ends to obtain a well-behaved moduli space. Suppose (A, u) is a pair of a connection on V and a map $U_+ \rightarrow \mathcal{M}(\Sigma, F)$. The *analytical energy* of the pair (A, u) is defined as

$$\mathfrak{E}(A, u) := \int_X |F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A'_t} h'|^2 \text{dvol}_X + \int_{U_+} |du|_{J_{(s,\theta)}}^2 ds d\theta. \quad (3.5)$$

where $|du|_{J_{(s,\theta)}}^2 = \Omega(\frac{\partial u}{\partial s}, J_{(s,\theta)} \frac{\partial u}{\partial s}) + \Omega(\frac{\partial u}{\partial \theta}, J_{(s,\theta)} \frac{\partial u}{\partial \theta})$.

Proposition 3.6. *There is a positive real number δ_0 such that the following holds. Suppose (A, u) is a pair of an $L^2_{2,loc}$ connection on V and a continuous map $U_+ \rightarrow \mathcal{M}(\Sigma, F)$ such that du belongs to the Sobolev space $L^2_{1,loc}$. Suppose (A, u) is a solution of the mixed equation in (3.4), satisfies the matching and Lagrangian boundary conditions, and $\mathfrak{E}(A, u)$ is finite. Then u is smooth and there is a section g of $V \times_{ad} \text{SU}(2)$ such that $\tilde{A} := g^* A$ is also smooth. Moreover, the following properties for any positive integer l hold.*

- (i) There is a representative α for an element of \mathfrak{C}_G such that the difference $a := \tilde{A} - \pi_{\#}^*(\alpha)$, defined on the end $(-\infty, -3] \times Y_{\#}$, is in L_l^2 .
- (ii) There is an element β of \mathfrak{C}_S such that $\lim_{s \rightarrow \infty} u(s, \theta) = \beta$ and du is in L_{l-1}^2 .
- (iii) On the mixed cylinder associated to (Y, E) (resp. (Y', E')), there is an element $q \in L(Y, E)$ (resp. $q' \in L(Y', E')$) with a representative connection B on E (resp. B' on E') such that $\tilde{A} - \pi^*(B)$ (resp. $\tilde{A} - (\pi')^*(B')$) is in L_{l, δ_0}^2 , $\lim_{\theta \rightarrow \infty} u = q$ (resp. $\lim_{\theta \rightarrow -\infty} u = q'$) and du is in L_{l-1, δ_0}^2 .

The weighted Sobolev norms in Theorem 3.6 are defined as follow. Let $\tau : [2, \infty) \times Y \rightarrow \mathbf{R}^{\geq 0}$ be given by projection to the second factor. Then the $L_{l, \delta}^2$ norm of a function f on $[2, \infty) \times Y$ is the L^2 norm of $e^{\tau} \cdot f$. Similarly, the $L_{l, \delta}^2$ norm on $(-\infty, -2] \times Y'$ is defined using $\tau' : (-\infty, -2] \times Y' \rightarrow \mathbf{R}^{\geq 0}$ given by the magnitude of projection to the second factor. These definitions extend to sections of bundles in the obvious way. The $L_{l-1, \delta}^2$ norm of du over the ends $[0, 2] \times [2, \infty)$ and $[0, 2] \times (-\infty, -2]$ are also defined in a similar fashion. The proof of the above theorem will be given in Section 5 based on results of [DFL].

Theorem 3.6 can be used as a guide to define a configuration space where the mixed equation for the special quintuple is defined. Fix $\alpha \in \mathfrak{C}_G$ and $\beta \in \mathfrak{C}_S$. We assume that a connection on $E_{\#}$ representing α is fixed, and with a slight abuse of notation, we denote this connection by α . Let $\mathbf{A}(\alpha, \beta)$ be the space of all pairs (A, u) which are in $L_{l, loc}^2$, satisfy the matching and Lagrangian conditions, satisfy (i) and (ii) of Theorem 3.6 for the given α and β . Moreover, property (iii) of Theorem 3.6 is satisfied for some choice of q, q' and their representatives B, B' (which might vary from one element of $\mathbf{A}(\alpha, \beta)$ to another one) with δ_0 being replaced with a positive constant $\delta < \delta_0$, which will be fixed later. In particular, any element of $\mathbf{A}(\alpha, \beta)$ has finite analytical energy.

Suppose $\mathcal{G}(V)$ is the space of all sections g of $V \times_{ad} \text{SU}(2)$ such that for an element $(A_0, u_0) \in \mathbf{A}(\alpha, \beta)$, the 1-form $(\nabla_{A_0} g)g^{-1}$ is in $L_{l, loc}^2$ and its restriction to the end $(-\infty, -3] \times Y_{\#}$ is in L_l^2 . Moreover, there are $\mathfrak{g} \in \mathcal{G}(E)$ and $\mathfrak{g}' \in \mathcal{G}(E')$ such that the 1-forms $(\nabla_{A_0} g)g^{-1} - (\nabla_{A_0} \mathfrak{g})\mathfrak{g}^{-1}$ and $(\nabla_{A_0} g)g^{-1} - (\nabla_{A_0} \mathfrak{g}')\mathfrak{g}'^{-1}$ on $[2, \infty) \times Y$ and $(-\infty, -2] \times Y'$ are in $L_{l, \delta}^2$. Here we regard \mathfrak{g} and \mathfrak{g}' as gauge transformations over $[2, \infty) \times Y$ and $(-\infty, -2] \times Y'$ by pulling them back using the projection maps π and π' . There is an obvious map $f : \mathcal{G}(V) \rightarrow \mathcal{G}(E) \times \mathcal{G}(E')$. The group $\mathcal{G}(V)$ acts on $\mathbf{A}(\alpha, \beta)$ and the quotient space is denoted by $\mathbf{B}(\alpha, \beta)$. We may use the Sobolev norms to topologize the space $\mathbf{B}(\alpha, \beta)$ in the obvious way. In particular, if $[A_i, u_i] \in \mathbf{B}(\alpha, \beta)$ is convergent to $[A_0, u_0] \in \mathbf{B}(\alpha, \beta)$, then the points $q_i \in L(Y, E)$ and $q'_i \in L(Y', E')$ associated to $[A_i, u_i]$ are convergent to $q_0 \in L(Y, E)$ and $q'_0 \in L(Y', E')$ associated to $[A_0, u_0]$.

Remark 3.7. Note that the space $\mathbf{B}(\alpha, \beta)$ is essentially independent of the choice of a representative for α because any element of $\mathcal{G}(E_{\#})$ and the involution ι can be extended into V .

The spaces $\mathbf{A}(\alpha, \beta)$ and $\mathbf{B}(\alpha, \beta)$ are smooth infinite dimensional spaces. To state this claim in a more precise way, we need to introduce some Banach spaces.

Definition 3.8. Let $\tau : X \rightarrow \mathbf{R}$ be a smooth function on X whose restrictions to $(-\infty, -3] \times Y_{\#}$, $[2, \infty) \times Y$ and $(-\infty, -2] \times Y'$ are respectively equal to 0, projection to the first factor and the magnitude of the projection to the first factor. For a vector bundle E on X , the weighted Sobolev space $L_{k, \delta}^2(X, E)$ is defined as the space of sections s of E such that $e^{\tau} s$ is in the Sobolev space $L_k^2(X, E)$. For a vector

bundle E over U_+ , the weighted sobolev space $L_{k,\delta}^2(U_+, E)$ is defined in a similar way. Thus, roughly speaking, an element of $L_{k,\delta}^2(U_+, E)$ is in L_k^2 and is required to have exponential decay along the ends $[0, 2] \times [2, \infty)$ and $[0, 2] \times (-\infty, -2]$.

Definition 3.9. Let $(A, u) \in \mathbf{A}(\alpha, \beta)$ be a mixed pair which is asymptotic to (B, q) and (B', q') on the mixed ends associated to (Y, E) and (Y', E') . Define $E_{(A,u)}^k$ as the space of all

$$(\zeta, \nu) \in L_{k,loc}^2(X, \Lambda^1 \otimes V) \times L_{k,loc}^2(U_+, u^*TM(\Sigma, F))$$

such that

(i) $\zeta|_{(-\infty, -3] \times Y_\#}$ and $\nu|_{[3, \infty) \times [-1, 1]}$ have finite L_k^2 norms.

(ii) There are $b \in \mathcal{H}_h^1(Y; B)$ and $b' \in \mathcal{H}_{h'}^1(Y'; B')$ such that

$$\zeta - \pi^*(b)|_{[2, \infty) \times Y} \quad \text{and} \quad \zeta - \pi^*(b')|_{(-\infty, -2] \times Y'}$$

have finite $L_{k,\delta}^2$ norms. Let s and s' be tangent vectors to $\mathcal{M}(\Sigma, F)$ at the points q and q' given by restriction of b and b' to the boundary. Then

$$\nu - \pi^*(s)|_{[0, 2] \times [2, \infty)} \quad \text{and} \quad \nu - \pi^*(s')|_{[0, 2] \times (-\infty, -2]}$$

also have finite $L_{k,\delta}^2$ norms.

(iii) $*\zeta|_{U_\partial \times \Sigma} = 0$, $d_{A_\theta} \zeta_\theta = 0$ and $[\zeta_\theta] = \nu(0, \theta)$ where A_θ and ζ_θ are restrictions of A and ζ to $\{(0, \theta)\} \times \Sigma \subset X$, and $[\zeta_\theta]$ is the element of $\mathcal{H}^1(\Sigma; A_\theta)$ represented by ζ_θ .

(iv) $\nu|_{\eta_+} \in u^*TL(Y, E)$, $\nu|_{\eta'_+} \in u^*TL(Y', E')$.

The proof of the following proposition is discussed in Subsection 4.1.

Proposition 3.10. *The space $\mathbf{A}(\alpha, \beta)$ is a Banach manifold and $\mathcal{G}(V)$ is a Banach Lie group which acts smoothly on $\mathbf{A}(\alpha, \beta)$, and the stabilizer of any element of $\mathbf{A}(\alpha, \beta)$ is ± 1 . The quotient space $\mathbf{B}(\alpha, \beta)$ is also a Banach manifold. Let $(A, u) \in \mathbf{A}(\alpha, \beta)$ be a mixed pair which is asymptotic to (B, q) and (B', q') on the mixed ends associated to (Y, E) and (Y', E') . Then the tangent space to the point $[A, u]$ of $\mathbf{B}(\alpha, \beta)$ can be identified with the kernel of the surjective operator*

$$d_A^* : E_{(A,u)}^l \rightarrow L_{l-1,\delta}^2(X, V).$$

For any element $[A, u]$ of the configuration space $\mathbf{B}(\alpha, \beta)$, define the *topological energy* of $[A, u]$ as

$$\mathcal{E}(A, u) := \frac{1}{8\pi^2} \int_X \text{tr} \left((F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A'_t} h') \wedge (F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A'_t} h') \right) + \frac{1}{4\pi^2} \int_{U_+} u^* \Omega.$$

Note that if $[A, u]$ satisfy the mixed equation in (3.4), then $\mathfrak{E}(A, u) = 8\pi^2 \mathcal{E}(A, u)$. Thus, the topological energy is non-negative for the solutions of (3.4). As it is justified by the following lemma, topological energy can be regarded as a soft variation of analytical energy.

Lemma 3.11. *The topological energy $\mathcal{E}(A, u)$ depends only on the connected component of $\mathbf{B}(\alpha, \beta)$ that contains (A, u) .*

Proof. It suffices to show that for a 1-parameter family $(A(s), u(s))$ of elements of $\mathbf{A}(\alpha, \beta)$, depending smoothly on s , $\mathcal{E}(A(s), u(s))$ is constant with respect to s . Since this is equivalent to vanishing of $\frac{d\mathcal{E}(A(s), u(s))}{ds}$, the claim follows if the expression

$$\int_X \text{tr} \left((F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A'_t} h') \wedge (d_A \zeta + *_3 \text{Hess}_{A_t} h(\zeta_t) + *_3 \text{Hess}_{A'_t} h'(\zeta'_t)) \right) + \int_{U_+} d_{\nu} u^* \Omega \quad (3.12)$$

vanishes for any $(A, u) \in \mathbf{A}(\alpha, \beta)$ and $(\zeta, \nu) \in E^l_{(A, u)}$. Here ζ_t and ζ'_t are respectively restrictions of ζ to $\{t\} \times Y_0$ and $\{t\} \times -Y'_0$, and $d_{\nu} u^* \Omega$ is the exterior derivative of the 1-form

$$\Omega(\nu, \frac{\partial u}{\partial s}) ds + \Omega(\nu, \frac{\partial u}{\partial \theta}) d\theta.$$

Without loss of generality, in the following we may assume that the restrictions of A to $\mathbf{R} \times Y_0$ and $\mathbf{R} \times Y'_0$ are in temporal gauge.

We start by analyzing the first integral of (3.12) over the sub-manifold $\mathbf{R} \times Y_0$ of X . Note that h' vanishes on this space. Therefore, the integrand over $\mathbf{R} \times Y_0$ can be simplified to

$$\text{tr} \left(F_A \wedge d_A \zeta + dt \wedge \frac{dA_t}{dt} \wedge *_3 \text{Hess}_{A_t} h(\zeta_t) + *_3 \nabla_{A_t} h \wedge dt \wedge \frac{d\zeta_t}{dt} \right).$$

By applying Stokes theorem, Bianchi identity and Lemma 2.13, the integral of the above expression over $\mathbf{R} \times Y_0$ is equal to

$$\begin{aligned} & \int_{\mathbf{R} \times Y_0} d \text{tr}(F_A \wedge \zeta) + \int_{\mathbf{R}} dt \left(\int_{Y_0} \text{tr}(*_3 \text{Hess}_{A_t} h(\frac{dA_t}{dt}) \wedge \zeta_t + *_3 \nabla_{A_t} h \wedge \frac{d\zeta_t}{dt} \right) = \\ & = \int_{\mathbf{R} \times \partial Y_0} \text{tr}(F_A \wedge \zeta) + \lim_{t \rightarrow \infty} \int_{Y_0} \text{tr}(F_{A_t} \wedge \zeta_t) + \int_{\mathbf{R}} dt \frac{d}{dt} \left(\int_{Y_0} \text{tr}(*_3 \nabla_{A_t} h \wedge \zeta_t) \right) \\ & = \int_{\mathbf{R} \times \partial Y_0} \text{tr}(F_A \wedge \zeta) + \lim_{t \rightarrow \infty} \int_{Y_0} \text{tr}((F_{A_t} + *_3 \nabla_{A_t} h) \wedge \zeta_t). \end{aligned}$$

Note that we did not include the integrals of $\text{tr}(F_{A_t} \wedge \zeta_t)$ and $\text{tr}(*_3 \nabla_{A_t} h \wedge \zeta_t)$ over Y_0 as $t \rightarrow -\infty$ in the second and the third identities because of the decay of ζ_t on the gauge theoretical end. As $t \rightarrow \infty$, A_t and ζ_t are convergent to B and an element of $\mathcal{H}_h^1(Y; B)$. In particular, the integral of $\text{tr}((F_{A_t} + *_3 \nabla_{A_t} h) \wedge \zeta_t)$ over Y_0 as $t \rightarrow \infty$ is trivial. Consequently, the contribution of $\mathbf{R} \times Y_0$ to (3.12) equals the integral of $\text{tr}(F_A \wedge \zeta)$ over $\mathbf{R} \times \partial Y_0$. A similar claim holds about $\mathbf{R} \times -Y'_0$.

The first integrand of (3.12) over $U_- \times \Sigma$ simplifies to $\text{tr}(F_A \wedge d_A \zeta)$. Thus, by Stokes theorem and Bianchi identity this integral is equal to:

$$\int_{\partial(U_- \times \Sigma)} \text{tr}(F_A \wedge \zeta).$$

Assuming that the restriction of A to $U_{\partial} \times \Sigma$ has the form $A_{\theta} + \phi ds + \psi d\theta$, we can summarize our simplifications as

$$\begin{aligned} \int_X \operatorname{tr} \left((F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A'_t} h') \wedge (d_A \zeta + *_3 \operatorname{Hess}_{A_t} h(\zeta_t) + *_3 \operatorname{Hess}_{A'_t} h'(\zeta'_t)) \right) &= \int_{U_{\partial} \times \Sigma} \operatorname{tr}(F_A \wedge \zeta) \\ &= \int_{-\infty}^{\infty} d\theta \left(\int_{\Sigma} \operatorname{tr}(\partial_{\theta} A_{\theta} \wedge \zeta_{\theta}) - \operatorname{tr}(d_{A_{\theta}} \psi \wedge \zeta_{\theta}) \right) \\ &= \int_{-\infty}^{\infty} d\theta \int_{\Sigma} \operatorname{tr}(\partial_{\theta} A_{\theta} \wedge \zeta_{\theta}) \end{aligned} \quad (3.13)$$

The second identity is a consequence of the Stokes theorem and the assumption $d_{A_{\theta}} \zeta_{\theta} = 0$. Another application of the Stokes theorem also shows that the second integral in (3.12) can be simplified to

$$- \int_{U_{\partial}} \iota_{\nu} u^* \Omega = - \int_{-\infty}^{\infty} d\theta \int_{\Sigma} \operatorname{tr} \left(\frac{\partial u}{\partial \theta} \wedge \nu(0, \theta) \right). \quad (3.14)$$

Using the matching conditions for (A, u) and (ζ, ν) , the expressions (3.13) and (3.14) cancel out each other and (3.12) vanishes. \square

In fact, Lemma 3.11 can be strengthened as follows. The proof of this lemma will be given in Subsection 4.5.

Lemma 3.15. *If $[A, u], [A', u'] \in \mathbf{B}(\alpha, \beta)$, then $2(\mathcal{E}(A, u) - \mathcal{E}(A', u'))$ is an integer. Moreover, if $\mathcal{E}(A, u) = \mathcal{E}(A', u')$, then $[A, u]$ and $[A', u']$ belong to the same connected component of $\mathbf{B}(\alpha, \beta)$.*

Let $\mathbf{M}(\alpha, \beta)$ be the subspace of $\mathbf{B}(\alpha, \beta)$ given by the solutions of (3.4). The local behavior of this moduli space around a solution (A, u) is governed by the *linearization* of the mixed equation. Define a linear operator

$$L_{A,u} : E_{(A,u)}^l \rightarrow L_{l-1,\delta}^2(X, \Lambda^+ \otimes V) \oplus L_{l-1,\delta}^2(U_+, u^* T\mathcal{M}(\Sigma, F)),$$

as

$$L_{(A,u)}(\zeta, \nu) := (d_A^+ \zeta + (*_3 \operatorname{Hess}_{A_t} h(\zeta_t))^+ + (*_3 \operatorname{Hess}_{A'_t} h'(\zeta'_t))^+, \mathcal{D}_u(\nu)), \quad (3.16)$$

where ζ_t (resp. ζ'_t) is the restriction of ζ to $\{t\} \times Y_0$ (resp. $\{t\} \times -Y'_0$), and analogous to (2.34), \mathcal{D}_u is the linearization of the Cauchy-Riemann operator

$$\mathcal{D}_u(\nu) := \nabla_{\theta} \nu - J_{s,\theta}(u) \nabla_s \nu - (\nabla_{\nu} J_{s,\theta}) \frac{du}{ds},$$

with ∇ being the Levi-Civita connection again. Using Proposition 3.10, the linearization of the mixed equation is given by the restriction of $L_{(A,u)}$ to the kernel of the operator d_A^* . It is more convenient to combine these operators and define the *mixed operator*

$$\mathcal{D}_{(A,u)} : E_{(A,u)}^l \rightarrow L_{l-1,\delta}^2(X, V) \oplus L_{l-1,\delta}^2(X, \Lambda^+ \otimes V) \oplus L_{l-1,\delta}^2(U_+, u^* T\mathcal{M}(\Sigma, F))$$

as $\mathcal{D}_{(A,u)} := (d_A^*, L_{(A,u)})$. Since the operator d_A^* is surjective, the kernels and co-kernels of the linearization of the mixed equation and $\mathcal{D}_{(A,u)}$ can be identified with each other.

Definition 3.17. An element $[A, u] \in \mathbf{B}(\alpha, \beta)$ is *regular* if $\mathcal{D}_{(A,u)}$ is surjective.

The proof of the following proposition will be given in Subsection 4.2:

Proposition 3.18. *After possibly decreasing the constant δ_0 of Theorem 3.6, the following claim holds for any $\delta < \delta_0$. Suppose $\mathbf{A}(\alpha, \beta)$ is defined using δ and $[A, u]$ is a smooth element of $\mathbf{A}(\alpha, \beta)$ that satisfies property (iii) of Theorem 3.6. Then the operator $\mathcal{D}_{(A,u)}$ is Fredholm. If (A, u) represents a regular point of the mixed moduli space $\mathbf{M}(\alpha, \beta)$, then $\mathbf{M}(\alpha, \beta)$ is a smooth manifold of dimension $\text{index}(\mathcal{D}_{(A,u)})$ in a neighborhood of $[A, u]$.*

For the rest of this section, we assume that the constant δ used in the definition of $\mathbf{A}(\alpha, \beta)$ is given by Proposition 3.18.

To each $\alpha \in \mathcal{C}_S \cong \mathcal{C}_G$, we can associate an element of the moduli space $\mathbf{M}(\alpha, \alpha)$ as we explain now. Fix a representative connection for α which is also denoted by α . By definition there are connections B and B' on (Y, E) and (Y', E') which satisfy the equations $\phi_h(B) = 0$ and $\phi_{h'}(B') = 0$ of Subsection 2.3, and the restrictions of B and B' to collar neighborhoods of the boundaries of Y and Y' are given by the pullbacks of α . Then the pullbacks of B to $\mathbf{R} \times Y_0$, B' to $\mathbf{R} \times Y'_0$ and α to $\Sigma \times U_-$ give rise to a connection A_α on X . We also define $u_\alpha : U_+ \rightarrow \mathcal{M}(\Sigma, F)$ to be the constant map given by α . The pair (A_α, u_α) , which is called the *constant pair*, clearly satisfies the mixed equation in (3.4) and hence it represents an element of the moduli space $\mathbf{M}(\alpha, \alpha)$. Notice that the topological energy of any constant pair is zero. In fact, constant pairs are characterized as solutions of (3.4) with vanishing topological energy (or equivalently analytical energy). The proof of the following proposition will be given in Subsection 4.3.

Proposition 3.19. *The index of the mixed operator $\mathcal{D}_{(A_\alpha, u_\alpha)}$ is 0. Moreover, the kernel and the cokernel of the operator $\mathcal{D}_{(A_\alpha, u_\alpha)}$ are trivial.*

The following proposition generalizes the index computation of Proposition 3.19 to the case of arbitrary mixed pairs.

Proposition 3.20. *The index of the mixed operator associated to a mixed pair $(A, u) \in \mathbf{A}(\alpha, \beta)$ satisfies*

$$\text{index}(\mathcal{D}_{(A,u)}) = 8\mathcal{E}(A, u) + \varepsilon(\beta) - \varepsilon(\alpha), \quad (3.21)$$

where the constants $\varepsilon(\alpha)$ and $\varepsilon(\beta)$ are given by Proposition 2.53.

The proof of the above proposition will be given in Subsection 4.5. Notice that the first part of Proposition 3.19 is a special case of this proposition. However, the proof of Proposition 3.20 relies on Proposition 3.19 as an essential input. The other input is the *mixed shifting operator*, which is introduced in Subsection 4.4.

For any integer d , let $\mathbf{M}(\alpha, \beta)_d$ denote the subspace of $\mathbf{M}(\alpha, \beta)$ consisting of index d solutions. Suppose (A, u) represents an element of $\mathbf{M}(\alpha, \beta)_0$. Since the topological energy of (A, u) is non-negative, the index formula implies that $\varepsilon(\alpha) \geq \varepsilon(\beta)$, and the equality holds if and only if $\alpha = \beta$. The latter claim holds because any element of $\mathbf{M}(\alpha, \beta)$ with vanishing topological energy is a constant pair. In summary, $\mathbf{M}(\alpha, \beta)_0$ is non-empty only if $\varepsilon(\alpha) > \varepsilon(\beta)$ or $\alpha = \beta$. In the latter case, there is exactly one element in $\mathbf{M}(\alpha, \alpha)_0$ which is regular.

We have shown that the constant solutions of the mixed solution are regular. However, not all elements of $\mathbf{M}(\alpha, \beta)$ are regular. In Section 6.2, we introduce perturbations of mixed equation by deforming the family of almost complex structures $\{J_{(s,\theta)}\}_{(s,\theta) \in U_+}$ and adding a term to the ASD equation:

$$\begin{cases} F_A^+ + (*_3 \nabla_{A_t} h)^+ + (*_3 \nabla_{A'_t} h')^+ + \eta(A) = 0 \\ \frac{\partial u}{\partial s} + J_{(s,\theta)} \frac{\partial u}{\partial \theta} = 0 \end{cases} \quad (3.22)$$

Here the term $\eta(A)$ is invariant with respect to the action of $\mathcal{G}(V)$. Thus, the solutions of the above equations determine a subspace of $\mathbf{B}(\alpha, \beta)$, denoted by $\mathbf{M}_\eta(\alpha, \beta)$.

Proposition 3.23. *There are secondary perturbations of the mixed equation satisfying the following properties.*

- (i) *There is a compact subset $K_- \subset U_-$ away from the matching line U_∂ such that $\eta(A)$ depends on $A|_{K_-}$ and is supported in K_- . There is a compact subset $K_+ \subset U_+$ away from the matching line U_∂ such that the deformation of $J_{(s,\theta)}$ is trivial on the complement of K_+ . In particular, the deformed almost complex structure agrees with the standard complex structure J_* in a neighborhood of the matching line.*
- (ii) *The moduli spaces with expected dimension at most 3 are regular.*
- (iii) *Any element of $\mathbf{M}_\eta(\alpha, \beta)$ has non-negative topological energy. The moduli space $\mathbf{M}_\eta(\alpha, \beta)_0$ is non-empty only if $\varepsilon(\alpha) > \varepsilon(\beta)$ or $\alpha = \beta$. Moreover, $\mathbf{M}_\eta(\alpha, \alpha)_0$ consists of only one element for each $\alpha \in \mathfrak{C}_S$.*

This proposition is proved in Section 6.2 after introducing an appropriate family of perturbation terms η . For any solution (A, u) of (3.22) we have

$$\mathfrak{E}(A, u) = 8\pi^2 \mathcal{E}(A, u) + 2\|\eta(A)\|_{L^2(X)}^2. \quad (3.24)$$

Proposition 3.25. *Let η be given by Proposition 3.23. Then the moduli spaces $\mathbf{M}_\eta(\alpha, \beta)_d$ with $d \leq 3$ are orientable d -dimensional manifolds.*

Proposition 3.26. *The perturbation η in Proposition 3.23 can be chosen such that the following holds.*

- (i) *The moduli spaces of the form $\mathbf{M}_\eta(\alpha, \beta)_0$ are compact.*
- (ii) *The moduli spaces of the form $\mathbf{M}_\eta(\alpha, \beta)_1$ can be compactified into compact 1-manifolds by adding points in correspondence to the 0-dimensional spaces*

$$\mathbf{M}_\eta(\alpha, \gamma)_0 \times \check{\mathbf{M}}_S(\gamma, \beta)_p, \quad \check{\mathbf{M}}_G(\alpha, \gamma)_p \times \mathbf{M}_\eta(\gamma, \beta)_0, \quad (3.27)$$

where $\gamma \in \mathfrak{C}_G \cong \mathfrak{C}_S$, and in both cases p denotes a path of index 1.

Moreover, the orientations of the moduli spaces $\mathbf{M}_\eta(\alpha, \beta)_d$ with $d \leq 1$ provided by Proposition 3.25 can be chosen such that the induced orientation on the boundary components of the compactified moduli space $\mathbf{M}_\eta(\alpha, \beta)_1$ (using outward-normal-first convention) agree with the product orientation on the first term in (3.27) and disagrees with the induced orientation on the second term in (3.27).

The essential step in the orientability of mixed moduli spaces is discussed in Subsection 4.7. The proof of the compactness claims in Proposition 3.26 uses results of [DFL], and is given in Subsection 5.4. The rest of the above two propositions is verified in Section 6.2.

We define a homomorphism $\mathbf{N} : C_G(Y_\#, E_\#) \rightarrow C_S((Y, E), (Y', E'))$ using 0-dimensional moduli spaces $\mathbf{M}_\eta(\alpha, \beta)_0$. First we pick the perturbation η such that the claims of Propositions 3.23 and 3.26 hold. Then define

$$\mathbf{N}(\alpha) := \sum_{\beta \in \mathfrak{C}_S} \# \mathbf{M}_\eta(\alpha, \beta)_0 \cdot \beta, \quad (3.28)$$

where $\# \mathbf{M}_\eta(\alpha, \beta)_0$ denotes the signed count of the points in the 0-dimensional moduli space $\mathbf{M}_\eta(\alpha, \beta)_0$. Our main theorem is a consequence of the following result.

Theorem 3.29. *The map \mathbf{N} is an isomorphism and a chain map.*

Proof. By Proposition 3.23, $\mathbf{M}_\eta(\alpha, \beta)_0$ is non-empty only if $\varepsilon(\alpha) > \varepsilon(\beta)$ or $\alpha = \beta$. In the latter case, $\mathbf{M}_\eta(\alpha, \beta)_0$ consists of only one element. Thus \mathbf{N} is an isomorphism. The chain map property of \mathbf{N} follows from a standard argument using the second part of Proposition 3.26. \square

4 Linear analysis

In this section, we verify several claims in Sections 2 and 3 related to the linear analysis of the mixed equation. During this section (Y, E) and (Y', E') are fixed as in Subsection 2.1, and we fix Lagrangian 3-manifolds associated to these pairs that have transversal intersection and the claim of Lemma 2.56 holds. We continue to drop h and h' from our notations for the 3-manifolds Lagrangians, and denote them by $L(Y, E)$ and $L(Y', E')$.

4.1 The configuration space of mixed pairs

In Subsection 2.2, we introduced the space of connections $\mathcal{A}(\Sigma, F)$, which is an affine space modeled on the Banach space $\mathfrak{B} := L_{l-1}^2(\Sigma, \Lambda^1 \otimes F)$. For any positive constant ε , we write $\mathfrak{B}_{<\varepsilon}$ for the subspace of elements of \mathfrak{B} with L^2 norm less than ε . We write $\mathcal{A}_{\text{fl}}(\Sigma, F)$ for the subspace of $\mathcal{A}(\Sigma, F)$ given by flat connections. If we want to be specific about the Sobolev exponent in the definition of $\mathcal{A}(\Sigma, F)$ and $\mathcal{A}_{\text{fl}}(\Sigma, F)$, we denote them by $\mathcal{A}^{l-1}(\Sigma, F)$ and $\mathcal{A}_{\text{fl}}^{l-1}(\Sigma, F)$. The space of L_l^2 gauge transformations of F are also denoted by $\mathcal{G}_l(F)$. The following lemma provides an exponential map for the tangent vectors of $\mathcal{A}(\Sigma, F)$, which is invariant with respect to $\mathcal{G}(F)$ and induces an exponential map on $\mathcal{A}_{\text{fl}}(\Sigma, F)$.

Lemma 4.1. *There are a positive constant ε and a smooth map $E : \mathcal{A}(\Sigma, F) \times \mathfrak{B}_{<\varepsilon} \rightarrow \mathcal{A}(\Sigma, F)$ satisfying the following properties.*

- (i) E is $\mathcal{G}(F)$ -equivariant where we use the diagonal action on $\mathcal{A}(\Sigma, F) \times \mathfrak{B}_{<\varepsilon}$.
- (ii) $E(\sigma, 0) = \sigma$.

(iii) For any $\sigma \in \mathcal{A}(\Sigma, F)$, the differential $D_{(\sigma,0)}\mathbb{E}|_{\{0\} \times \mathfrak{B}} : \mathfrak{B} \rightarrow \mathfrak{B}$ is identity. The map \mathbb{E} determines a diffeomorphism from $\{\sigma\} \times \mathfrak{B}_{<\varepsilon}$ to a neighborhood of σ .

(iv) For any $\alpha \in \mathcal{A}_{\mathfrak{H}}(\Sigma, F)$ and $c \in \mathfrak{B}_{<\varepsilon}$ with $d_\alpha c = 0$, the connection $\mathbb{E}(\sigma, c)$ is flat. Moreover, if $c \in \mathcal{H}^1(\Sigma; \alpha)$ and $c' := c + d_\alpha \zeta$ with $\zeta \in L_1^2(\Sigma, F)$, then $\mathbb{E}(\alpha, c') := g^* \mathbb{E}(\alpha, c)$ where g is obtained from exponentiating ζ .

We may restrict a map \mathbb{E} provided by Lemma 4.1 to the configuration of flat connections $\mathcal{A}_{\mathfrak{H}}(\Sigma, F)$, and then use gauge equivariance of \mathbb{E} to obtain a map from $T_\varepsilon \mathcal{M}(\Sigma, F)$ to $\mathcal{M}(\Sigma, F)$, where $T_\varepsilon \mathcal{M}(\Sigma, F)$ denotes tangent vectors to $\mathcal{M}(\Sigma, F)$ with length at most ε . In fact, it is useful to fix one such map before constructing \mathbb{E} . To do this, let $e : T\mathcal{M}(\Sigma, F) \rightarrow \mathcal{M}(\Sigma, F)$ be the exponential map with respect to the chosen metric on $\mathcal{M}(\Sigma, F)$.

Lemma 4.2. *There is a constant κ such that the following holds. Suppose U_κ denotes the subspace of L_1^2 connections σ on F with $\|F_\sigma\|_{L^2} < \kappa$. Then there is a smooth $\mathcal{G}(F)$ -equivariant map*

$$P : U_\kappa \rightarrow \mathcal{A}_{\mathfrak{H}}^1(\Sigma, F) \times L_2^2(\Sigma, F)$$

such that if $P(\sigma) = (\alpha, \zeta)$, then $\sigma = \alpha + *d_\alpha \zeta$. Moreover, if σ is in L_k^2 for $k \geq 1$, then $\alpha \in L_k^2$ and $\zeta \in L_{k+1}^2$.

Proof. Consider the $\mathcal{G}(\Sigma, F)$ -equivariant map

$$\Psi : \mathcal{A}_{\mathfrak{H}}^1(\Sigma, F) \times L_2^2(\Sigma, F) \rightarrow \mathcal{A}^1(\Sigma, F)$$

given by $\Psi(\alpha, \zeta) = \alpha + *d_\alpha \zeta$. Inverse function theorem and Uhlenbeck compactness theorem imply that there are $\kappa > 0$ and a $\mathcal{G}_2(F)$ -invariant neighborhood V of $\mathcal{A}_{\mathfrak{H}}^1(\Sigma, F) \times \{0\}$ such that Ψ induces a diffeomorphism from V to U_κ . Then we define $P : U_\kappa \rightarrow \mathcal{A}_{\mathfrak{H}}^1(\Sigma, F) \times L_2^2(\Sigma, F)$ to be the inverse of this map. Now suppose $\sigma \in U_\kappa$ is an L_k^2 connection with $k \geq 2$ and $P(\sigma) = (\alpha, \zeta)$. There is an L_2^2 automorphism g of F such that $\alpha' = g^* \alpha$ is a smooth flat connection. Moreover, $F_{g^* \sigma} = g F_\sigma g^{-1}$ is in L_1^2 , and if $\zeta' := g^*(\zeta)$, then

$$d_{\alpha'} * d_{\alpha'} \zeta' = - * d_{\alpha'} \zeta' \wedge * d_{\alpha'} \zeta' + F_{g^* \sigma}.$$

By applying elliptic regularity for the Laplacian operator $d_{\alpha'} * d_{\alpha'}$ twice, we may conclude that ζ' is in L_3^2 and hence $g^* \sigma$ is in L_2^2 . This implies that g is in fact an L_3^2 gauge transformation of F , α is an L_2^2 flat connection and ζ is in L_3^2 . Iterations of the above argument shows that g is in fact an L_{k+1}^2 gauge transformation, α is an L_k^2 flat connection and ζ is in L_{k+1}^2 . \square

Proof of Lemma 4.1. First we define $\mathbb{E}(\alpha, c)$ in the case that α belongs to $\mathcal{A}_{\mathfrak{H}}(\Sigma, F)$. The 1-form c can be uniquely decomposed as

$$c = c_0 + d_\alpha \zeta + *d_\alpha \zeta'$$

with $c_0 \in \mathcal{H}^1(\Sigma; \alpha)$, $\zeta, \zeta' \in L_1^2(\Sigma, F)$. The 1-form c_0 determines an element of $T_{[\alpha]} \mathcal{M}(\Sigma, F)$ and

$$\gamma(t) := e([\alpha], tc) \in \mathcal{M}(\Sigma, F)$$

defines a path from $[\alpha]$ to $e([\alpha], c)$. Let $\tilde{\gamma} : [0, 1] \rightarrow \mathcal{A}_{\mathfrak{H}}(\Sigma, F)$ be the unique path satisfying

- (i) $\tilde{\gamma}(t)$ is a flat connection representing $\gamma(t)$;
- (ii) $d_{\tilde{\gamma}(t)}^* \left(\frac{d}{dt} \tilde{\gamma}(t) \right) = 0$.

Let also g be the gauge transformation in $\mathcal{G}(F)$ given by exponentiating ζ . Then we define

$$E(\alpha, c) = g^* \tilde{\gamma}(1) + *d_\alpha \zeta'.$$

Thus, we obtain a map $E(\alpha, c)$, for flat α , that satisfies properties (i)-(iv). Compactness of $\mathcal{M}(\Sigma, F)$ and the inverse function theorem can be used to find ε such that for any $\alpha \in \mathcal{A}_{\text{fl}}(\Sigma, F)$ and $t \in [0, 1]$ the map

$$c \rightarrow \alpha + c + t(E(\alpha, c) - \alpha - c)$$

sends $\mathfrak{B}_{<\varepsilon}$ to a neighborhood of $\alpha \in \mathcal{A}(\Sigma, F)$ by a diffeomorphism.

Next, we extend $E(\sigma, c)$ to the case that σ is an arbitrary element of $\mathcal{A}(\Sigma, F)$. Suppose $\tau : [0, 1] \rightarrow [0, 1]$ is a smooth function, which is equal to 1 in a neighborhood of 0 and evaluates to 0 in a neighborhood of 1. Let U_κ be given by Lemma 4.2. Suppose $\sigma \in U_\kappa$ and $P(\sigma) = (\alpha, \zeta)$. For any $c \in \mathfrak{B}_{<\varepsilon}$ define

$$E(\sigma, c) = \sigma + c + \tau(\kappa^{-1} \|F(\sigma)\|_{L^2})(E(\alpha, c) - \alpha - c). \quad (4.3)$$

and extend (4.3) to the case that $\sigma \in \mathcal{A}(\Sigma, F) \setminus U_\kappa$ as $E(\sigma, c) = \sigma + c$. \square

Next, we need deformations of e , which are well-behaved with respect to $L(Y, E)$ and $L(Y', E')$.

Lemma 4.4. *For any $-1 \leq s \leq 1$, there is a smooth map $e_s : T\mathcal{M}(\Sigma, F) \rightarrow \mathcal{M}(\Sigma, F)$, depending smoothly on s , such that the following properties hold.*

- (i) $e_0 = e$.
- (ii) e_s maps the zero section of $T\mathcal{M}(\Sigma, F)$ to $\mathcal{M}(\Sigma, F)$ by the identity map
- (iii) The derivative of e_s at any point x in the zero section and along the fiber $T_x\mathcal{M}(\Sigma, F)$ is given by the identity map.
- (iv) e_1 maps the subspace $TL(Y, E)$ of $T\mathcal{M}(\Sigma, F)$ to $L(Y, E)$ and e_{-1} maps the subspace $TL(Y', E')$ of $T\mathcal{M}(\Sigma, F)$ to $L(Y', E')$.

By applying the argument of Lemma 4.1 to the family of maps e_s provided by the above lemma, we may construct a family of maps $E_s : \mathcal{A}(\Sigma, F) \times \mathfrak{B}_{<\varepsilon} \rightarrow \mathcal{A}(\Sigma, F)$, which satisfies the properties (i)-(iv) of Lemma 4.1. Moreover, if α is a flat connection on F representing an element of $L(Y, E)$ and c is a d_α -closed 1-form representing a tangent vector to $L(Y, E)$, then $E_1(\sigma, c)$ also represents an element in $L(Y, E)$. The map E_{-1} has a similar property with respect to $L(Y', E')$.

Proof. Fix a metric on $\mathcal{M}(\Sigma, F)$ such that $L(Y, E)$ is totally geodesic with respect to this metric. Then the exponential map with respect to this metric gives e_1 . A homotopy from this metric and the standard Riemannian metric on $\mathcal{M}(\Sigma, F)$ can be used in a similar way to define the maps e_t for $t \in [0, 1]$. The maps e_t for $t \in [-1, 0]$ can be constructed in an analogous way. \square

Our next goal is to give a chart for a neighborhood of a mixed pair $(A, u) \in \mathbf{A}(\alpha, \beta)$. Before giving a description of such a chart, we need to fix another additional piece of data. The mixed pair (A, u) is convergent to pairs (B, q) and (B', q') as $\theta \rightarrow \infty$ and $\theta \rightarrow -\infty$, where $q \in L(Y, E)$, $q' \in L(Y', E')$, and B, B' are respectively connections on E, E' representing q, q' . The restrictions of B, B' to Σ is denoted by α, α' . Let $b \in \mathcal{H}_h^1(Y; B)$ and c denote the restriction of b to Σ . Then $\alpha_b := E_1(\alpha, c)$ is a flat connection on F which represents an element of $L(Y, E) \subset \mathcal{M}(\Sigma, F)$. After possibly decreasing ε , we fix a connection B_b , for $|b| < \varepsilon$, such that

(i) B_b depends smoothly on b ;

(ii) $B_0 = B$;

(iii) B_b represents an element of $L(Y, E)$, and its restriction to the boundary is equal to α_b .

Similarly, we fix a smooth family of connections $\{B_{b'}\}$ for $b' \in \mathcal{H}_h^1(Y'; B')$ with $|b'| < \varepsilon$.

Suppose $B_{(A,u)}^l$ is the space of all $\zeta \in L_{l,loc}^2(X, \Lambda^1 \otimes V)$, $\nu \in L_{l,loc}^2(U_+, u^*T\mathcal{M}(\Sigma, F))$, which satisfy the following properties.

(i) $\zeta|_{(-\infty, -3] \times Y_\#}$ and $\nu|_{[3, \infty) \times [-1, 1]}$ have finite L_l^2 norms.

(ii) There are $b \in \mathcal{H}_h^1(Y; B)$ and $b' \in \mathcal{H}_h^1(Y'; B')$ such that

$$\zeta - \pi^*(b)|_{Y \times [2, \infty)} \quad \text{and} \quad \zeta - \pi^*(b')|_{Y' \times (-\infty, -2]}$$

have finite $L_{l,\delta}^2$ norms where δ is a small positive constant, which will be fixed in the next subsection. Let s and s' be tangent vectors to $\mathcal{M}(\Sigma, F)$ at the points q and q' given by restrictions of b and b' to the boundary. Then

$$\nu - \pi^*(s)|_{[0, 2] \times [2, \infty)} \quad \text{and} \quad \nu - \pi^*(s')|_{[0, 2] \times (-\infty, -2]}$$

also have finite $L_{l,\delta}^2$ norms.

(iii) $d_{A_\theta} \zeta_\theta = 0$ and $[\zeta_\theta] = \nu(0, \theta)$ where A_θ and ζ_θ are restrictions of A and ζ to $\{(0, \theta)\} \times \Sigma \subset X$, and $[\zeta_\theta]$ is the element of $\mathcal{H}^1(\Sigma; A_\theta)$ represented by ζ_θ .

(iv) $\nu|_{\eta_+} \in u^*TL(Y, E)$, $\nu|_{\eta'_+} \in u^*TL(Y', E')$.

Then $B_{(A,u)}^l$ is a Banach space where the norm is defined as

$$\begin{aligned} |(\zeta, \nu)|_{B_{(A,u)}^l} := & \|\zeta\|_{L_l^2(X^\circ)} + \|\nu\|_{L_l^2(U_+^\circ)} + \|\zeta - \pi^*(b)\|_{L_{l,\delta}^2(Y \times [2, \infty))} + \|\zeta - \pi^*(b')\|_{L_{l,\delta}^2(Y' \times (-\infty, -2])} + \\ & + \|\nu - \pi^*(s)\|_{L_{l,\delta}^2([0, 2] \times [2, \infty))} + \|\nu - \pi^*(s')\|_{L_{l,\delta}^2([0, 2] \times (-\infty, -2])} + |s| + |s'| \end{aligned} \quad (4.5)$$

with

$$X^\circ := X \setminus Y \times [2, \infty) \setminus Y' \times (-\infty, -2], \quad U_+^\circ := U_+ \setminus [0, 2] \times [2, \infty) \setminus [0, 2] \times (-\infty, -2].$$

In the following, we fix a constant κ_0 such that if $|(\zeta, \nu)|_{B_{(A,u)}^l} < \kappa_0$, then for any $(s, \theta) \in U_-$, the restriction of ζ to $\{(s, \theta)\} \times \Sigma$ belongs to $\mathfrak{B}_{<\varepsilon}$ and for any $(s, \theta) \in U_-$, $\nu(s, \theta) \in T_\varepsilon \mathcal{M}(\Sigma, F)$.

A neighborhood of (A, u) in $\mathbf{A}(\alpha, \beta)$ can be parametrized by the product of a small ball centered at the origin in $B_{(A,u)}^l$ and the group $\mathcal{G}(E) \times \mathcal{G}(E')$, as it is explained in the following. First we fix a smooth function $\tau : U_+ \rightarrow \mathbf{R}$ which in a neighborhood of η_+ is equal to 1, in a neighborhood of η'_+ is equal to -1 , on $[0, 2] \times [2, \infty)$ is equal to 1, and on $[0, 2] \times (-\infty, -2]$ is equal to -1 . Moreover, $\tau(s, \theta)$ on $[0, 2] \times [2, \infty)$ and $[0, 2] \times (-\infty, -2]$ depends only on s and on $[3, \infty) \times [-1, 1]$ depends only on θ . We also fix a smooth *cutoff* function $\rho : [-2, 0] \rightarrow [0, 1]$ which is equal to 0 in a neighborhood of -2 and is equal to 1 in a neighborhood of 0. For (ζ, ν) in $B_{(A,u)}^l$ with norm less than κ_0 define

$$\widehat{A}_\zeta = A + \zeta + \rho(s)(E_{\tau(0,\theta)}(A_{s,\theta}, \zeta_{s,\theta}) - \zeta_{s,\theta}), \quad u_\nu(s, \theta) = e_{\tau(s,\theta)}(\nu(s, \theta)).$$

where $A_{s,\theta}, \zeta_{s,\theta}$ are the restrictions of A, ζ to $\{(s, \theta)\} \times \Sigma$. With a slight abuse of notation, ρ in the definition of \widehat{A}_ζ denotes the induced function $X \rightarrow [0, 1]$ which vanishes outside of $[-2, 0] \times \mathbf{R} \times \Sigma$ and equals $\rho(s)$ for $(s, \theta, x) \in [-2, 0] \times \mathbf{R} \times \Sigma$. Then \widehat{A}_ζ is respectively asymptotic to the connections

$$\widehat{B}_b := B + b + \rho(s)(E_1(B(s), b(s)) - b(s)), \quad \widehat{B}'_{b'} := B' + b' + \rho(s)(E_{-1}(B'(s), b'(s)) - b'(s)),$$

as $\theta \rightarrow \infty$ and $-\infty$. Here $B(s)$ and $b(s)$ are the restrictions of B and b to $\{s\} \times \Sigma \subset Y$, and $B'(s)$ and $b'(s)$ are defined similarly. The map ρ is interpreted as a function on Y and Y' by composing the function $\rho : X \rightarrow [0, 1]$ with the inclusion of Y and Y' as $Y \times \{2\}$ and $Y' \times \{-2\}$ in X .

The connections \widehat{B}_b and $\widehat{B}'_{b'}$ do not necessarily represent elements of $L(Y, E)$ and $L(Y', E')$. We fix this issue by modifying A_ζ^0 as

$$A_\zeta := \widehat{A}_\zeta + \varphi_+ \cdot (B_b - \widehat{B}_b) + \varphi_- \cdot (B'_{b'} - \widehat{B}'_{b'}).$$

Here $\varphi_+ : X \rightarrow \mathbf{R}$ (resp. $\varphi_- : X \rightarrow \mathbf{R}$) is a fixed cutoff function which is equal to 1 on $Y \times [3, \infty)$ (resp. $Y' \times (-\infty, -3]$) and 0 on the complement of $Y \times (2, \infty)$ (resp. $Y' \times (-\infty, -2)$). Given $(\mathfrak{g}, \mathfrak{g}') \in \mathcal{G}(E) \times \mathcal{G}(E')$, we define

$$A_{\zeta, \mathfrak{g}, \mathfrak{g}'} := A_\zeta - \varphi_+ \cdot (\nabla_{B_b} \mathfrak{g}) \mathfrak{g}^{-1} - \varphi_- \cdot (\nabla_{B'_{b'}} \mathfrak{g}') \mathfrak{g}'^{-1}.$$

The connection $A_{\zeta, \mathfrak{g}, \mathfrak{g}'}$ is asymptotic to $\mathfrak{g}^* B_b$ and $\mathfrak{g}'^* B'_{b'}$ as $\theta \rightarrow \infty$ and $\theta \rightarrow -\infty$. We define a map P from the product of the ball of radius κ_0 centered at the origin in $B_{(A,u)}^l$ and $\mathcal{G}(E) \times \mathcal{G}(E')$ to $\mathbf{A}(\alpha, \beta)$ by mapping $(\zeta, \nu, \mathfrak{g}, \mathfrak{g}')$ to $P(\zeta, \nu, \mathfrak{g}, \mathfrak{g}') := (A_{\zeta, \mathfrak{g}, \mathfrak{g}'}, u_\nu)$. The map P gives a chart for a neighborhood of (A, u) in $\mathbf{A}(\alpha, \beta)$. By a slight abuse of notation, $P(\zeta, \nu)$ in what follows denotes $P(\zeta, \nu, 1, 1)$. It is a straightforward (but daunting) task to check that the transition maps associated to these charts for different (A, u) in $\mathbf{A}(\alpha, \beta)$ are smooth.

The above discussion can be modified using *Coulomb gauge fixing condition* to define a chart for the configuration space $\mathbf{B}(\alpha, \beta)$, which is obtained from $\mathbf{A}(\alpha, \beta)$ by taking the quotient with respect to the action of the gauge group $\mathcal{G}(V)$. The following proposition is a consequence of Coulomb gauge fixing for the action of $\mathcal{G}(V)$ on $\mathbf{A}(\alpha, \beta)$.

Proposition 4.6. *For any $(A, u) \in \mathbf{A}(\alpha, \beta)$, there is a constant κ_0 such that the following holds. Let \mathfrak{U}_{κ_0} denote the space of $(\zeta, \nu) \in B_{(A,u)}^l$ such that $\|(\zeta, \nu)\|_{B_{(A,u)}^l} < \kappa_0$ and*

$$*\zeta|_{U_\partial \times \Sigma} = 0, \quad d_A^* \zeta = 0. \quad (4.7)$$

Then the map

$$(g, (\zeta, \nu)) \in \mathcal{G}(V) \times \mathfrak{U}_{\kappa_0} \rightarrow g^*P(\zeta, \nu)$$

gives a diffeomorphism onto a neighborhood of (A, u) in $\mathbf{A}(\alpha, \beta)$.

This proposition together with a standard argument can be used to show that $\mathbf{B}(\alpha, \beta)$ is a Banach manifold modeled on the closed subspace of $B_{(A,u)}^l$ consisting of the elements which satisfy (4.7). Recall that $E_{(A,u)}^l$ is the subspace of $B_{(A,u)}^l$ given by elements which satisfy the first identity in (4.7), and we equip this space with the Banach space structure using the norm in (4.5). Thus, the above proposition implies that a neighborhood of $[A, u]$ in $\mathbf{B}(\alpha, \beta)$ can be parametrized by the kernel of d_A^* acting on $E_{(A,u)}^l$. To complete the proof of Proposition 3.10, we need to show that the operator d_A^* is surjective. To see this note that if ξ is in the L^2 -orthogonal of the image of d_A^* , then ξ is in the kernel of d_A . Since A is irreducible, ξ has to be zero, which verifies the claim.

4.2 Fredholm property of the mixed operator

In this subsection, we study the Fredholm properties of the mixed operator $\mathcal{D}_{(A,u)}$. As it is mentioned in Section 3, the domain of $\mathcal{D}_{(A,u)}$ is $E_{(A,u)}^l$ (equipped with the norm in (4.5)) and its target is given by

$$L_{l-1,\delta}^2(X, (\Lambda^+ \oplus \Lambda^0) \otimes V) \oplus L_{l-1,\delta}^2(U_+, u^*T\mathcal{M}(\Sigma, F)). \quad (4.8)$$

For our purposes, it is useful to consider another operator $\mathcal{D}_{(A,u)}^*$. The following definition is the counterpart of Definition 3.9, and it provides a function space which serves as the domain of $\mathcal{D}_{(A,u)}^*$.

Definition 4.9. Let $(A, u) \in \mathbf{A}(\alpha, \beta)$ be a mixed pair which is asymptotic to (B, q) and (B', q') on the mixed ends associated to (Y, E) and (Y', E') . For any positive integer k , define $K_{(A,u)}^k$ as the space of all

$$(\mu, \xi, z) \in L_{k,loc}^2(X, (\Lambda^+ \oplus \Lambda^0) \otimes V) \oplus L_{k,loc}^2(U_+, u^*T\mathcal{M}(\Sigma, F))$$

such that

- (i) $(\mu, \xi)|_{(-\infty, -3] \times Y_{\#}}$ and $z|_{[3, \infty) \times [-1, 1]}$ have finite L_k^2 norms.
- (ii) The restrictions of (μ, ξ) to $[2, \infty) \times Y$ and $(-\infty, -2] \times Y'$ and the restrictions of z to $[0, 2] \times [2, \infty)$ and $[0, 2] \times (-\infty, -2]$ have finite $L_{k,\delta}^2$ norms.
- (iii) The restriction of μ to $U_{\partial} \times \Sigma$ has the form $\frac{1}{2}d\theta \wedge c$, where c is a section of $\Lambda^1\Sigma \otimes F$ over $U_{\partial} \times \Sigma$. Moreover, if c_{θ} denotes the restriction of c to $\{(0, \theta)\} \times \Sigma \subset X$, then $d_{A_{\theta}}c_{\theta} = 0$ and $z(0, \theta)$ is equal to the element of $T_{u(0,\theta)}\mathcal{M}(\Sigma, F)$ represented by c_{θ} .
- (iv) The $(0, 1)$ -form $zd\theta + J_{s,\theta}zds$ maps $T\eta_+$ to $u^*TL(Y, E)$ and $T\eta'_+$ to $u^*TL(Y', E')$.

The norm on $K_{(A,u)}^k$ is given by the weighted Sobolev norm $L_{k,\delta}^2$.

For $(\mu, \xi, z) \in K_{(A,u)}^l$, define

$$\mathcal{D}_{(A,u)}^*(\mu, \xi, z) := (\mathcal{D}_A^*(\mu, \xi), \mathcal{D}_u^*(z)), \quad (4.10)$$

where \mathcal{D}_A^* and \mathcal{D}_u^* are formal adjoints of the ASD operator and the Cauchy-Riemann operator. Thus, these are the unique operators satisfying

$$\int_X \langle \zeta, \mathcal{D}_A^*(\mu, \xi) \rangle = \int_X \langle \mathcal{D}_A(\zeta), (\mu, \xi) \rangle, \quad \int_{U_+} \langle \nu, \mathcal{D}_u^*(z) \rangle = \int_{U_+} \langle \mathcal{D}_u(\nu), z \rangle, \quad (4.11)$$

for any ξ, ζ, μ , which are respectively smooth sections of $V, \Lambda^1 \otimes V, \Lambda^+ \otimes V$ compactly supported in the interior of X , and any ν, z , which are smooth sections of $u^*T\mathcal{M}(\Sigma, F)$ compactly supported in the interior of U_+ . To be more specific, the L^2 pairing for the second term in (4.11) is defined using the metric

$$\int_{U_+} \langle \nu, \nu' \rangle := \int_{U_+} \Omega(\nu, J_{s,\theta} \nu') ds \wedge d\theta,$$

where ν and ν' are sections of $u^*T\mathcal{M}(\Sigma, F)$. We have

$$\mathcal{D}_A^*(\mu, \zeta) = d_A \xi + d_A^* \mu + \text{Hess}_{A_t} h(*_3 \mu_t) + \text{Hess}_{A'_t} h'(*_3 \mu'_t),$$

with μ_t, μ'_t being the restrictions of μ to $\{t\} \times Y_0, \{t\} \times -Y'_0$. Using this notation, we may write the self-dual 2-form μ on $\mathbf{R} \times Y_0$ and $\mathbf{R} \times -Y'_0$ as $\mu_t - *_3 \mu_t \wedge dt$ and $\mu'_t - *_3 \mu'_t \wedge dt$. The target of the operator $\mathcal{D}_{(A,u)}^*$ is $L_{l-1,\delta}^2(X, \Lambda^1 \otimes V) \oplus L_{l-1,\delta}^2(U_+, u^*T\mathcal{M}(\Sigma, F))$ where our convention for the weighted Sobolev space $L_{l-1,\delta}^2$ is fixed in Definition 3.8. Proposition 3.18 is a consequence of the following theorem.

Theorem 4.12. *There is δ_0 such that the following holds. For $\delta < \delta_0$, suppose $\mathbf{A}(\alpha, \beta)$ is defined using δ and $[A, u]$ is a smooth element of $\mathbf{A}(\alpha, \beta)$ that satisfies property (iii) of Theorem 3.6 for δ_0 . Then the operators $\mathcal{D}_{(A,u)}$ and $\mathcal{D}_{(A,u)}^*$ are Fredholm. Furthermore, the cokernel (resp. the kernel) of $\mathcal{D}_{(A,u)}^*$ can be identified with the kernel (resp. the cokernel) of $\mathcal{D}_{(A,u)}$. In particular, $\text{index}(\mathcal{D}_{(A,u)}) = -\text{index}(\mathcal{D}_{(A,u)}^*)$.*

In order to fix the constant δ_0 in Theorem 4.12, we need to look more closely at the mixed operator on the mixed ends. This will be addressed in Subsection 4.2.1, where we also review some of the results of [DFL] relevant to the Fredholm property of mixed operators. We will come back to the proof of Theorem 4.12 in Subsection 4.2.2.

4.2.1 Mixed cylinders and mixed operators

Suppose (Y, E) is as in the previous sections and I is a Riemannian connected 1-dimensional manifold. Thus, I is either an open interval in \mathbf{R} or S^1 with a fixed length. The *cylinder quintuple* associated to I is given as

$$\mathbf{c}_I := (I \times Y, E \times I, [0, 2] \times I, \mathcal{M}(\Sigma, F), L(Y, E)).$$

We fix the product metric on $I \times Y$ and a family of almost complex structures $\{J_s\}_{s \in [0,2]}$ on $\mathcal{M}(\Sigma, F)$ such that $J_s = J_*$ for $s < 1$. This family induces a family of almost complex structures parametrized by $[0, 2] \times I$, which is constant with respect to the second component. For instance, the restriction of a mixed

pair for the special quintuple to the mixed end associated to Y determines a mixed pair for the cylinder quintuples $\mathfrak{c}_{(2,\infty)}$.

As in the case of special quintuples, we may associate a mixed operator $\mathcal{D}_{(A,u)}$ to any mixed pair (A, u) on the cylinder quintuple \mathfrak{c}_I . The domain $E_{(A,u)}^k(I)$ of this operator consists of

$$\zeta \in L_{k,loc}^2(I \times Y, \Lambda^1 \otimes E), \quad \nu \in L_{k,loc}^2([0, 2] \times I, u^*T\mathcal{M}(\Sigma, F)), \quad (4.13)$$

such that $*\zeta|_{\Sigma \times I} = 0$, and for any $\theta \in I$, we have

$$\nu(2, \theta) \in T_{u(2,\theta)}L(Y, E), \quad d_A\zeta|_{\Sigma \times \{(0,\theta)\}} = 0, \quad [\zeta|_{\Sigma \times \{(0,\theta)\}}] = \nu(0, \theta) \quad (4.14)$$

In the case that I is an infinite interval, we demand that an element $(\zeta, \nu) \in E_{(A,u)}^k(I)$ has a finite weighted Sobolev norm with respect to the weight $e^{\delta\tau}$. Here δ is a real number, and $\tau : I \times Y \sqcup [0, 2] \times I \rightarrow \mathbf{R}$, is the projection map to I . For any $(\zeta, \nu) \in E_{(A,u)}^k(I)$, we have

$$\mathcal{D}_{(A,u)}(\zeta, \nu) = (d_A^*\zeta, d_A^+\zeta + (*_3\text{Hess}_{A_\theta}h(\zeta_\theta))^+, \mathcal{D}_u\nu).$$

where the Cauchy-Riemann operator \mathcal{D}_u is defined as in (2.34). The target of $\mathcal{D}_{(A,u)}$ is the space

$$L_{k-1,\delta}^2(I \times Y, (\Lambda^0 \oplus \Lambda^+) \otimes E) \oplus L_{k-1,\delta}^2([0, 2] \times I, u^*T\mathcal{M}(\Sigma, F)). \quad (4.15)$$

where the wighted Sobolev space is defined again using the weight $e^{\delta\tau}$. Of course, if I is a finite interval the weight does not play any role and we may replace the weighted Sobolev space $L_{k-1,\delta}^2$ with L_{k-1}^2 .

There is a useful reparametrization of the target of $\mathcal{D}_{(A,u)}$ in the case of cylinder quintuples. Any section ζ of $(\Lambda^0 \oplus \Lambda^+) \otimes E$ over $I \times Y$ has the form $(\varphi, \frac{1}{2}(d\theta \wedge b + *_3b))$ where b is a section of $\Lambda^1(Y) \otimes E$ over $I \times Y$. In particular, we may associate $b - \varphi d\theta$, a section of $\Lambda^1 \otimes E$, to ζ . This allows us to identify the target of $\mathcal{D}_{(A,u)}$ in (4.15) with

$$L_{k-1,\delta}^2(I \times Y, \Lambda^1 \otimes E) \oplus L_{k-1,\delta}^2([0, 2] \times I, u^*T\mathcal{M}(\Sigma, F)). \quad (4.16)$$

We will use this reparametrization of the target of $\mathcal{D}_{(A,u)}$ in the rest of this subsection.

Theorem 4.17. *Suppose $I = (a, b)$, $J = (c, d)$ are finite intervals with $a < c < d < b$. Suppose (A, u) is a smooth mixed pair associated to the cylinder quintuple \mathfrak{c}_I .*

- (i) *Suppose $k \geq 1$ is an integer, $(\zeta, \nu) \in E_{(A,u)}^1(I)$ and $\mathcal{D}_{(A,u)}(\zeta, \nu)$ is in L_{k-1}^2 . Then $(\zeta, \nu) \in E_{(A,u)}^k(J)$. There is also a constant C , independent of (ζ, ν) , such that*

$$\|(\zeta, \nu)\|_{L_k^2(J)} \leq C \left(\|\mathcal{D}_{(A,u)}(\zeta, \nu)\|_{L_{k-1}^2(I)} + \|(\zeta, \nu)\|_{L^2(I)} \right). \quad (4.18)$$

- (ii) *Suppose $\zeta \in L^2(I \times Y, \Lambda^1 \otimes E)$ and $\nu \in L^2([0, 2] \times I, u^*T\mathcal{M}(\Sigma, F))$ satisfy*

$$|\langle (\zeta, \nu), \mathcal{D}_{(A,u)}(\zeta', \nu') \rangle| \leq \kappa \|(\zeta', \nu')\|_{L^2}$$

for any smooth element (ζ', ν') of $E_{(A,u)}^1(I)$ with compact support and for a fixed constant κ . Then $(\zeta, \nu) \in E_{(A,u)}^1(J)$.

Proof. This theorem in the absence of the perturbation term $(*_3 \text{Hess}_{A_\theta} h(\zeta_\theta))^+$ is proved in [DFL, Theorem 5]. This special case and the property of cylinder functions mentioned in part (iii) of Proposition 6.1 allows us to conclude the general case. \square

Remark 4.19. The mixed operators associated to special quintuples satisfy a uniform version of Theorem 4.17. To be more precise, suppose (A_i, u_i) is a sequence of smooth mixed pairs associated to the cylinder quintuple \mathfrak{c}_I that are C^∞ -convergent to (A, u) . Then there is a constant C such that for any i and any $(\zeta, \nu) \in E_{(A_i, u_i)}^k(I)$, we have

$$\|(\zeta, \nu)\|_{L_k^2(J)} \leq C \left(\|\mathcal{D}_{(A_i, u_i)}(\zeta, \nu)\|_{L_{k-1}^2(I)} + \|(\zeta, \nu)\|_{L^2(I)} \right).$$

As in Theorem 4.17, this is again a consequence of the results of [DFL] and Proposition 6.1. (See [DFL, Remarks 5.73 and 5.76].)

The required result from [DFL] in the proof of Theorem 4.17 uses a description of the mixed operator in terms of a *dimensionally reduced mixed operator*. First we review this description in a simpler case. Any $q \in L(Y, E)$ determines a mixed pair associated to the cylinder quintuple \mathfrak{c}_I . Suppose B_q is a connection on E that represents q , and its restriction to the boundary is α_q . Let A_q be the pullback of B_q to $I \times Y$ and $u_q : [0, 2] \times I \rightarrow \mathcal{M}(\Sigma, F)$ be the constant map to q . The pair (A_q, u_q) defines a mixed pair for the cylinder quintuple \mathfrak{c}_I , which can be regarded as the counterpart of constant pairs for special quintuples. In fact, the restriction of a constant pair to the mixed end associated to Y determines an element of the form (A_q, u_q) for the interval $I = (2, \infty)$. For any

$$(\varphi, b, \nu) \in \Omega^0(Y, E) \oplus \Omega^1(Y, E) \oplus \Omega^0([0, 1], T_q \mathcal{M}(\Sigma, F)), \quad (4.20)$$

define

$$\mathfrak{D}_q(\varphi, b, \nu) = (d_{B_q}^* b, -*_3 d_{B_q} b + \text{Hess}_{B_q} h(b) + d_{B_q} \varphi, J_s \frac{d\nu}{ds}),$$

which is again an element of $\Omega^0(Y, E) \oplus \Omega^1(Y, E) \oplus \Omega^0([0, 1], T_q \mathcal{M}(\Sigma, F))$. Then we have

$$\mathcal{D}_{(A_q, u_q)} = \frac{d}{d\theta} - \mathfrak{D}_q.$$

Here we again use the identification of a 1-form on $I \times Y$ with a map from I to the space of sections of $\Lambda^0 \oplus \Lambda^1$ on Y .

Proposition 4.21. *There is a positive constant δ_0 such that if $0 < \delta < \delta_0$ or $-\delta_0 < \delta < 0$, then the operator*

$$\mathcal{D}_{(A_q, u_q)} : E_{(A_q, u_q)}^1(\mathbf{R}) \rightarrow L_\delta^2(\mathbf{R} \times Y, \Lambda^1 \otimes E) \oplus L_\delta^2([0, 2] \times \mathbf{R}, u^* T \mathcal{M}(\Sigma, F)).$$

is an isomorphism. Moreover, there is C such that for any $(\zeta, \nu) \in E_{(A_q, u_q)}^1(\mathbf{R})$, we have

$$C^{-1} \|(\zeta, \nu)\|_{L_{1,\delta}^2} \leq \|\mathcal{D}_{(A_q, u_q)}(\zeta, \nu)\|_{L_\delta^2} \leq C \|(\zeta, \nu)\|_{L_{1,\delta}^2}.$$

Proof. Suppose \mathcal{H}_q is the completion of the space of (φ, b, ν) as in (4.20) with respect to the L^2 norm

$$\langle (\varphi_0, b_0, \nu_0), (\varphi_1, b_1, \nu_1) \rangle_{L^2} := \int_Y \text{tr}(\varphi_0 \wedge * \varphi_1 + b_0 \wedge * b_1) + \int_0^1 \Omega(\nu_0(s), J_s \nu_1(s)) ds. \quad (4.22)$$

Let \mathcal{W}_q denote the L^2_1 completion of the space of all triples (φ, b, ν) as in (4.20) such that

$$*b|_\Sigma = 0, \quad d_{\alpha_q} b|_\Sigma = 0, \quad [b|_\Sigma] = \nu(0), \quad \nu(1) \in T_q L(Y, E). \quad (4.23)$$

Given a 1-parameter family $\{(\varphi_\theta, b_\theta)\}_{\theta \in \mathbf{R}}$, we may define a 1-form on $\mathbf{R} \times Y$ as $b_\theta - \varphi_\theta d\theta$. Using this identification, we have

$$E^1_{(A_q, u_q)}(\mathbf{R}) = L^2_{1, \delta}(\mathbf{R}, \mathcal{H}_q) \cap L^2_\delta(\mathbf{R}, \mathcal{W}_q),$$

and

$$L^2_\delta(\mathbf{R} \times Y, \Lambda^1 \otimes E) \oplus L^2_\delta([0, 2] \times \mathbf{R}, T_q \mathcal{M}(\Sigma, F)) = L^2_\delta(\mathbf{R}, \mathcal{H}_q).$$

It is shown in [DFL] that $\mathfrak{D}_q : \mathcal{H}_q \rightarrow \mathcal{H}_q$ is an (unbounded) self-adjoint Fredholm operator with domain \mathcal{W}_q and a discrete spectrum that has a finite intersection with any finite interval. The kernel of \mathfrak{D}_q can be identified with $T_q L(Y, E)$. Moreover, the operator $\mathcal{D}_{(A_q, u_q)}$ is invertible if and only if δ is not in the spectrum of \mathfrak{D}_q (see [DFL, Proposition 5.79]). Thus, it suffices to show that

$$\delta_0 := \inf_{B_q} \{\delta_q \mid \delta_q \text{ is the smallest magnitude of a non-zero eigenvalue of } \mathfrak{D}_q\} > 0 \quad (4.24)$$

Although \mathfrak{D}_q is defined in terms of B_q , it essentially depends only on q , the gauge equivalence class of B_q up to conjugation. In fact, the Hilbert spaces \mathcal{H}_q and \mathcal{W}_q define Hilbert space bundles \mathbf{H} and \mathbf{W} on $L(Y, E)$. (It is clear that \mathbf{H} is locally trivial, and local trivializations of \mathbf{W} is given by [DFL, Proposition 5.27].) Then $\{\mathfrak{D}_q\}_q$ define a smooth family of Fredholm operators from the fibers of \mathbf{W} to the fibers of \mathbf{H} . The claim in (4.24) follows because the dimension of the kernels of these operators is independent of q and $L(Y, E)$ is compact. \square

Corollary 4.25. *Suppose δ_0 is as in Proposition 4.21 and $0 < \delta < \delta_0$. Suppose*

$$\zeta \in L^2_{1, \text{loc}}(\mathbf{R} \times Y, \Lambda^1 \otimes E), \quad \nu \in L^2_{1, \text{loc}}([0, 2] \times \mathbf{R}, T_q \mathcal{M}(\Sigma, F)), \quad (4.26)$$

such that $\zeta|_{\Sigma \times \mathbf{R}} = 0$, and the identities in (4.14) hold for any $\theta \in \mathbf{R}$ and the mixed pair (A_q, u_q) . Suppose also $(\zeta, \nu) \in L^2_{-\delta}$ and $(\zeta', \nu') := \mathcal{D}_{(A_q, u_q)}(\zeta, \nu) \in L^2_\delta$, where $L^2_{-\delta}$ is the weighted Sobolev norm defined using the negative exponent $-\delta$. Then there is (ζ_1, ν_1) , which is the pullback of an element of the kernel of \mathfrak{D}_q , such that $(\zeta - \zeta_1, \nu - \nu_1)$ has finite $L^2_{1, \delta}$ norm.*

Proof. Proposition 4.21 implies that there is $(\zeta_0, \nu_0) \in E^1_{(A_q, u_q)}(\mathbf{R})$ such that $\mathcal{D}_{(A_q, u_q)}(\zeta_0, \nu_0) = (\zeta', \nu')$. In particular, $(\zeta_1, \eta_1) := (\zeta - \zeta_0, \nu - \nu_0)$ belongs to the kernel of $\mathcal{D}_{(A_q, u_q)}$. Moreover, $e^{-|\theta|\delta}(\zeta_1, \eta_1)$ has a finite L^2 norm. There is a complete eigenspace decomposition $\{f_i\}_i$ associated to the operator \mathfrak{D}_q which provides an orthonormal basis for \mathcal{H} . Using this eigenspace decomposition, we have

$$(\zeta_1, \nu_1) = \sum_i c_i e^{\lambda_i \theta} f_i \quad (4.27)$$

where λ_i is the eigenvalue of f_i and $c_i \in \mathbf{R}$. Our assumption on $e^{-|\theta|\delta}(\zeta_1, \eta_1)$ implies that $c_i = 0$ unless $\lambda_i = 0$. This gives the claim. \square

In fact, we shall need a generalization of Corollary 4.25 where (A_q, u_q) is replaced with a more general mixed pair (A, u) .

Corollary 4.28. *Suppose δ_0 is as in Proposition 4.21 and $0 < \delta < \delta_0$. For $q \in L(Y, E)$, suppose (A, u) is a smooth mixed pair for the cylinder quintuple $\mathfrak{c}_{(0, \infty)}$ such that $A - A_q \in L_{1, \delta}^2$, u converges to q as $\theta \rightarrow \infty$ and du is in $L_{1, \delta}^2$. Suppose*

$$\zeta \in L_{1, \text{loc}}^2((0, \infty) \times Y, \Lambda^1 \otimes E), \quad \nu \in L_{1, \text{loc}}^2([0, 2] \times (0, \infty), u^*T\mathcal{M}(\Sigma, F)), \quad (4.29)$$

such that $*\zeta|_{\Sigma \times (0, \infty)} = 0$, and the identities in (4.14) hold for the mixed pair (A, u) and any $\theta \in (0, \infty)$. Suppose also $(\zeta, \nu) \in L_{1, -\delta}^2$ and $(\zeta', \nu') := \mathcal{D}_{(A, u)}(\zeta, \nu) \in L_{\delta}^2$. Then there is (ζ_1, ν_1) , which is the pullback of an element of the kernel of \mathfrak{D}_q , such that $(\zeta - \zeta_1, \nu - \nu_1) \in E_{(A, u)}^1((0, \infty))$.

Proof. Suppose (B_θ, u_θ) denotes the restriction of A, u to $\{\theta\} \times Y$ and $[0, 2] \times \{\theta\}$. Analogous to \mathcal{H}_q and \mathcal{W}_q in the proof of Proposition 4.21, we may use (B_θ, u_θ) to define the completions \mathcal{H}_θ and \mathcal{W}_θ of

$$\Omega^0(Y, E) \oplus \Omega^1(Y, E) \oplus \Omega^0([0, 1], u_\theta^*T\mathcal{M}(\Sigma, F)).$$

We may use local trivializations of the Hermitian bundles $(T\mathcal{M}(\Sigma, F), \Omega, J_s)$ in a neighborhood of q , to identify \mathcal{H}_θ and \mathcal{H}_q in the obvious way. This allows us to drop θ from our notation for \mathcal{H}_θ , and denote it by \mathcal{H} .

To prove the claim, it suffices to show that there is (ζ_1, ν_1) as above such that for some $T_0 > 0$ the restriction of $(\zeta - \zeta_1, \nu - \nu_1)$ to $(T_0, \infty) \times Y$ and $[0, 2] \times (T_0, \infty)$ is in $L_{1, \delta}^2$. In particular, by taking T_0 large enough, we may assume that (B_θ, u_θ) is in a neighborhood of (B_q, u_q) such that we can apply [DFL, Proposition 5.27] and show that there are isomorphisms

$$Q_\theta : \mathcal{H} \rightarrow \mathcal{H}$$

such that Q_θ maps \mathcal{W}_q to \mathcal{W}_θ . Moreover, Q_θ maps the subspace of L_k^2 elements of \mathcal{H}_θ isomorphically onto the subspace of L_k^2 elements of \mathcal{H}_q , and satisfies

$$C_k^{-1} \|(\varphi, b, \nu)\|_{L_k^2} \leq \|Q_\theta(\varphi, b, \nu)\|_{L_k^2} \leq C \|(\varphi, b, \nu)\|_{L_k^2} \quad (4.30)$$

for a constant C_k independent of θ and for any (φ, b, ν) . In fact, the operator norm of $Q_\theta - \text{Id}$ with respect to the L_k^2 norm is bounded by $C_k e^{-\delta\theta}$. The map $\theta \rightarrow Q_\theta$ as a map from (T_0, ∞) to the space $B(\mathcal{H})$ of bounded operators of \mathcal{H} is smooth and its derivatives satisfy the analogue of (4.30). In particular, the operators Q_θ can be put together to define

$$\begin{aligned} \mathbf{Q} : L_{k, \text{loc}}^2((T_0, \infty) \times Y, \Lambda^1 \otimes E) \oplus L_{k, \text{loc}}^2([0, 2] \times (T_0, \infty), T_q\mathcal{M}(\Sigma, F)) &\rightarrow \\ \rightarrow L_{k, \text{loc}}^2((T_0, \infty) \times Y, \Lambda^1 \otimes E) \oplus L_{k, \text{loc}}^2([0, 2] \times (T_0, \infty), u^*T\mathcal{M}(\Sigma, F)) &\quad (4.31) \end{aligned}$$

for any k [DFL, Lemma 5.67].

The operator \mathbf{Q} maps the domain and the target of $\mathcal{D}_{(A_q, u_q)}$ respectively to the the domain and the target of $\mathcal{D}_{(A, u)}$, and we have

$$\mathbf{Q}^{-1} \circ \mathcal{D}_{(A, u)} \circ \mathbf{Q} = \frac{d}{d\theta} - \mathfrak{D}_q - S_\theta, \quad (4.32)$$

where $S_\theta : \mathcal{W}_q \rightarrow \mathcal{H}_q$ is a bounded linear operator whose norm is bounded by $Ce^{-\delta\theta}$.

Using (4.32), we may write

$$\mathcal{D}_{(A_q, u_q)}(\mathbf{Q}^{-1}(\zeta, \nu)) = \mathbf{Q}^{-1}(\zeta', \nu') + \mathbf{S}\mathbf{Q}^{-1}(\zeta, \nu).$$

where \mathbf{S} is defined using the operators S_θ . By assumption the first term on the left hand side has a finite L_δ^2 norm and the second term has a finite L^2 norm. Using a cutoff function $\rho : (T_0, \infty) \rightarrow \mathbf{R}$ which vanishes for $\theta < T_0 + \frac{1}{2}$ and is equal to 1 for $\theta > T_0 + 1$, we may extend $\mathbf{Q}(\zeta, \nu)$ to

$$(\tilde{\zeta}, \tilde{\nu}) \in L_{1,loc}^2(\mathbf{R} \times Y, \Lambda^1 \otimes E) \oplus L_{1,loc}^2([0, 2] \times \mathbf{R}, T_q \mathcal{M}(\Sigma, F)),$$

such that $\mathcal{D}_{(A_q, u_q)}(\tilde{\zeta}, \tilde{\nu})$ has a finite $L_{-\delta/2}^2$ norm. (In fact, $L_{-\delta/2}^2$ can be replaced with L^2 .) By applying the argument in the proof of Corollary 4.25, we may conclude that $(\tilde{\zeta}, \tilde{\nu})$ has finite $L_{1, -\delta/2}^2$ norm. Thus, the same claim holds for (ζ, ν) . By iterating the same argument, we can show that now that $\mathcal{D}_{(A_q, u_q)}(\mathbf{Q}^{-1}(\zeta, \nu))$ has a finite $L_{\delta/2}^2$ norm. Using Corollary 4.25 again we may conclude that there is (ζ_1, ν_1) , which is the pullback of an element of the kernel of \mathfrak{D}_q , such that $\mathbf{Q}^{-1}(\zeta, \nu) - (\zeta_1, \nu_1)$ has finite $L_{\delta/2}^2$ norm. Iterating this argument once more, we conclude that $\mathbf{Q}^{-1}(\zeta, \nu) - (\zeta_1, \nu_1)$ has finite L_δ^2 norm. Our assumption on \mathbf{Q} implies that $(\zeta, \nu) - (\zeta_1, \nu_1)$ also has finite L_δ^2 norm. \square

Remark 4.33. The analogues of the results of this subsection hold for the adjoint of the mixed operator. In fact, the adjoint of the mixed operator for cylinder quintuples have a similar form as the mixed operator (see [DFL, Section 5] for more details), and the results of this section would immediately imply the corresponding results for the adjoint of the mixed operator.

4.2.2 Proof of Theorem 4.12

In this subsection, we prove Theorem 4.12 on Fredholmness of mixed operators associated to the special quintuple where δ_0 is given by Proposition 4.21. Suppose (A, u) is a smooth mixed operator that satisfies the assumption of Theorem 4.12. Let also X_T denote the compact subspace of X given as the complement of the subspaces $(T, \infty) \times Y$, $(-\infty, -T) \times Y'$ and $(-\infty, -T) \times Y_\#$ in X . Similarly, let U_T be the compact subspace of U_+ given as the complement of $[0, 2] \times (T, \infty)$, $[0, 2] \times (-\infty, -T)$ and $(T, \infty) \times [-1, 1]$.

Lemma 4.34. *For (A, u) as above and any $k \geq 1$, there are constants C and T_0 such that the following holds. Suppose $(\zeta, \nu) \in E_{(A, u)}^1$ and $\mathcal{D}_{(A, u)}(\zeta, \nu) \in L_{k-1, \delta}^2$. Then $(\zeta, \nu) \in E_{(A, u)}^k$, and we have*

$$\|(\zeta, \nu)\|_{E_{(A, u)}^k} \leq C(\|\mathcal{D}_{(A, u)}(\zeta, \nu)\|_{L_{k-1, \delta}^2} + \|\zeta\|_{L^2(X_{T_0})} + \|\nu\|_{L^2(U_{T_0})} + |s| + |s'|). \quad (4.35)$$

A similar result holds for $\mathcal{D}_{(A, u)}^*$.

Proof. Theorem 4.17 and standard regularity results about the linearized ASD and CR equations imply that $(\zeta, \nu) \in L_{k, loc}^2$. Moreover, for any $T_0 > 1$, there is C such that

$$\|\zeta\|_{L_k^2(X_{T_0+1})} + \|\nu\|_{L_k^2(U_{T_0+1})} \leq C(\|\mathcal{D}_{(A, u)}(\zeta, \nu)\|_{L_{k-1, \delta}^2} + \|\zeta\|_{L^2(X_{T_0+2})} + \|\nu\|_{L^2(U_{T_0+2})}). \quad (4.36)$$

Next, we obtain control over the decay of (ζ, ν) on the mixed end associated to Y . Suppose (A, u) is asymptotic to (B_q, q) on this mixed end for $q \in L(Y, E)$. We use a similar construction as in the proof of Corollary 4.28. Suppose (B_θ, u_θ) , \mathcal{H} and \mathcal{W}_θ are as in there, and for any $\theta \in (T_0, \infty)$, let the isomorphism $Q_\theta : \mathcal{H} \rightarrow \mathcal{H}$ be given by [DFL, Proposition 5.27]. Suppose \mathbf{Q} is obtained from Q_θ analogous to (4.31). Let $(\widehat{\zeta}, \widehat{\nu})$ be the result of applying \mathbf{Q}^{-1} to the restriction of (ζ, ν) to $(T_0, \infty) \times Y$ and $[0, 2] \times (T_0, \infty)$. We have

$$\mathcal{D}_{(A_q, u_q)}(\widehat{\zeta}, \widehat{\nu}) = \mathbf{Q}^{-1} \mathcal{D}_{(A, u)}(\zeta, \nu) + \mathbf{S} \mathbf{Q}^{-1}(\zeta, \nu),$$

where $\mathcal{D}_{(A_q, u_q)}$ is the mixed operator associated to the pullback of (B_q, q) on the cylinder quintuple, and \mathbf{S} is given by a family of operators S_θ defined as in (4.32). In particular, for a given positive constant ε , we may assume that T_0 is chosen such that

$$\|\mathcal{D}_{(A_q, u_q)}(\widehat{\zeta}, \widehat{\nu})\|_{L^2_{k-1, \delta}(T_0, \infty)} \leq C_0 \|\mathcal{D}_{(A, u)}(\zeta, \nu)\|_{L^2_{k-1, \delta}(T_0, \infty)} + \varepsilon \|(\zeta, \nu)\|_{E^k_{(A, u)}(T_0, \infty)}, \quad (4.37)$$

where C is a constant independent of (ζ, ν) . Here $\|\mathcal{D}_{(A, u)}(\zeta, \nu)\|_{L^2_{k-1, \delta}(T_0, \infty)}$ denotes the $L^2_{k, \delta}$ norm of the restriction of $\mathcal{D}_{(A, u)}(\zeta, \nu)$ to $(T_0, \infty) \times Y$.

Suppose (ζ, ν) is asymptotic to (b, s) on the mixed end associated to Y . Theorem 4.17 for the pair (A_q, u_q) implies that there is a constant C such that for any T , we have

$$\begin{aligned} \|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2_k(T-1, T+1)} &\leq \\ C(\|\mathcal{D}_{(A_q, u_q)}(\widehat{\zeta}, \widehat{\nu})\|_{L^2_{k-1}(T-2, T+2)} &+ \|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2(T-2, T+2)}). \end{aligned}$$

A weighted sum of these inequalities imply that

$$\|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2_{k, \delta}(T_0+1, \infty)} \leq C(\|\mathcal{D}_{(A_q, u_q)}(\widehat{\zeta}, \widehat{\nu})\|_{L^2_{k-1, \delta}(T_0, \infty)} + \|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2_\delta(T_0, \infty)}).$$

The last term in the above inequality can be controlled by $\mathcal{D}_{(A_q, u_q)}(\widehat{\zeta}, \widehat{\nu})$. In fact, by multiplying $(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)$ by a cutoff function $g : (T_0, \infty) \rightarrow \mathbf{R}$ satisfying $g(\theta) = 1$ for $\theta \geq T_0 + 1$ and $g(\theta) = 0$ for $\theta \leq T_0 + 1/2$, we may regard it as an element of $E^k_{(A_q, u_q)}(\mathbf{R})$. In particular, applying Proposition 4.21 implies that

$$\begin{aligned} \|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2_{k, \delta}(T_0+1, \infty)} &\leq \\ C(\|\mathcal{D}_{(A_q, u_q)}(\widehat{\zeta}, \widehat{\nu})\|_{L^2_{k-1, \delta}(T_0, \infty)} &+ \|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2(T_0, T_0+1)} + |s|). \end{aligned}$$

Our assumption on the exponential decay of (A, u) over the mixed end and the properties of the map \mathbf{Q} imply that

$$\|(\zeta - \pi^* b, \nu - \pi^* s)\|_{L^2_{k, \delta}(T_0+1, \infty)} \leq C(\|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2_{k, \delta}(T_0+1, \infty)} + |s|).$$

In particular, from the previous two inequalities and (4.37), we conclude that

$$\begin{aligned} \|(\zeta, \nu)\|_{E^k_{(A, u)}(T_0+1, \infty)} &\leq C(\|\mathcal{D}_{(A_q, u_q)}(\widehat{\zeta}, \widehat{\nu})\|_{L^2_{k-1, \delta}(T_0, \infty)} + \|(\widehat{\zeta} - \pi^* b, \widehat{\nu} - \pi^* s)\|_{L^2(T_0, T_0+1)} + |s|) \\ &\leq C(C_0 \|\mathcal{D}_{(A, u)}(\zeta, \nu)\|_{L^2_{k-1, \delta}} + \varepsilon \|(\zeta, \nu)\|_{E^k_{(A, u)}(T_0, \infty)} + \|\zeta\|_{L^2(X_{T_0+1})} + \|\nu\|_{L^2(U_{T_0+1})} + |s|). \end{aligned}$$

By picking ε small enough, we may rearrange the terms and use (4.36) to remove $\varepsilon \|(\zeta, \nu)\|_{E_{(A,u)}^k(T_0, \infty)}$ from the above inequality. In summary, we have

$$\|(\zeta, \nu)\|_{E_{(A,u)}^k(T_0+1, \infty)} \leq C(\|\mathcal{D}_{(A,u)}(\zeta, \nu)\|_{L_{k-1, \delta}^2} + \|\zeta\|_{L^2(X_{T_0+2})} + \|\nu\|_{L^2(U_{T_0+2})} + |s|). \quad (4.38)$$

We obtain a similar inequality for the mixed end associated to Y' .

Non-degeneracy of $\alpha \in \mathfrak{C}_G$, $\beta \in \mathfrak{C}_S$ and standard results about solutions of the ASD and the CR equations on cylinders imply that

$$\begin{aligned} \|\zeta\|_{L_k^2(-\infty, -T_0-1) \times Y_{\#}} + \|\nu\|_{L_k^2(T_0+1, \infty) \times [-1, 1]} \leq \\ C(\|\mathcal{D}_{(A,u)}(\zeta, \nu)\|_{L_{k-1, \delta}^2} + \|\zeta\|_{L^2(X_{T_0+2})} + \|\nu\|_{L^2(U_{T_0+2})} + |s|). \end{aligned}$$

In fact, this inequality can be verified following a similar strategy analogous to (4.38). Combining this inequality, (4.38) and its counterpart for Y' gives us the desired result after replacing T_0 with $T_0 + 2$. The proof of the analogous result for the adjoint operator $\mathcal{D}_{(A,u)}^*$ where we replace Theorem 4.17 and Proposition 4.21 with the corresponding result for the adjoint operator (see Remark 4.33). \square

As a consequence of Lemma 4.34, the operators $\mathcal{D}_{(A,u)}$ and $\mathcal{D}_{(A,u)}^*$ have finite dimensional kernels and closed images. Moreover, in order to show that the cokernel of $\mathcal{D}_{(A,u)}$ is finite dimensional, it suffices to show that for $k = 1$, the cokernel of $\mathcal{D}_{(A,u)}$ can be identified with the kernel of $\mathcal{D}_{(A,u)}^*$. An element of the cokernel of $\mathcal{D}_{(A,u)}$ in this case is given by (μ, ξ, z) such that for any $(\zeta, \nu) \in E_{(A,u)}^1$, we have

$$\langle (\mu, \xi, z), \mathcal{D}_{(A,u)}(\zeta, \nu) \rangle_{L^2} := 2 \int_X \langle (\mu, \xi), \mathcal{D}_A(\zeta) \rangle + \int_{U_+} \langle z, \mathcal{D}_u(\nu) \rangle = 0. \quad (4.39)$$

Moreover, ξ , μ and z belong to L_{loc}^2 , the restrictions of ξ and μ to $(-\infty, -3] \times Y_{\#}$ and z to $[3, \infty) \times [-1, 1]$ are in L^2 , the restrictions of ξ , μ and z to the mixed ends are in $L_{-\delta}^2$. In particular, an element of this space is allowed to have an exponential growth by a controlled quantity over the mixed ends. We included a factor 2 in our convention for the L^2 pairing so that after integration by parts the boundary terms behave in the desired form.

Theorem 4.17 and standard results on Fredholm theory of the adjoints of linearized ASD and CR operators imply that (μ, ξ, z) is in fact in $L_{1, loc}^2$ and properties (i) (for $k = 1$), (iii) and (iv) of Definition 4.9 hold for (μ, ξ, z) . The proof of property (i) of Definition 4.9 for (μ, ξ, z) uses the fact that α and β are non-degenerate elements of \mathfrak{C}_G and \mathfrak{C}_S . Now applying (4.39) to ζ and ν which are compactly supported in the interior of X and U_+ implies that $\mathcal{D}_{(A,u)}^*(\mu, \xi, z) = 0$. In particular, the inequality in (4.18) in Theorem 4.17 can be used to show that the restrictions of ξ , μ and z to the mixed ends are in fact in $L_{1, -\delta}^2$. So, we can apply Corollary 4.28 to the restrictions of (μ, ξ, z) to the mixed ends and conclude that the following holds. There are $\mathfrak{b} \in \mathcal{H}_h^1(Y; B)$ and $\mathfrak{b}' \in \mathcal{H}_h^1(Y'; B')$ such that

$$\mu - \frac{1}{2}(d\theta \wedge \pi^*(\mathfrak{b}) + *_3\pi^*(\mathfrak{b}))|_{Y \times [2, \infty)} \quad \text{and} \quad \mu - \frac{1}{2}(d\theta \wedge \pi^*(\mathfrak{b}') + *_3\pi^*(\mathfrak{b}'))|_{Y' \times (-\infty, -2]}$$

have finite $L_{1, \delta}^2$ norms. Let \mathfrak{s} and \mathfrak{s}' be tangent vectors to $\mathcal{M}(\Sigma, F)$ at the points q and q' given by restriction of \mathfrak{b} and \mathfrak{b}' to the boundary. Then

$$z - \pi^*(\mathfrak{s})|_{[0, 2] \times [2, \infty)} \quad \text{and} \quad z - \pi^*(\mathfrak{s}')|_{[0, 2] \times (-\infty, -2]}$$

also have finite $L^2_{1,\delta}$ norms. The Stokes' theorem together with $\mathcal{D}^*_{(A,u)}(\mu, \xi, z) = 0$ shows that for an arbitrary element (ζ, ν) of $E^1_{(A,u)}$ which is asymptotic to (b, s) on the mixed end associated to Y and is asymptotic to (b', s') on the mixed end associated to Y' , we have

$$\langle (\mu, \xi, z), \mathcal{D}_{(A,u)}(\zeta, \nu) \rangle_{L^2} = \langle (b, s), (\mathfrak{b}, \mathfrak{s}) \rangle_{L^2} - \langle (b', s'), (\mathfrak{b}', \mathfrak{s}') \rangle_{L^2}.$$

In particular, (4.39) implies that $(\mathfrak{b}, \mathfrak{s}) = 0$ and $(\mathfrak{b}', \mathfrak{s}') = 0$. Thus, property (ii) of Definition 4.9 holds for (μ, ξ, z) , and hence $(\mu, \xi, z) \in K^1_{(A,u)}$. In summary, cokernel of $\mathcal{D}_{(A,u)}$ can be identified with the kernel of $\mathcal{D}^*_{(A,u)}$. In particular, it is finite dimensional. This completes the proof of the claim that $\mathcal{D}_{(A,u)}$ is Fredholm. Similarly, one can show that the cokernel of $\mathcal{D}^*_{(A,u)}$ can be identified with the kernel of $\mathcal{D}_{(A,u)}$ and $\mathcal{D}^*_{(A,u)}$ is Fredholm.

4.3 Proof of Proposition 3.19

The goal of this subsection is to show that the mixed operator $\mathcal{D}_{(A_\alpha, u_\alpha)}$ associated to a constant solution (A_α, u_α) is an isomorphism. We start with the following general result about the elements of $E^1_{(A_\alpha, u_\alpha)}$.

Lemma 4.40. *For any $(\zeta, \nu) \in E^1_{(A_\alpha, u_\alpha)}$, the expression*

$$\int_X \text{tr}(d_{A_\alpha}^{h,h'} \zeta \wedge d_{A_\alpha}^{h,h'} \zeta) + \int_{U_+} \Omega(d\nu, d\nu) \quad (4.41)$$

vanishes, where

$$d_{A_\alpha}^{h,h'} \zeta := d_{A_\alpha} \zeta + *_3 \text{Hess}_{B_\alpha} h(\zeta_t) + *_3 \text{Hess}_{B'_\alpha} h'(\zeta'_t). \quad (4.42)$$

with B_α (resp. B'_α) being the the restrictions of A_α to $\{t\} \times Y_0 \subset X$ (resp. $\{t\} \times -Y'_0 \subset X$), which is independent of t . The term $d\nu$ in (4.41) denotes the exterior derivative of $\nu : U_+ \rightarrow T_\alpha \mathcal{M}(\Sigma, F)$.

In (4.42), $*_3 \text{Hess}_{B_\alpha} h(\zeta_t)$ and $*_3 \text{Hess}_{B'_\alpha} h'(\zeta'_t)$ are defined as in (3.16), and in what follows, they are respectively denoted by $H(\zeta_t)$ and $H(\zeta'_t)$.

Proof. Stokes theorem and the decay constraints on ν , given as part of the definition of $E^1_{(A_\alpha, u_\alpha)}$, imply that

$$\begin{aligned} \int_{U_+} \Omega(d\nu, d\nu) &= \int_{U_\partial} \Omega(\nu, d\nu) + \int_{\eta_+ \sqcup \eta'_+} \Omega(\nu, d\nu) \\ &= \int_{-\infty}^{\infty} d\theta \int_\Sigma \text{tr}(\nu(0, \theta) \wedge \partial_\theta \nu(0, \theta)) \end{aligned} \quad (4.43)$$

The second identity is due to the Lagrangian boundary condition satisfied by ν . We shall show that the contribution from the first integral in (4.41) cancels out the integral in (4.43).

We decompose the domain of the first integral in (4.41) into $\mathbf{R} \times Y_0$, $\mathbf{R} \times -Y'_0$ and $U_- \times \Sigma$, and study the contributions from each of them separately. The restriction of ζ to $\{t\} \times Y_0$ has the form

$$\zeta = \zeta_t + \phi_t dt \quad (4.44)$$

where ϕ_t is a 0-form with values in E . Therefore, we have

$$d_{A_\alpha}^{h,h'} \zeta = d_{B_\alpha} \zeta_t + *_3 \text{Hess}_{B_\alpha} h(\zeta_t) + dt \wedge \left(\frac{d\zeta_t}{dt} - d_{B_\alpha} \phi_t \right) \quad (4.45)$$

where $d_{B_\alpha} \zeta_t$ and $d_{B_\alpha} \phi_t$ denote the three dimensional exterior derivatives of ζ_t and ϕ_t with respect to B_α . Identity (4.45) implies that

$$\text{tr}(d_{A_\alpha}^{h,h'} \zeta \wedge d_{A_\alpha}^{h,h'} \zeta) = 2dt \wedge \text{tr} \left(\left(\frac{d\zeta_t}{dt} - d_{B_\alpha} \phi_t \right) \wedge (d_{B_\alpha} \zeta_t + H(\zeta_t)) \right). \quad (4.46)$$

Using Stokes theorem and Lemma 2.13, we have

$$2 \int_{Y_0} \text{tr} \left(\frac{d\zeta_t}{dt} \wedge (d_{B_\alpha} \zeta_t + H(\zeta_t)) \right) = \frac{d}{dt} \int_{Y_0} \text{tr}(\zeta_t \wedge (d_{B_\alpha} \zeta_t + H(\zeta_t))) - \int_{\partial Y_0} \text{tr}(\zeta_t \wedge \frac{d}{dt} \zeta_t). \quad (4.47)$$

Stokes theorem, vanishing of $H(\zeta_t)$ on ∂Y and Proposition 2.10 give

$$\begin{aligned} 2 \int_{Y_0} \text{tr}(d_{B_\alpha} \phi_t \wedge (d_{B_\alpha} \zeta_t + H(\zeta_t))) &= 2 \int_{\partial Y_0} \text{tr}(\phi_t \wedge d_{B_\alpha} \zeta_t) \\ &= \int_{\partial Y_0} \text{tr}(\phi_t \wedge d_{B_\alpha} \zeta_t) + \int_{\partial Y_0} \text{tr}(\zeta_t \wedge d_{B_\alpha} \phi_t). \end{aligned} \quad (4.48)$$

We can use (4.46), (4.47) and (4.48) and the exponential convergence of ζ to an element of $\mathcal{H}_h^1(Y; B)$ to conclude that

$$\int_{\mathbf{R} \times Y_0} \text{tr}(d_{A_\alpha}^{h,h'} \zeta \wedge d_{A_\alpha}^{h,h'} \zeta) = \int_{\partial(\mathbf{R} \times Y_0)} \text{tr}(\zeta \wedge d_{A_\alpha} \zeta). \quad (4.49)$$

A similar argument shows that

$$\int_{\mathbf{R} \times -Y'_0} \text{tr}(d_{A_\alpha}^{h,h'} \zeta \wedge d_{A_\alpha}^{h,h'} \zeta) = \int_{\partial(\mathbf{R} \times -Y'_0)} \text{tr}(\zeta \wedge d_{A_\alpha} \zeta). \quad (4.50)$$

Since A_α is flat on $U_- \times \Sigma$, Stokes theorem implies that

$$\int_{U_- \times \Sigma} \text{tr}(d_{A_\alpha}^{h,h'} \zeta \wedge d_{A_\alpha}^{h,h'} \zeta) = \int_{\partial(U_- \times \Sigma)} \text{tr}(\zeta \wedge d_{A_\alpha} \zeta) \quad (4.51)$$

By adding up the above three equations, we have the following simple form for the first term in (4.41):

$$\begin{aligned} \int_X \text{tr}(d_{A_\alpha}^{h,h'} \zeta \wedge d_{A_\alpha}^{h,h'} \zeta) &= \int_{U_\partial \times \Sigma} \text{tr}(\zeta \wedge d_{A_\alpha} \zeta) \\ &= - \int_{-\infty}^{\infty} d\theta \int_{\Sigma} \text{tr}(a(\theta) \wedge \partial_\theta a(\theta)) \end{aligned} \quad (4.52)$$

To clarify the notation in the second line, note that the restriction of ζ to $U_\partial \times \Sigma$ has the form $\zeta = a(\theta) + \psi(\theta)d\theta$ where $a(\theta)$ and $\psi(\theta)$ are respectively 1- and 0-forms on Σ with values in F . The second identity is a consequence of the assumption that for each θ the 2-dimensional exterior derivative $d_\alpha a(\theta)$ vanishes. Since $a(\theta) - v(0, \theta)$ is d_α -exact, identities (4.43) and (4.52) imply that (4.41) vanishes. \square

Now we assume that (ζ, ν) belongs to the kernel of the mixed operator $\mathcal{D}_{(A_\alpha, u_\alpha)}$:

$$d_{A_\alpha}^* \zeta = 0, \quad d_{A_\alpha}^+ \zeta + (*_3 \text{Hess}_{B_\alpha} h(\zeta_t))^+ + (*_3 \text{Hess}_{B'_\alpha} h'(\zeta'_t))^+ = 0, \quad \partial_\theta \nu - J_{s,\theta} \partial_s \nu = 0.$$

Thus we have

$$\text{tr}(d_{A_\alpha}^{h,h'} \zeta \wedge d_{A_\alpha}^{h,h'} \zeta) = |d_{A_\alpha}^{h,h'}(\zeta)|^2 \text{dvol}_X,$$

and

$$\Omega(d\nu, d\nu)(s, \theta) = 2|\partial_s \nu(s, \theta)|^2 ds \wedge d\theta.$$

We conclude from these identities and the vanishing of (4.41) that

$$d_{A_\alpha}^{h,h'} \zeta = 0, \quad d\nu = 0. \quad (4.53)$$

In particular, ν is constant, which implies that $\nu = 0$ due to its decay on the symplectic end. In particular, ζ has exponential decay on the mixed ends associated to Y and Y' .

Proposition 4.54. *For ζ as above, there is an $L_{1,\delta}^2$ section η of the bundle V over X such that $\zeta = d_{A_\alpha} \eta$.*

Proof. We construct η on the subspaces $\mathbf{R} \times Y_0$, $\mathbf{R} \times -Y'_0$ and $U_- \times \Sigma$ separately. For $(\tau, y) \in \mathbf{R} \times Y_0$, let

$$\eta_1(\tau, y) := \int_{-\infty}^{\tau} \phi_t(y) dt,$$

where ϕ_t is given in (4.44). Vanishing of $d_{A_\alpha}^{h,h'} \zeta$ implies that $\frac{d\zeta_t}{dt} = d_{B_\alpha} \phi_t$. This observation and the decay of ζ on the end that $t \rightarrow -\infty$ imply that $d_{A_\alpha} \eta_1$ is equal to ζ over the subspace $\mathbf{R} \times Y_0$. Similarly, we define η_2 on $\mathbf{R} \times -Y'_0$. On the subspace $U_- \times \Sigma$, we have $d_{A_\alpha} \zeta = 0$. The element of $\mathcal{H}^1(\Sigma; \alpha)$ represented by the 1-form $\zeta(s, \theta)$ for $(s, \theta) \in U_-$ is independent of the choice of (s, θ) . In particular, this cohomology class is trivial because of the decay assumption on ζ as $s \rightarrow -\infty$. Thus for any (s, θ) , there is a unique $\eta_3(s, \theta)$ such that $d_\alpha \eta_3(s, \theta) = \zeta(s, \theta)$. It is also straightforward to see $d_{A_\alpha} \eta_3 = \zeta$ because $d_{A_\alpha} \eta_3 - \zeta$ is d_{A_α} -closed and its restriction to $\{(s, \theta)\} \times \Sigma$ for any (s, θ) vanishes. Since the restriction of A_α to the overlaps of $\mathbf{R} \times Y$ and $U_- \times \Sigma$ (resp. $\mathbf{R} \times Y$ and $U_- \times \Sigma$) is still irreducible, the sections η_1 (resp. η_2) and η_3 agree on the overlap regions. In particular, we obtain a section η of V over X such that $\zeta = d_{A_\alpha} \eta$. It is straightforward to check that η is in $L_{1,loc}^2$. For any $\theta \in [2, \infty)$, if η_θ is the restriction of η to $\{\theta\} \times Y$, then $d_{B_\alpha} \eta_\theta$ equals the restriction of ζ to $\{\theta\} \times Y$. Since B_α is irreducible, we may conclude that η on the mixed end associated to Y belongs to $L_{1,\delta}^2$ because ζ satisfies a similar exponential decay. Similar argument shows the decay of ζ on the mixed end associated to Y' and the gauge theoretical end. \square

The identity $\zeta = d_{A_\alpha} \eta$, Stokes theorem and the boundary condition $*\zeta|_{U_\partial \times \Sigma} = 0$ implies that

$$\int_X \langle \zeta, \zeta \rangle = \int_X \langle d_{A_\alpha}^* d_{A_\alpha} \eta, \eta \rangle.$$

Since $d_{A_\alpha}^* \zeta = 0$, we conclude that $\zeta = 0$. Thus, the kernel of $\mathcal{D}_{(A_\alpha, u_\alpha)}$ is trivial.

Next, we show that the cokernel of the operator $\mathcal{D}_{(A_\alpha, u_\alpha)}$ is trivial. Let $(\mu, \xi, z) \in K_{(A_\alpha, u_\alpha)}^l$ belongs to the kernel of $\mathcal{D}_{(A_\alpha, u_\alpha)}^*$. This implies that z is a map from U_+ to $\mathcal{H}^1(\Sigma; \alpha)$. These terms satisfy

$$d_{A_\alpha} \xi + d_{A_\alpha}^* \mu + \text{Hess}_{B_\alpha} h(*_3 \mu_t) + \text{Hess}_{B'_\alpha} h'(*_3 \mu'_t) = 0, \quad \partial_\theta(J_{s, \theta} z) - \partial_s z = 0. \quad (4.55)$$

Moreover, (μ, ξ, z) satisfy the conditions spelled out in Definition 4.9.

First we show that $d_{A_\alpha} \xi$ vanishes, which immediately implies that $\xi = 0$, because A_α is an irreducible connection. In fact, we have the following identities for the L^2 norm of $d_{A_\alpha} \xi$:

$$\begin{aligned} \int_X \langle d_{A_\alpha} \xi, d_{A_\alpha} \xi \rangle &= \int_X \text{tr}(d_{A_\alpha} \xi \wedge *d_{A_\alpha}^* \mu) + \int_{-\infty}^{\infty} \int_{Y_0} \text{tr}(d_{A_\alpha} \xi_t \wedge *_3 \text{Hess}_{A_t} h(*_3 \mu_t)) dt \\ &\quad + \int_{-\infty}^{\infty} \int_{-Y'_0} \text{tr}(d_{A_\alpha} \xi_t \wedge *_3 \text{Hess}_{A'_t} h'(*_3 \mu'_t)) dt \\ &= \int_X \text{tr}(d_{A_\alpha} \xi \wedge d_{A_\alpha} \mu) - \int_{-\infty}^{\infty} \int_{Y_0} \text{tr}(\xi_t \cdot d_{A_\alpha}(*_3 \text{Hess}_{A_t} h(*_3 \mu_t))) dt \\ &\quad - \int_{-\infty}^{\infty} \int_{-Y'_0} \text{tr}(\xi_t \cdot d_{A_\alpha}(*_3 \text{Hess}_{A'_t} h'(*_3 \mu'_t))) dt \\ &= \int_{U_\partial \times \Sigma} \text{tr}(\xi \cdot d_{A_\alpha} \mu) - \int_X \text{tr}(\xi \cdot [F_{A_\alpha}, \mu]) + \int_{-\infty}^{\infty} \int_{Y_0} \text{tr}(\xi_t \cdot [F_{A_\alpha}, *_3 \mu_t]) dt \\ &\quad + \int_{-\infty}^{\infty} \int_{-Y'_0} \text{tr}(\xi_t \cdot [F_{A_\alpha}, *_3 \mu'_t]) dt. \end{aligned}$$

We use Stokes theorem in the last two identities, and Proposition 2.10 is used in the third identity. The assumption on the restriction of μ to $U_\partial \times \Sigma$ and the assumption that (A_α, u_α) is a constant solution to the mixed equation imply that the last expression is zero. Thus ξ vanishes.

We introduced $d_{A_\alpha}^{h, h'}$ in (4.42), as a deformation of the exterior derivative operator d_α acting on sections of $\Lambda^1 \otimes V$, and now we define a similar operator for sections of $\Lambda^2 \otimes V$. For a section κ of $\Lambda^2 \otimes V$ over X , let

$$d_{A_\alpha}^{h, h'} \kappa := d_{A_\alpha} \kappa - *_3 \text{Hess}_{B_\alpha} h((\iota_{\partial_t} \kappa)_t) \wedge dt - *_3 \text{Hess}_{B'_\alpha} h'((\iota_{\partial_t} \kappa)'_t) \wedge dt, \quad (4.56)$$

where $(\iota_{\partial_t} \kappa)_t$ is obtained by contracting $\kappa|_{\mathbf{R} \times Y_0}$ with respect to ∂_t and then restricting it to $\{t\} \times Y_0$. The 1-form $(\iota_{\partial_t} \kappa)'_t$ is defined similarly by replacing Y_0 with $-Y'_0$. A straightforward calculation using Proposition 2.10 shows that $d_{A_\alpha}^{h, h'} d_{A_\alpha}^{h, h'} \zeta = 0$ for a section of $\Lambda^1 \otimes V$. As a consequence of the first identity in (4.55) and the vanishing of ξ , $d_{A_\alpha}^{h, h'} \mu$ vanishes.

Lemma 4.57. *There is $(\zeta, \nu) \in E_{(A_\alpha, u_\alpha)}^l$ such that:*

$$2\mu = d_{A_\alpha}^{h, h'} \zeta, \quad z d\theta + J_{s, \theta} z ds = d\nu. \quad (4.58)$$

This lemma allows us to conclude the triviality of the kernel of $\mathcal{D}_{(A_\alpha, u_\alpha)}^*$ as in the case of the kernel of $\mathcal{D}_{(A_\alpha, u_\alpha)}$ in the following way. On one hand, the expression in (4.41) vanishes for the pair (ζ, ν) produced by the lemma because $(\zeta, \nu) \in E_{(A_\alpha, u_\alpha)}^l$. On the other hand, $d_{A_\alpha}^{h, h'} \zeta$ is self-dual and $d\nu$ at (s, θ) is a

$(0, 1)$ -form with respect to $J_{s,\theta}$, and a similar argument as in the previous case shows that the expression in (4.41) is non-positive and it is equal to zero if and only if ζ and ν vanish. This shows that (μ, ξ, z) is trivial.

Proof. The pair of μ and $\eta := zd\theta + J_{s,\theta}zds$ satisfies:

- (i) $\mu \in L_{l,\delta}^2(X, \Lambda^2 \otimes V)$ and $\eta \in L_{k,\delta}^2(U_+, \Lambda^1 \otimes T_\alpha \mathcal{M}(\Sigma, F))$;
- (ii) $d_{A_\alpha}^{h,h'} \mu = 0$ and $d\eta = 0$;
- (iii) At any point $(0, \theta) \in U_\partial$, we have $2\iota_{\partial_\theta} \mu(0, \theta)$ is d_α -closed and represents the same cohomology class as $\iota_{\partial_\theta} \eta(0, \theta)$.

We prove a more general result showing that for any μ and η as above there is $(\zeta, \nu) \in E_{(A_\alpha, u_\alpha)}^l$ such that (4.58) holds.

The transversality of the Lagrangians $L(Y, E)$ and $L(Y', E')$ implies that there are $c \in T_\alpha L(Y, E)$ and $c' \in T_\alpha L(Y', E')$ such that

$$\int_{-\infty}^{\infty} \iota_{\partial_\theta} \eta(0, \theta) d\theta = c - c'. \quad (4.59)$$

Note that our assumption on z implies that the integral on the left exists. Then c and c' determine $b \in \mathcal{H}_h^1(Y; B_\alpha)$ and $b' \in \mathcal{H}_{h'}^1(Y'; B'_\alpha)$. We define a section ζ_0 of $\Lambda^1 \otimes V$ which is supported in $(2, \infty) \times Y$, and over this subspace of X it is given by

$$\zeta_0(\theta, y) = 2f(\theta) \left(\frac{1}{2} \pi^*(b) - \int_\theta^\infty (\iota_{\partial_\theta} \mu(y, \tau)) d\tau \right),$$

where $f : \mathbf{R} \rightarrow \mathbf{R}$ is a smooth function that is equal to 1 on $(3, \infty)$ and vanishes on $(-\infty, 5/2)$. Similarly define

$$\nu_0(s, \theta) = f(\theta) \left(c - \int_\theta^\infty \iota_{\partial_\theta} \eta(s, \tau) d\tau \right).$$

Then $(\zeta_0, \nu_0) \in E_{(A_\alpha, u_\alpha)}^l$. Moreover, the identities in (4.55) and the decay assumptions on μ and z imply that over the space $Y \times (3, \infty)$ we have $2\mu = d_{A_\alpha}^{h,h'} \zeta_0$ and $\eta = d\nu_0$. Let $f' : \mathbf{R} \rightarrow \mathbf{R}$ be another bump function that equals 1 on $(-\infty, -3)$ and vanishes over $(-5/2, \infty)$. Define

$$\zeta'_0(\theta, y) = 2f'(\theta) \left(\frac{1}{2} \pi^*(b') + \int_{-\infty}^\theta (\iota_{\partial_\theta} \mu(y, \tau)) d\tau \right), \quad \nu'_0(s, \theta) = f'(\theta) \left(c' + \int_{-\infty}^\theta \iota_{\partial_\theta} \eta(s, \tau) d\tau \right).$$

We again have $(\zeta'_0, \nu'_0) \in E_{(A_\alpha, u_\alpha)}^l$. Moreover, the pair

$$\mu_0 := \mu - d_{A_\alpha}^{h,h'} (\zeta_0 + \zeta'_0), \quad \eta_0 := \eta - d(\nu_0 + \nu'_0),$$

satisfies properties (i)-(iii) stated above, μ_0 vanishes on $(3, \infty) \times Y$ and $(-\infty, -3) \times Y'$, and η_0 vanishes on $[0, 2] \times (3, \infty)$ and $[0, 2] \times (-\infty, -3)$. Moreover, there is a section λ of F over $\Sigma \times [-2, 0]$ such that for any $x \in \Sigma$, $s \in [-2, 0]$ and $s' \in [0, 2]$, we have

$$\int_{-\infty}^{\infty} \iota_{\partial_\theta} \mu_0|_{\{(s,\theta)\} \times \Sigma} d\theta = d_\alpha \lambda(x, s), \quad \int_{-\infty}^{\infty} \iota_{\partial_\theta} \eta_0(s', \theta) d\theta = 0. \quad (4.60)$$

The second identity in (4.60) for $s' = 0$ follows readily from (4.59). We obtain the identity for all values of s' using the assumption that η_0 is closed. The first identity in (4.60) for $s = 0$ follows from the second one and property (iii) of (μ_0, η_0) . This can be extended to all values of s using $d_{A_\alpha}^{h, h'} \mu = 0$ and the Stokes' theorem.

Next, we modify μ_0 and η_0 such that in addition to the properties mentioned in the previous paragraph, they vanish in a neighborhood of the matching line U_∂ . Fix a bump function $f_\partial : \mathbf{R} \rightarrow \mathbf{R}$ that is equal to 1 on the interval $[-1, 1]$ and vanishes outside the interval $[-2, 2]$. Let also $h : \mathbf{R} \rightarrow \mathbf{R}$ be a compactly supported bump function with support in $[-2, 2]$ whose integral over \mathbf{R} equals 1. Define

$$\zeta_\partial(x, s, \theta) := 2f_\partial(s) \int_{-\infty}^{\theta} -h(\tau) d_\alpha \lambda(x, s) + \iota_{\partial_\theta} \mu_0(x, s, \tau) d\tau, \quad \nu_\partial(s, \theta) := f_\partial(s) \int_{-\infty}^{\theta} \iota_{\partial_\theta} \eta_0(s, \tau) d\tau$$

where $(x, s, \theta) \in \Sigma \times [-2, 0] \times [-3, 3]$ and $(s, \theta) \in [0, 2] \times [-3, 3]$. Extend ζ_∂ in the trivial way to the rest of X , and extend ν_∂ in the trivial way to the rest of U_+ . Then we can see $(\zeta_\partial, \nu_\partial)$ belongs to $E_{(A_\alpha, u_\alpha)}^l$ using the identities in (4.60). From the definition, it is clear that the support of ζ_∂ is contained in $\Sigma \times [-2, 0] \times [-3, 3]$ and the support of ν_∂ is contained in $[0, 2] \times [-3, 3]$. If we define

$$\mu_1 := \mu_0 - d_{A_\alpha}^{h, h'}(\zeta_\partial), \quad \eta_1 := \eta_0 - d\nu_\partial,$$

then (μ_1, η_1) satisfies (i)-(iii), μ_1 vanishes on $(3, \infty) \times Y$, $(-\infty, -3) \times X$ and the neighborhood $\Sigma \times [-1, 0] \times U_\partial$ of the matching line, and η_1 vanishes on $[0, 2] \times (3, \infty)$, $[0, 2] \times (-\infty, -3)$ and the neighborhood $[0, 1] \times U_\partial$ of the matching line.

The support of μ_1 is contained in an open subspace K_- of X which is diffeomorphic to $(-\infty, 3) \times Y_\#$ (see Figure 3). We may assume that the diffeomorphism from K_- to $(-\infty, 3) \times Y_\#$ is given by the identity map on $(-\infty, -3) \times Y_\#$, $(-\infty, 3) \times Y_0$ and $(-\infty, 3) \times -Y_0'$. We use this diffeomorphism to identify K_- with $(-\infty, 3) \times Y_\#$ and for any $(t, y_\#) \in (-\infty, 3) \times Y_\#$, we define

$$\zeta'_\#(t, y_\#) := 2 \int_{-\infty}^t (\iota_{\partial_\theta} \mu_1(\tau, y_\#)) d\tau$$

Then $d_{A_\alpha}^{h, h'} \zeta'_\# = \mu_1$. In particular, the restriction of $\zeta'_\#$ to $\{3\} \times Y_\#$ is in the kernel of $d_{B_\alpha} + *_3 \text{Hess}_{B_\alpha} h + *_3 \text{Hess}_{B_\alpha'} h'$. Non-degeneracy of $\alpha \in \mathfrak{C}_G$ implies that there is a section ϕ of $E_\#$ such that $\zeta'_\#(3, \cdot) = d_{B_\alpha} \phi$. Let $f_\# : (-\infty, 3) \rightarrow \mathbf{R}$ be a function which equals 0 on $(-\infty, 2)$ and equals 1 in a neighborhood of 3, and modify $\zeta'_\#$ as $\zeta_\# = \zeta'_\# - d_{A_\alpha}(f_\# \phi)$. Then $\zeta_\#$ vanishes in a neighborhood of $\{3\} \times Y_\#$ and we may extend it to X trivially. Now $(\zeta_\#, 0) \in E_{(A_\alpha, u_\alpha)}^l$ and $d_{A_\alpha}^{h, h'} \zeta_\# = \mu_1$. Similarly, we may find $\nu_\#$ such that $(0, \nu_\#) \in E_{(A_\alpha, u_\alpha)}^l$ and $d\nu_\# = \eta_1$. Consequently, the pair

$$\zeta := \zeta_0 + \zeta'_0 + \zeta_\partial + \zeta_\#, \quad \nu := \nu_0 + \nu'_0 + \nu_\partial + \nu_\#,$$

gives the desired claim. \square

4.4 Mixed shifting

We start this subsection by introducing a special type of mixed pairs.

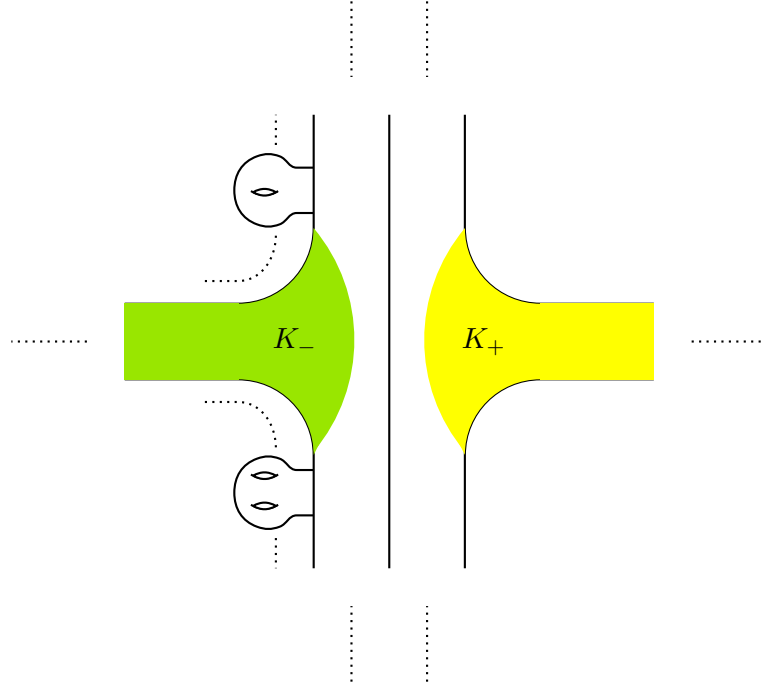


Figure 3: The support of μ_1 is contained in the set K_- and the support of η_1 is contained in K_+

Definition 4.61. An element $(A, u) \in \mathbf{A}(\alpha, \beta)$ is *symplectically constant* if (A, u) , restricted to the complement of $(-\infty, -3] \times Y_\#$, is equal to a constant pair (A_β, u_β) associated to β . In particular, the map u is a constant map to β .

Lemma 4.62. For any $(A, u) \in \mathbf{A}(\alpha, \beta)$, there is a path from (A, u) to a symplectically constant pair in $\mathbf{A}(\alpha, \beta)$.

To verify the lemma, it is helpful to give a different parametrization of the 4-manifold X and the 2-dimensional domain U_+ . Identify $\mathbf{R} \times [-1, 1]$ in the standard way with the subspace of \mathbf{C} given by numbers whose imaginary parts are in $[-1, 1]$. There is a diffeomorphism $\Phi_+ : U_+ \rightarrow ([0, \infty) \times [-1, 1]) \setminus \{\pm \mathbf{i}\}$ which satisfy the following conditions (see Figure 4).

- (i) The restriction of Φ_+ to the subspace $[3, \infty) \times [-1, 1]$ of U_+ is given by the identity map.
- (ii) On the subspace $[0, 2] \times [2, \infty)$ of U_+ , we have

$$\Phi_+(s, \theta) = \mathbf{i} + e^{-\theta + \pi \mathbf{i} \frac{s-2}{4}}.$$

- (iii) For any $(s, \theta) \in U_+$, $\Phi_+(s, -\theta)$ is equal to the complex conjugate of $\Phi_+(s, \theta)$. In particular, on the subspace $[0, 2] \times (-\infty, -2]$ of U_+ , we have:

$$\Phi_+(s, \theta) = -\mathbf{i} + e^{\theta + \pi \mathbf{i} \frac{2-s}{4}}.$$

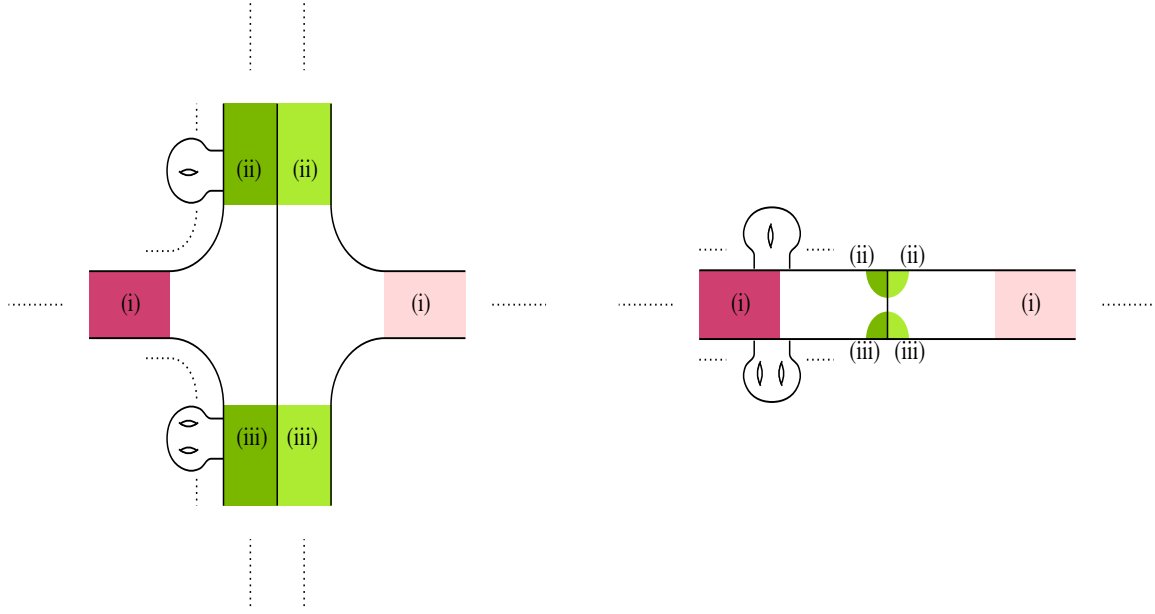


Figure 4: The old and the new parametrizations of U_+ and X : The diffeomorphisms Φ_+ and Φ_- map the spaces on the left to the spaces on the right while mapping each colored region with a label to a region with the same color and label.

There is also a diffeomorphism $\Phi_- : X \rightarrow ((-\infty, 0] \times Y_{\#}) \setminus (\{0\} \times Y_0 \cup \{0\} \times -Y'_0)$ such that the following hold.

- (i) The restriction of Φ_- to the subspace $U_- \times \Sigma$ of X is given by

$$\Phi_-(s, \theta, x) = (-\Phi_{+,s}(-s, \theta), \Phi_{+,\theta}(-s, \theta), x),$$

where $\Phi_+(s, \theta) = (\Phi_{+,s}(s, \theta), \Phi_{+,\theta}(s, \theta)) \in [0, \infty) \times [-1, 1]$.

- (ii) On the subspace $\mathbf{R} \times Y_0$ of X , we have

$$\Phi_-(\tau, y) = (f(\tau), y),$$

where $f : \mathbf{R} \rightarrow (-\infty, 0)$ is an increasing smooth function which is determined by the restriction of Φ_- to the subspace $\eta_- \times \Sigma$ and satisfies

$$f(\tau) = \begin{cases} \tau & \tau \leq -3, \\ -e^{-\tau} & \tau \geq 2. \end{cases}$$

- (iii) On the subspace $\mathbf{R} \times -Y'_0$ of X , we have

$$\Phi_-(\tau, y') = (f(\tau), y').$$

Before delving into the technical aspects of the proof of Lemma 4.62, we discuss the main idea of the construction of a path from a mixed pair $(A, u) \in \mathbf{A}(\alpha, \beta)$ to a symplectically constant pair. Using the above reparametrization, we may regard u as a map from $[0, \infty) \times [-1, 1]$ to $\mathcal{M}(\Sigma, F)$ and A as a connection on $(-\infty, 0] \times Y_{\#}$. (Strictly speaking, we have to remove $\pm \mathbf{i}$ from the domain of u and the subspace $\{0\} \times Y_0 \cup \{0\} \times -Y'_0$ from $(-\infty, 0] \times Y_{\#}$.) Let A_u be a connection on $[0, \infty) \times Y_{\#}$ such that for any $(s, \theta) \in [0, \infty) \times [-1, 1]$, the restriction of A_u to $\{(s, \theta)\} \times \Sigma \subset \{s\} \times Y_{\#}$ represents $u(s, \theta)$, its restriction to $\{0\} \times Y_{\#}$ agrees with the restriction of A to $\{0\} \times Y_{\#}$ and for any $s \in [0, \infty)$, the restriction of A_u to $\{s\} \times Y_0 \subset \{s\} \times Y_{\#}$ (resp. $\{s\} \times -Y'_0 \subset \{s\} \times Y_{\#}$) represents $u(s, 1) \in L(Y, E)$ (resp. $u(s, -1) \in L(Y', E')$). Then we may shift the mixed pair to the gauge theory side and define the pair (A_{τ}, u_{τ}) for any $\tau \in [0, \infty)$

$$A_{\tau}(s, y) := \begin{cases} A(s + \tau, y) & s \leq -\tau \\ A_u(s + \tau, y) & s > -\tau \end{cases}, \quad u_{\tau}(s, y) := u(s + \tau, y). \quad (4.63)$$

As τ tends to infinity, the pair (A_{τ}, u_{τ}) converges to a symplectically constant pair.

The above argument needs to be modified to guarantee that the mixed pairs (A_{τ}, u_{τ}) belong to the function space used in the definition of $\mathbf{A}(\alpha, \beta)$. Before applying the above shifting construction, we pick a path from (A, u) to a smooth mixed pair (A', u') which satisfies the following additional assumptions. (In the following, we use the old parametrization of the spaces X and U_+ .)

- (i) The restriction of u' to the subspace $[3, \infty) \times [-1, 1]$ of U_+ is the constant map to β .
- (ii) On a tubular neighborhood of η_+ (resp. η'_+) identified with $\eta_+ \times (\frac{1}{2}, 1]$ (resp. $\eta'_+ \times [-1, -\frac{1}{2})$) the map u is equal to the pullback of a smooth map from η_+ to $L(Y, E)$ (resp. η'_+ to $L(Y', E')$).
- (iii) The restriction to $[0, 2] \times [2, \infty)$ (resp. $[0, 2] \times (-\infty, -2]$) of u is the constant map to an element $q \in L(Y, E)$ (resp. $q' \in L(Y', E')$).
- (iv) There is a smooth function $w : [-2, 2] \rightarrow \mathcal{M}(\Sigma, F)$ such that for $(s, \theta) \in [0, 1] \times [-2, 2]$, $u(s, \theta) = w(\theta)$.
- (v) The restriction of A to the subspaces $[2, \infty) \times Y$ (resp. $(-\infty, -2] \times Y'$) is the pull-back of a connection B (resp. B') on E (resp. E') which is a representative for q (resp. q').
- (vi) The restriction of A to $[-1, 0] \times [-2, 2] \times \Sigma$ is the pullback of a smooth connection $B_{\#}$ on $[-2, 2] \times \Sigma$ with a vanishing $d\theta$ component. In particular, $B_{\#}|_{\{\theta\} \times \Sigma}$ is flat and is a representative for $w(\theta)$.

Next, we wish to lift u' to a connection A'_u on $[0, \infty) \times Y_{\#}$. In the following, we use the new reparametrization of X and U_+ . In particular, A' can be identified as a connection on $((-\infty, 0] \times Y_{\#}) \setminus (\{0\} \times Y_0 \cup \{0\} \times -Y'_0)$, which can be extended smoothly to a connection on $(-\infty, 0] \times Y_{\#}$. For any $(r, t) \in [0, \infty) \times [-1, 1]$, let $\alpha(r, t)$ be the unique connection on F which satisfies

- (i) $\alpha(r, t)$ is a flat connection representing $u(r, t)$;
- (ii) $\alpha(0, t) = A'|_{\Sigma \times \{(0, t)\}}$ for any t ;

$$(iii) \quad d_{\alpha(r,t)}^* \partial_r \alpha(r,t) = 0.$$

For any $r \in [0, \infty)$, we fix smooth connections $B(r)$ and $B'(r)$ on Y and Y' such that

- (i) $B(r)$ (resp. $B'(r)$) represents an element of $L(Y, E)$ (resp. $L(Y', E')$) whose restriction to the tubular neighborhood of the boundary of Y (resp. Y') is determined by $\alpha(r, 1)$ (resp. $\alpha(r, -1)$);
- (ii) the restriction of $B(0)$ to Y_0 (resp. $B'(0)$ to Y'_0) is equal to the restriction of A' to $\{0\} \times Y_0$ (resp. $\{0\} \times -Y'_0$).

The flat connections $\alpha(r, t)$ determine a smooth connection on $\Sigma \times [0, \infty) \times [-1, 1]$ with vanishing dr and dt components, and the connections $B(r)$, $B'(r)$ determine connections on $[0, \infty) \times Y_0$, $[0, \infty) \times -Y'_0$ with vanishing dr components. Gluing these connections determines the desired connection A'_u on $[0, \infty) \times Y_{\#}$. Now it is easy to see that the above shifting operation in (4.63) applied to A' and A'_u provides a smooth path in $\mathbf{A}(\alpha, \beta)$ from (A', u') to a symplectically constant pair. In fact, the same argument addresses the family version of Lemma 4.62.

Lemma 4.64. *For a compact space T , suppose $f : T \rightarrow \mathbf{B}(\alpha, \beta)$ is a smooth map. Then there is a smooth map $F : T \times [0, 1] \rightarrow \mathbf{B}(\alpha, \beta)$ such that for any $x \in T$, $F(x, 0) = f(x)$ and $F(x, 1)$ is a symplectically constant pair. Moreover, if $f(x)$ is already symplectically constant pair, then $F(x, t)$ is a symplectically constant pair for any t .*

4.5 Proof of Proposition 3.20

Suppose $A \in \mathcal{A}_G(\alpha, \beta)$ is a connection on the cylindrical manifold $\mathbf{R} \times Y$ and $(A', u) \in \mathbf{A}(\beta, \gamma)$ is a mixed pair. We assume that $\mathcal{A}_G(\alpha, \beta)$ and $\mathbf{A}(\beta, \gamma)$ are defined using the same representative for β . For any $T \in [3, \infty)$, we can glue A and (A', u) to define an element $(A_T, u) \in \mathbf{A}(\alpha, \gamma)$. The connection A_T is defined as follows.

- (i) On the cylinder $(-\infty, -2T] \times Y_{\#}$, A_T is equal to $\tau_{4T}^*(A)$, the translation of the restriction A over $(-\infty, 2T] \times Y_{\#}$ by $4T$.
- (ii) On the complement of $(-\infty, -T] \times Y_{\#}$, A_T is equal to A' .
- (iii) On the cylinder $(-2T, T) \times Y_{\#}$, A_T is equal to $\rho(\frac{s+2T}{T}) \cdot \tau_{4T}^*(A) + (1 - \rho(\frac{s+2T}{T})) \cdot A'$ where $\rho : [0, 3] \rightarrow [0, 1]$ is a fixed smooth function with $\rho(t) = 1$ if $t \leq 1$ and $\rho(t) = 0$ if $t \geq 2$.

By putting the connections A and $A'|_{(-\infty, -3] \times Y_{\#}}$ in the temporal gauge, the above gluing construction descends to gluing an element of $\mathcal{B}_G(\alpha, \beta)$ and $\mathbf{B}(\alpha, \beta)$.

Proposition 4.65. *The topological energy of (A_T, u) and the index of $\mathcal{D}_{(A_T, u)}$ are given by*

$$\mathcal{E}(A_T, u) = \mathcal{E}(A) + \mathcal{E}(A', u), \quad (4.66)$$

and

$$\text{index}(\mathcal{D}_{(A_T, u)}) = \text{index}(\mathcal{D}_A) + \text{index}(\mathcal{D}_{(A', u)}). \quad (4.67)$$

Proof. Lemma 3.11 implies that $\mathcal{E}(A_T)$ is independent of T . Thus the identity in (4.66) can be obtained by taking the limit $T \rightarrow \infty$. The additivity formula in (4.67) is the counterpart of the additivity of the index of the ASD operator with respect to gluing [Don02, Section 3.3] and a similar argument can be used to prove (4.67). \square

Proof of Proposition 3.20. Since the index of the mixed operator and topological energy of mixed pairs are locally constant, Lemma 4.62 implies that it suffices to prove Proposition 3.20 for symplectically constant pairs. A symplectically constant pair $(A, u) \in \mathbf{A}(\alpha, \beta)$ can be obtained by gluing a constant pair (A_β, u_β) and a connection $A \in \mathcal{A}_G(\alpha, \beta)$. Now Propositions 2.53, 3.19 and 4.65 give the index formula for mixed operators. \square

Proof of Lemma 3.15. Suppose $[A, u], [A', u'] \in \mathbf{B}(\alpha, \beta)$. Using Lemma 4.62 we may assume that $[A, u]$ and $[A', u']$ are symplectically constant pairs without changing their topological energies. Thus, after picking appropriate representatives for the connections A and A' , we may assume that they agree on the complement of $(-\infty, -3] \times Y_\#$. In particular, these two connections induce connections A_G and A'_G on $\mathbf{R} \times Y_\#$ which represent elements of $\mathcal{B}_G(\alpha, \beta)$. Characterization of the components of $\mathcal{B}_G(\alpha, \beta)$ implies that $2(\mathcal{E}(A_G) - \mathcal{E}(A'_G))$ is an integer, and hence a similar result holds for $2(\mathcal{E}(A, u) - \mathcal{E}(A', u'))$. Moreover, if $\mathcal{E}(A_G) = \mathcal{E}(A'_G)$, then A_G and A'_G can be connected to each other by a path of connections which is fixed on $[-3, \infty) \times Y_\#$. This induces a path between the mixed pairs $[A, u]$ and $[A', u']$. \square

The following is a consequence of Proposition 3.20 and Lemma 3.15.

Corollary 4.68. *For any smooth $(A, u) \in \mathbf{A}(\alpha, \alpha)$, the index $\mathcal{D}_{(A, u)}$ is a multiple of 4.*

There is a variant of Proposition 4.65 where a mixed pair $(A, u) \in \mathbf{A}(\alpha, \beta)$ is glued to a map $u' : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ representing a path from $\beta \in \mathfrak{C}_S$ to $\gamma \in \mathfrak{C}_S$. After arranging an appropriate chart for a neighborhood of β in $\mathcal{M}(\Sigma, F)$ and the Lagrangians $L(Y, E)$ and $L(Y', E')$, we may follow a similar process as in the previous case to define $(A, u_T) \in \mathbf{A}(\alpha, \gamma)$ for T large enough. The proof of the following proposition is similar to Proposition 4.65.

Proposition 4.69. *The topological energy of (A, u_T) and the index of the mixed operator $\mathcal{D}_{(A, u_T)}$ is given by*

$$\mathcal{E}(A, u_T) = \mathcal{E}(A, u) + \frac{1}{4\pi^2} \int_{\mathbf{R} \times Y} (u')^* \Omega \quad (4.70)$$

and

$$\text{index}(\mathcal{D}_{(A, u_T)}) = \text{index}(\mathcal{D}_{(A, u)}) + \text{index}(\mathcal{D}_{u'}). \quad (4.71)$$

4.6 Proof of Propositions 2.27 and 2.54

Proof of Proposition 2.27. Suppose $\alpha \in \mathfrak{C}_S$ and $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ is a smooth map that is the constant map to α on the complement of the compact region $[-1, 1] \times [-1, 1]$. This map determines an element γ_u of $\pi_1(\Omega(L, L'), o_\alpha)$ in an obvious way and we have

$$[\Omega](\gamma_u) = \int_{\mathbf{R} \times [-1, 1]} u^* \Omega, \quad \mu(\gamma_u) = \text{index}(\mathcal{D}_u). \quad (4.72)$$

We may glue u_α to the constant map (A_α, u_α) to define a mixed pair $(A_\alpha, u') \in \mathbf{A}(\alpha, \alpha)$ which satisfies the following properties by Propositions 3.19, 4.69 and the identities in (4.72)

$$\mathcal{E}(A_\alpha, u') = \frac{1}{4\pi^2}[\Omega](\gamma_u), \quad \text{index}(\mathcal{D}_{(A_\alpha, u')}) = \mu(\gamma_u). \quad (4.73)$$

As a consequence of Proposition 3.20 we can conclude that

$$\mu(\gamma_u) = \frac{2}{\pi^2}[\Omega](\gamma_u).$$

This implies that $(L(Y, E), L(Y', E'))$ is a monotone pair. The second identity in (4.73) and Corollary 4.68 imply that the minimal Maslov number of the pair is divisible by 4.

The minimal Maslov number of the pair $(L(Y, E), L(Y', E'))$ is in fact equal to 4. This follows from the well-known fact that $c_2(\mathcal{M}(\Sigma, F))$ is twice the generator of $H^2(\mathcal{M}(\Sigma, F), \mathbf{Z})$ [Ram73, AB83]. (This fact can be also derived from the arguments used in this section.) Since $\mathcal{M}(\Sigma, F)$ is simply connected, there is an element of $\pi_2(\mathcal{M}(\Sigma, F))$ whose pairing with $c_2(\mathcal{M}(\Sigma, F))$ is any given even integer. Thus we may change the Maslov number of $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ as above by any multiple of 4 after gluing to a sphere in $\mathcal{M}(\Sigma, F)$.

Assuming that \mathfrak{C}_S is non-empty, monotonicity of the pair $(L(Y, E), L(Y', E'))$ implies that each of $L(Y, E)$ and $L(Y', E')$ is an oriented monotone Lagrangian with minimal Maslov number 4. To avoid the assumption on \mathfrak{C}_S , we may assume $(Y', E') = (Y, E)$ and use two different perturbation terms h, h' for $L(Y, E)$ such that the corresponding Lagrangians intersect non-trivially and transversely. In any case, replacing $L(Y', E')$ with $L(Y, E)$ turns out to be unnecessary because $I_*(Y_\#, E_\#)$ is always non-trivial [KM04, KM10]. Next, let l be a loop in $L(Y, E)$. Let $v : D^2 \rightarrow \mathcal{M}(\Sigma, F)$ with $v|_{\partial D^2} = l$. Since the Maslov index of the disc v is an even integer, $TL(Y, E)$ is orientable. Thus $L(Y, E)$ and similarly $L(Y', E')$ are orientable. \square

Proof of Proposition 2.54. Suppose $\alpha, \beta \in \mathfrak{C}_{S,o}$ where o denotes a connected component of the path space $\Omega(L(Y, E), L(Y', E'))$, and $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ is a smooth map representing a path from α to β . Thus u satisfies

$$u(-s, \theta) = \alpha, \quad u(s, \theta) = \beta, \quad \forall (s, \theta) \in [1, \infty) \times [-1, 1] \quad (4.74)$$

and

$$u|_{\mathbf{R} \times \{1\}} \subset L(Y, E), \quad u|_{\mathbf{R} \times \{-1\}} \subset L(Y', E'). \quad (4.75)$$

Gluing the constant pair (A_α, u_α) to u produces a constant pair (A_α, u_T) which can be connected to a symplectically constant pair (A, u_β) by Lemma 4.62. The latter mixed pair is obtained by gluing a connection $A' \in \mathcal{A}_G(\alpha, \beta)$ to the constant pair (A_β, u_β) . A similar argument as above using Propositions 3.19, 4.65 and 4.69 shows

$$\text{index}(\mathcal{D}_u) = \text{index}(\mathcal{D}_{A'}).$$

The above identity implies that the relative grading of α and β with respect to deg_S and deg_G agree with each other. This completes the proof of Proposition 2.54. \square

4.7 Orientability of the mixed determinant lines

The smooth elements of $\mathbf{B}(\alpha, \beta)$ parametrize a family of Fredholm operators given by the mixed operators. Associated to this family of Fredholm operators, we can associate a determinant line bundle δ^M over the subspace of $\mathbf{B}(\alpha, \beta)$ given by smooth elements, where the fiber of δ^M over $[A, u]$ can be identified with

$$\Lambda^{\max} \ker(\mathcal{D}_{(A,u)}) \otimes (\Lambda^{\max} \operatorname{coker}(\mathcal{D}_{(A,u)}))^*.$$

We shall show in the next section that the elements of $\mathbf{M}(\alpha, \beta)$ are smooth. In particular, δ^M induces a line bundle on $\mathbf{M}(\alpha, \beta)$ whose restriction to the open subspace $\mathbf{M}^{\text{reg}}(\alpha, \beta)$ of regular elements of $\mathbf{M}(\alpha, \beta)$ can be naturally identified with the orientation bundle of the manifold $\mathbf{M}^{\text{reg}}(\alpha, \beta)$. Therefore, we are interested in trivializing δ^M to orient the moduli spaces of solutions to the mixed equation. Moreover, we use orientability of δ^M to verify the claim in Proposition 2.38.

To prove triviality of δ^M , it suffices to show that its restriction to any loop $\gamma : S^1 \rightarrow \mathbf{B}(\alpha, \beta)$ is orientable. Using Lemma 4.64, we may assume that γ parametrizes an S^1 -family of symplectically constant pairs. In particular, there is a connection A_β representing β and a loop $\gamma_G : S^1 \rightarrow \mathcal{B}_G(\alpha, \beta)_p$ such that γ is obtained by gluing γ_β to $[A_\beta, u_\beta]$ in the same way as in Subsection 4.5. The family version of (4.67) in Proposition 4.65, which can be proved again using essentially the same arguments as in [Don02, Section 3.3], implies that the restriction of δ^M to the family given by the loop γ can be identified with the tensor product $\delta_p^G|_{\gamma_G} \otimes \delta^M|_{[A_\beta, u_\beta]}$. In particular, orientability of δ_p^G implies that δ^M is orientable.

The above argument can be also used to fix an orientation of δ^M . First we fix an orientation of the line bundles δ_p^G over the configuration spaces $\mathcal{B}_G(\alpha, \beta)_p$. For any connected component of $\mathbf{B}(\alpha, \beta)$, we fix a symplectically constant pair $[A, u]$ which exists according to Lemma 4.62. Thus, (A, u) is obtained from gluing a connection $A_G \in \mathcal{A}_G(\alpha, \beta)_p$ over $\mathbf{R} \times Y_\#$ to a constant pair (A_β, u_β) . The fiber of δ^M over $[A, u]$ is isomorphic to $\delta_p^G|_{[A_G]} \otimes \delta^M|_{[A_\beta, u_\beta]}$, and the isomorphism is canonical up to multiplication by a positive constant. Since the kernel and the cokernel of the mixed operator associated to the mixed solution are trivial, $\delta^M|_{[A_\beta, u_\beta]}$ can be naturally identified with \mathbf{R} . Therefore, the fixed orientation of δ_p^G determines an orientation of $\delta^M|_{[A, u]}$. This orientation is independent of the choice of $[A, u]$. If $[A', u']$ is another element in the same connected component of $\mathbf{B}(\alpha, \beta)$, then there is a path $\gamma : [0, 1] \rightarrow \mathbf{B}(\alpha, \beta)$ from $[A, u]$ to $[A', u']$. Using Lemma 4.64, we may assume that γ is in fact a path in the subspace of symplectically constant pairs. Therefore, orientations of δ^M induced by $[A, u]$ and $[A', u']$ agree with each other.

We use a similar trick to show that the line bundles δ_p^S over the configuration spaces of strips $\mathcal{B}_S(\alpha, \beta)_p$ are trivial, and then fix a trivialization of these line bundles. Let $\gamma_S : S^1 \rightarrow \mathcal{B}_S(\alpha, \beta)_p$ be a loop. By changing this loop using a homotopy, we may assume that γ_S is represented by a smooth map $U : \mathbf{R} \times [-1, 1] \times S^1 \rightarrow \mathcal{M}(\Sigma, F)$ such that for $s \geq 1$, we have $U(-s, \theta, t) = \alpha$ and $U(s, \theta, t) = \beta$. We may glue this loop to the constant mixed pair (A_α, u_α) as in Subsection 4.5 to define $\gamma^T : S^1 \rightarrow \mathbf{A}(\alpha, \beta)$ for T large enough. The family version of Proposition 4.69 implies that $\delta^M|_{\gamma^T}$ is isomorphic to the tensor product $\delta^M|_{[A_\alpha, u_\alpha]} \otimes \delta_p^S|_\gamma$. In particular, $\delta_p^S|_\gamma$ is trivial, which verifies our claim.

We may fix an orientation of δ_p^S in the same way as in the case of δ^M . Given a strip $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ satisfying (4.74) and (4.75), we may glue the constant mixed pair (A_α, u_α) to u to define

another mixed pair (A_α, u') . Since $\delta^M|_{[A_\alpha, u']}$ and $\delta^M|_{[A_\alpha, u_\alpha]} \otimes \delta_p^S|_u$ are isomorphic and $\delta^M|_{[A_\alpha, u']}$ and $\delta^M|_{[A_\alpha, u_\alpha]}$ have fixed orientations, we obtain an orientation of $\delta_p^S|_u$. This induces a well-defined orientation of δ_p^S . These orientations are compatible with the strip gluing maps in (2.36) because the fixed orientations on δ_p^G are compatible with the cylinder gluing maps in (2.55). In summary, we obtain a coherent system of orientations for the line bundles δ_p^S .

Remark 4.76. We use gluing theory of various indices to define orientations of the line bundles δ^M and δ_p^S in terms of the orientations of the line bundles δ_p^G . Recall that we had a degree of freedom to orient δ_p^G . To define this orientation, we fixed an orientation of $\delta_{p_0}^G$ where p_0 is a path from a fixed $\alpha_0 \in \mathfrak{C}_G$ to itself, whose index has the form $8k + 4$. One such path p_0 can be fixed as follows. Let $s : S^2 \rightarrow \mathcal{M}(\Sigma, F)$ represent an element of $\pi_2(\mathcal{M}(\Sigma, F))$ which is associated to one of the connected components of Σ and is introduced at the end of Subsection 2.2. Gluing s to the constant strip mapped to α_0 determines $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ with $\text{index}(\mathcal{D}_u) = 4$. Applying the mixed shifting operation of Subsection 4.4 gives a connection on $\mathbf{R} \times Y_\#$, which represents the desired path p_0 . The sphere gluing map $\Psi_{p,s}$ in Remark 2.39 allows us to define an orientation of the index of \mathcal{D}_u . Then using the construction of this subsection in the reverse order, we may fix an orientation of $\delta_{p_0}^G$. Using this orientation of $\delta_{p_0}^G$, one may easily see that the induced coherent system of orientations of the line bundles δ_p^S is compatible with $\Psi_{p,s}$ when s is the above element of $\pi_2(\mathcal{M}(\Sigma, F))$.

5 Non-linear analysis

In this section we shall prove Proposition 3.6 and part of Proposition 3.26. Our primary tools are the compactness and regularity theorems of [DFL] together with some standard results about the solutions of ASD and pseudo-holomorphic curve equations. As in the previous section, (Y, E) and (Y', E') are fixed as in Subsection 2.1, and we fix Lagrangian 3-manifolds associated to these pairs that have transversal intersection and the claim of Lemma 2.56 holds. We continue to drop h and h' from our notations for the 3-manifolds Lagrangians, and denote them by $L(Y, E)$ and $L(Y', E')$.

5.1 Review of the results of [DFL] on regularity and compactness

The following compactness theorem from [DFL, Theorem 3] can be regarded as a common generalization of Uhlenbeck and Gromov compactness theorems to moduli spaces of solutions to the mixed equation.

Theorem 5.1. *Suppose $\mathfrak{q} = (X, V, S, \mathcal{M}(\Sigma, F), \mathbb{L})$ is a quintuple as in Subsection 3.1. There is a constant \bar{h} such that the following holds. Suppose $\{(A_i, u_i)\}$ is a sequence of solution of the mixed equation (3.2) associated to \mathfrak{q} such that*

$$\mathcal{E}(A_i, u_i) \leq \kappa$$

for a fixed constant κ . Then there are

- (i) a subsequence $\{(A_i^\pi, u_i^\pi)\}$ of $\{(A_i, u_i)\}$,
- (ii) a solution of the mixed equation (A_0, u_0) for the quintuple \mathfrak{q} ,

(iii) finite sets $\sigma_- \subset \text{int}(X)$, $\sigma_\partial \subset \gamma$ and $\sigma_+ \subset S \setminus \gamma$,

such that the following holds.

(i) The pair (A_0, u_0) satisfies the energy bound

$$\mathcal{E}(A_0, u_0) \leq \limsup_i \mathcal{E}(A_i, u_i).$$

If any of the sets σ_- , σ_∂ and σ_+ is nonempty, then the above inequality can be improved by subtracting \hbar from the right hand side.

(ii) u_i^π is C^∞ -convergent to u_0 on any compact subspace of $S \setminus (\sigma_+ \cup \sigma_\partial)$.

(iii) There are gauge transformations g_i^π defined over $X \setminus (\sigma_\partial \times \Sigma \cup \sigma_-)$ such that $(g_i^\pi)^* A_i^\pi$ is C^∞ convergent to A_0 on any compact subspace of $X \setminus (\sigma_\partial \times \Sigma \cup \sigma_-)$.

In the above theorem, one should think about σ_- , σ_+ and σ_∂ as the sets where the bubbling phenomenon happens. We have bubbling of the ASD equation on σ_- , bubbling of the holomorphic curve equation on σ_+ and mixed bubbling on σ_∂ . We need a slightly more general version of this compactness theorem where the mixed equation is perturbed by terms similar to the perturbation terms that appear in (3.22). To be more specific, we consider a generalization of the mixed equation where holomorphic equation part of the mixed equation is defined using a family of domain dependent almost complex structures and the ASD equation is deformed by holonomy perturbations. We shall make the precise type of such perturbations clear in the subsequent section. For now, we just point out that we only consider perturbations that in a neighborhood of γ in S the almost complex structure is the standard one J_* , and in a neighborhood of $\gamma \times \Sigma$ in X , the holonomy perturbation of the ASD equation is trivial. We call any such perturbation a *standard perturbation* of the mixed equation, which is *trivial in a neighborhood of the matching line*.

Theorem 5.2. Suppose $\mathfrak{q} = (X, V, S, \mathcal{M}(\Sigma, F), \mathbb{L})$ is given as in Theorem 5.1. Suppose the mixed equation associated to \mathfrak{q} is deformed by a standard perturbation, which is trivial in a neighborhood of the matching line, and $\{(A_i, u_i)\}$ is a sequence of solutions to the perturbed mixed equation. Then the same claim as in Theorem 5.1 holds except that the last part of the claim should be replaced with

(iii)' There are gauge transformations g_i^π defined over $X \setminus (\sigma_\partial \times \Sigma \cup \sigma_-)$ such that for any p , the connections $(g_i^\pi)^* A_i^\pi$ are L_1^p convergent to A_0 on any compact subspace of $X \setminus (\sigma_\partial \times \Sigma \cup \sigma_-)$. This convergence can be improved to C^∞ if σ_- is empty.

Proof. Theorem 5.1 has a local nature. First, one obtains a compactness theorem for *nice* neighborhoods of points in $\text{int}(X)$, γ and $S \setminus \gamma$. (By a neighborhood around a point $p \in \gamma$, we mean the disjoint union of a neighborhood of $\{p\} \times \Sigma$ in X and a neighborhood of p in S .) A neighborhood around a given point is nice if $\|F_{A_i}\|_{L^2}$ and $\|\nabla u_i\|_{L^2}$ are universally bounded by a specific constant \hbar in the neighborhood. Then a patching argument as in [DK90, Chapter 4] allows us to obtain the global compactness theorem. As a result the argument of the proof of Theorem 5.1 can be easily adapted to prove this variation. For points in γ , we may use the assumption to find neighborhoods where the ASD equation is not deformed and the

chosen family of complex structures on $\mathcal{M}(\Sigma, F)$ is the constant family given by J_* . For points in $S \setminus \gamma$, we may find neighborhoods contained in $S \setminus \gamma$ where we can use the Gromov compactness theorem for the pseudo-holomorphic curve equation with respect to a domain dependent almost complex structure (see, for example, [Gro85]). For points in $\text{int}(X)$, we may use compactness theorem for the deformation of the ASD equation (see [Uhl82a, Uhl82b, DK90, Kro05]). Here due to the non-local nature of holonomy perturbations one can only obtain L_1^p convergence in the presence of bubbles. A detailed treatment of this issue can be found in [Kro05] (in the more general case of $\mathbf{PU}(N)$ -connections.) \square

Next, we turn to regularity of solutions of the mixed equation. First we focus on quintuples which capture all novel issues for the moduli of solutions to the mixed equation. Suppose B_r is the unit disc of radius r centered at the origin in the (s, θ) -plane, and $D_+(r)$, $D_-(r)$ denote the intersections with the half planes $s \geq 0$ and $s \leq 0$. Let also $U_\partial(r)$ denote the intersection of $D_+(r)$ and $D_-(r)$. Consider the quintuple

$$\mathfrak{Q}(r) := (D_-(r) \times \Sigma, D_-(r) \times F, D_+(r), \mathcal{M}(\Sigma, F), \emptyset).$$

The standard metric on $D_-(r)$ and the fixed metric on Σ induce the product metric on $D_-(r) \times \Sigma$. Suppose (A, u) is a solution of the mixed equation (3.4) associated to the quintuple $\mathfrak{Q}(r)$ such that A satisfies the Coulomb gauge fixing condition

$$d_{A_0}^*(A - A_0) = 0, \quad *(A - A_0)|_{U_\partial(r) \times \Sigma} = 0. \quad (5.3)$$

Here A_0 is an auxiliary smooth connection on $D_-(r) \times \Sigma$. The following is Theorem 1 in [DFL].

Theorem 5.4. *Suppose $p > 2$ and (A, u) is an L_1^p solution of the mixed equation associated to $\mathfrak{Q}(r)$ satisfying (5.3). Then (A, u) is smooth.*

Suppose that (A, u) is a solution of the mixed equation given as in the statement of Proposition 3.6. Let x be a point in the matching line U_∂ , and $D_-(r) \times \Sigma$ (resp. $D_+(r)$) is a neighborhood of $\{x\} \times \Sigma$ (resp. x) which embeds into X (resp. S). For a fixed $2 < p < 4$, we may find an L_2^p gauge transformation h and a smooth connection A_0 on $D_-(r) \times \Sigma$ such that h^*A satisfies (5.3) with respect to the connection A_0 . Then $(h^*A|_{D_-(r) \times \Sigma}, u|_{D_+(r)})$ satisfies the assumptions of Theorem 5.4, and hence this pair is smooth. Standard regularity results for the solutions of the ASD equation (perturbed by a holonomy perturbation) and pseudo-holomorphic curves (with respect to a domain dependent almost complex structure) allow us to obtain similar results for the interior points of X and S . Thus, u is smooth and for any point $x \in X$, there is a gauge transformation g on a neighborhood of x such that $\tilde{A} := g^*A$ is smooth. We may use the patching argument of [DK90] to obtain a global gauge transformation g on X such that g^*A is smooth.

In the process of the construction of the gauge transformation g , we may obtain a stronger regularity result on the the gauge theoretic end of X . Since the restriction of A to $(-\infty, -3] \times Y_\#$ is an ASD connection, we may find a gauge transformation h on this end and a connection α representing an element of \mathfrak{C}_G such that $h^*A - \pi_\#^*(\alpha)$ is in L_l^2 for any l [Don02, Chapter 4]. Since the Lagrangians $L(Y, E)$ and $L(Y', E')$ intersect transversely, there is $\beta \in \mathfrak{C}_S$ such that $u(s, \theta) \rightarrow \beta$ as $s \rightarrow \infty$ and the restriction of du to the symplectic end $[3, \infty) \times [-1, 1]$ is in L_{l-1}^2 for the given l [Flo88c]. This verifies all parts of Proposition 3.6 except the last part about the behavior of the solutions of the mixed equation on the mixed ends, which will be taken up in Subsection 5.3. In fact, the same argument proves the generalization of these parts of Proposition 3.6 in the case that the mixed equation in (3.4) is perturbed by a standard perturbation, which is trivial in a neighborhood of the matching line.

5.2 Mixed Chern-Simons functional

We start this part by defining the 3-dimensional analogue of the configuration space of mixed pairs. Suppose c_0 denotes one of the connected components of $L(Y, E)$. Let $\mathbf{A}_{c_0}(Y, E)$ be the space of all pairs (B, q) where B is an L_t^2 connections on the bundle E over Y , and $q : [0, 2] \rightarrow \mathcal{M}(\Sigma, F)$ is an L_t^2 path such that the restriction of B to $\partial Y = \Sigma$ is flat and represents the flat connection $q(0)$, and $q(2)$ belongs to the connected component c_0 of $\mathcal{M}(\Sigma, F)$. The space of L_{t+1}^2 automorphisms of the bundle E acts on $\mathbf{A}_{c_0}(Y, E)$ in the obvious way, and we let $\mathbf{B}_{c_0}(Y, E)$ be the quotient space. A pair (B, q) representing an element of $\mathbf{B}_{c_0}(Y, E)$ is called a *flat mixed pair* if q is a constant map to an element $z \in L(Y, E)$ and B represents z . In particular, B satisfies the equation

$$*_3 F_B + \nabla_B h = 0,$$

and the subspace of $\mathbf{B}_{c_0}(Y, E)$ given by flat mixed pairs can be identified with the connected component c_0 of $L(Y, E)$.

Fix an arbitrary flat mixed pair $(B_0, q_0) \in \mathbf{A}_{c_0}(Y, E)$. Given $(B, q) \in \mathbf{A}_{c_0}(Y, E)$, there is a connection A on $[-1, 1] \times Y$ and a map $u : [0, 2] \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ such that

- (i) $A|_{\{-1\} \times Y} = B$ and $A|_{\{1\} \times Y} = B_0$;
- (ii) $u|_{[0, 2] \times \{-1\}} = q$ and $u|_{[0, 2] \times \{1\}} = q_0$;
- (iii) $q|_{\{2\} \times I} \subset L(Y, E)$;
- (iv) For any $\theta \in [-1, 1]$, $A|_{\Sigma \times \{\theta\}}$ is flat and represents $u(0, \theta)$.

Define the *mixed Chern-Simons* of the pair (B, q) as

$$CS_h(B, q) = \frac{1}{8\pi^2} \int_{[-1, 1] \times Y} \text{tr}((F_A + *_3 \nabla_{A_t} h) \wedge (F_A + *_3 \nabla_{A_t} h)) + \frac{1}{4\pi^2} \int_{[0, 2] \times [-1, 1]} u^* \Omega, \quad (5.5)$$

where A_t denotes the restriction of A to $\{t\} \times Y_0$. A priori, the value of $CS_h(B, q)$ depends on (A, u) . However, the following lemma asserts that $CS_h(B, q)$ does not change by a local deformation of (A, u) .

Lemma 5.6. *The mixed Chern-Simons functional $CS_h(B, q)$ depends only on the homotopy class of the path $\theta \in [-1, 1] \rightarrow [A|_{\{\theta\} \times Y}, u|_{[0, 2] \times \{\theta\}}] \in \mathbf{B}_{c_0}(Y, E)$ among the paths from $[B, q]$ to c_0 .*

Proof. As in Lemma 3.11, if we vary (A, u) while preserving $(A|_{\{\pm 1\} \times Y}, u|_{[0, 2] \times \{\pm 1\}})$, the expression on the left hand side of (5.5) does not change. Next, note that for any path $\gamma : [-1, 1] \rightarrow L(Y, E)$ there is a pair (A, u) satisfying (i)-(iv) such that for $(s, \theta) \in [0, 2] \times [-1, 1]$, $u(s, \theta) = \gamma(\theta)$ and $(A|_{\{\theta\} \times Y}, u|_{[0, 2] \times \{\theta\}})$ is a flat mixed pair. For any such pair, the left hand side of (5.5) vanishes. These two observations allow us to complete the proof. \square

Remark 5.7. We may alter the definition of $CS_h(B, q)$ by dropping the terms involving h from the first integrand in (5.5). Then the change in $CS_h(B, q)$ equals $\frac{1}{4\pi^2} (\tilde{h}(B_0) - \tilde{h}(B))$ which depends only $[B, q] \in \mathbf{B}_{c_0}(Y, E)$. Thus, in studying the dependence of $CS_h(B, q)$ on the homotopy class of the path

from $[B, q]$ to c_0 , which is our next goal, we may work with this alternative definition of $CS_h(B, q)$ that has a simpler form. However, we keep working with our original definition $CS_h(B, q)$, which has a more canonical role.

To see how $CS_h(B, q)$ depends on the homotopy class of the path from $[B, q]$ to c_0 , it is helpful to introduce an alternative characterization of the mixed Chern-Simons functional. Suppose the pair (\tilde{Y}, \tilde{E}) of a closed Riemannian 3-manifold and an $SO(3)$ -bundle is defined in the same way as $(Y_\#, E_\#)$ in Subsection 2.1 except that we replace (Y', E') with (Y, E) . In particular, we have the 3-manifold decomposition

$$\tilde{Y} = Y_0 \cup [-2, 2] \times \Sigma \cup -Y_0. \quad (5.8)$$

In (5.8), we slightly diverged from our convention in Subsection 2.1. The intersection of Y_0 (resp. $-Y_0$) with $[-2, 2] \times \Sigma$ is $\{-2\} \times \Sigma$ (resp. $\{2\} \times \Sigma$), and we identify Y with the subset $Y_0 \cup [-2, 0] \times \Sigma$ of \tilde{Y} . There is an obvious orientation reversing involution $\iota : \tilde{Y} \rightarrow \tilde{Y}$, and the Riemannian metric on \tilde{Y} is invariant with respect to this involution.

The function $h : \mathcal{B}(Y, E) \rightarrow \mathbf{R}$ induces a function \tilde{h} on the space of connections on \tilde{E} whose value at a connection B on \tilde{E} depends only on the gauge equivalence class of B over Y_0 and $-Y_0$. Any connection \tilde{B} on \tilde{Y} induces connections B and B' on Y_0 and $-Y_0$, and $\tilde{h}(\tilde{B})$ is equal to $h(B) - h(B')$. We perturb the Chern-Simons functional of the closed pair (\tilde{Y}, \tilde{E}) using \tilde{h} . This induces a perturbation of the flat equation on the space of connections on \tilde{E} as

$$\phi_h(\tilde{B}) = *_3 F_{\tilde{B}} + \nabla_{\tilde{B}} \tilde{h} \quad (5.9)$$

in the same way as in (2.44), which satisfies $\phi_h \circ \iota^* = -\iota^* \circ \phi_h$. The solutions of (5.9), $\mathfrak{C}(\tilde{Y}, \tilde{E})$, can be identified with the intersection of $L(Y, E)$ with itself, and hence, is equal to $L(Y, E)$.

Let (A, u) be a pair as above which connects (B, q) to a flat mixed pair (B_0, q_0) . We may use (A, u) to define a connection \tilde{A} on $[-1, 1] \times \tilde{Y}$. First we regard A as a connection on $[-1, 1] \times Y \subset [-1, 1] \times \tilde{Y}$. By applying a gauge transformation to A if necessary, we can assume that the $d\theta$ component of A on $[-1, 1] \times \{0\} \times \Sigma \subset [-1, 1] \times Y$ and the ds component of A on $[-1, 1] \times [-2, 0] \times \Sigma \subset [-1, 1] \times \tilde{Y}$ vanish. (As before, s denotes the coordinate on the interval $[-2, 0]$ and θ denotes the coordinate on $[-1, 1]$.) For any $(s, \theta) \in [0, 2] \times [-1, 1]$, let $\alpha(s, \theta)$ be the connection on F such that

- (i) $\alpha(s, \theta)$ is a flat connection representing $u(s, \theta)$;
- (ii) $\alpha(0, \theta) = A|_{\{0\} \times \Sigma \times \{\theta\}}$ for any θ ;
- (iii) $d_{\alpha(s, \theta)}^* \partial_s \alpha(s, \theta) = 0$.

For any $\theta \in [-1, 1]$, $\alpha(2, \theta) \in L(Y, E)$, and we fix a representative connection $B'(\theta)$ on $-Y_0 \subset \tilde{Y}$ for $\alpha(2, \theta)$ such that $B'(\theta)$ satisfies (5.9) over $-Y_0$, it has a vanishing ds component along the boundary of $-Y_0$ and $B'(1) = \iota^* B_0$. The connections A , $\alpha(s, t)$ and $B'(t)$ determine a connection \tilde{A} on $[-1, 1] \times \tilde{Y}$ which has vanishing ds and $d\theta$ components on $[0, 2] \times \Sigma \times [-1, 1]$ and vanishing $d\theta$ component on $[-1, 1] \times -Y_0$. Although \tilde{A} might not be smooth, it is in L_1^p for any p .

Lemma 5.10. *Suppose $CS_h(B, q)$ is defined using (A, u) . Then we have*

$$CS_h(B, q) = \frac{1}{8\pi^2} \int_{[-1,1] \times \tilde{Y}} \text{tr} \left((F_{\tilde{A}} + *_3 \nabla_{\tilde{A}_\theta} \tilde{h}) \wedge (F_{\tilde{A}} + *_3 \nabla_{\tilde{A}_\theta} \tilde{h}) \right). \quad (5.11)$$

Here \tilde{A}_θ denotes the restriction of \tilde{A} to $\{\theta\} \times Y$.

Proof. It is clear that the restriction of the integrand in (5.11) vanishes over $[-1, 1] \times -Y_0$, and its restriction over $I \times Y$ integrates to the first term on the right hand side of (5.5). Thus (5.11) follows if we show that

$$\int_{[-1,1] \times [0,2] \times \Sigma} \text{tr}(F_{\tilde{A}} \wedge F_{\tilde{A}}) = 2 \int_{[0,2] \times [-1,1]} u^* \Omega. \quad (5.12)$$

The restriction of $F_{\tilde{A}}$ to $[-1, 1] \times [0, 2] \times \Sigma$ of $[-1, 1] \times \tilde{Y}$ is given by $ds \wedge \partial_s \alpha(s, \theta) + d\theta \wedge \partial_\theta \alpha(s, \theta)$. Now it is easy to verify (5.12) using (iii) and the fact that $\partial_s \alpha(s, \theta)$ and $\partial_\theta \alpha(s, \theta)$ are $d_{\alpha(s, \theta)}$ closed. \square

Let $\tilde{B} = \tilde{A}_{-1}$ and $\tilde{B}_0 = \tilde{A}_1$ be given as in Lemma 5.10, and $\tilde{b} = \tilde{B} - \tilde{B}_0$. A straightforward examination of (5.11) shows that we may integrate out the variable θ in (5.11). After using the assumption that $*_3 \nabla_{\tilde{B}_0} \tilde{h}$ equals $-F_{\tilde{B}_0}$, we have

$$CS_h(B, q) = \frac{1}{8\pi^2} \int_{\tilde{Y}} \text{tr}(2\tilde{b} \wedge F_{\tilde{B}_0} + \tilde{b} \wedge d_{\tilde{B}_0} \tilde{b} + \frac{2}{3} \tilde{b} \wedge \tilde{b} \wedge \tilde{b}) + \frac{1}{4\pi^2} (\tilde{h}(\tilde{B}) - \tilde{h}(\tilde{B}_0)). \quad (5.13)$$

Note that \tilde{B} is essentially determined by (B, q) . The restriction of \tilde{B} to Y is B , and its restriction to $[0, 2] \times \Sigma$ is given by the flat connections $\alpha(s)$ for $s \in [0, 2]$, which are uniquely determined by

- (i) $\alpha(s)$ is a flat connection representing $q(s)$;
- (ii) $\alpha(0) = B|_{\{0\} \times \Sigma}$;
- (iii) $d_{\alpha(s)}^* \partial_s \alpha(s) = 0$.

The only ambiguity in determining \tilde{B} is to pick a representative B' for $\alpha(2)$ over $-Y_0$. In particular, any two different choices for \tilde{B} are related to each other by applying an element of $\mathcal{G}(\tilde{E})$ and then possibly the involution ι , defined as in Subsection 2.4 using the decomposition of \tilde{Y} as $Y \cup -Y$. In particular, \tilde{B} as an element of $\mathcal{B}(Y, E)/\iota$, completed with respect to the Sobolev norm L_1^2 , is well-defined.

The following proposition shows that $2CS_h$ as a map $\mathbf{B}_{c_0}(Y_0, E) \rightarrow \mathbf{R}/\mathbf{Z}$ is well-defined.

Proposition 5.14. *The value of $2CS_h(B, q)$ mod integers depends only on $[B, q] \in \mathbf{B}_{c_0}(Y, E)$ and is independent of the choice of the mixed pair (A, u) .*

Proof. Suppose (A, u) and (A', u') are two pairs as above connecting a flat mixed pair (B_0, q_0) to (B, q) and (g^*B, q) for a gauge transformation $g \in \mathcal{G}(E)$. Suppose also \tilde{A} and \tilde{A}' are defined in the same way as above. Suppose \tilde{B} and \tilde{B}' denote the restrictions of \tilde{A} and \tilde{A}' to $\{-1\} \times \tilde{Y}$. Then \tilde{B} and \tilde{B}' over the

subspaces Y and $-Y$ of \tilde{Y} are gauge equivalent to each other using elements of $\mathcal{G}(E)$. Thus \tilde{B} and \tilde{B}' are equivalent to each other either using a determinant one gauge transformation or the composition of a gauge transformation with an involution ι , defined as in Subsection 2.4 using the decomposition of \tilde{Y} as $Y \cup -Y$. In the first case, we can glue \tilde{A} and \tilde{A}' to define an $SO(3)$ connection \hat{A} on $S^1 \times \tilde{Y}$ whose w_2 is the pullback of $w_2(\tilde{E})$. We also have

$$\begin{aligned} CS_h(B, q) - CS_h(g^*B, q) &= \frac{1}{8\pi^2} \int_{S^1 \times \tilde{Y}} \text{tr} \left((F_{\hat{A}} + *_3 \nabla_{\hat{A}_\theta} \tilde{h}) \wedge (F_{\hat{A}} + *_3 \nabla_{\hat{A}_\theta} \tilde{h}) \right) \\ &= \frac{1}{8\pi^2} \int_{S^1 \times \tilde{Y}} \text{tr} (F_{\hat{A}} \wedge F_{\hat{A}}). \end{aligned} \quad (5.15)$$

The last identity is a consequence of the Stokes theorem. In general, (5.15) equals $-p_1(\hat{E})/4$, where \hat{E} denotes the underlying $SO(3)$ -bundle of \tilde{E} . Therefore, (5.15) is an integer because $w_2(\hat{A})$ is the pullback of $w_2(\tilde{E})$. A similar argument can be applied to the second case with the difference that $w_2(\hat{A})$ is the sum of the Poincaré dual of Σ_0 (one of the connected components of Σ) and the pullback of $w_2(\tilde{E})$. Thus (5.15) is a half integer in the second case. \square

The following lemma, which gives a relation between the mixed Chern-Simons functional and its gradient, plays a crucial role in showing that the solutions of the mixed equation satisfies an exponential decay on cylindrical ends.

Lemma 5.16. *There is a positive constant κ and a neighborhood U of the space of flat mixed pairs of $\mathbf{B}_{c_0}(Y, E)$ such that the following holds. There is a real lift $\widetilde{CS}_h(B, q)$ of $CS_h(B, q)$ for any $[B, q] \in U$ such that*

$$\widetilde{CS}(B, q) \leq \kappa (\|F_B + *_3 \nabla_B h\|_{L^2(Y)}^2 + \|dq\|_{L^2([0,1])}^2). \quad (5.17)$$

This lemma is an (infinite dimensional) instance of a general property for Morse-Bott functions. Suppose $f : M \rightarrow \mathbf{R}$ is a function on a finite dimensional manifold and $\mathfrak{C} \subset M$ is a submanifold of M which gives a connected component of the critical points of f consisting of Morse-Bott critical points. Then there are a constant C and a neighborhood U of \mathfrak{C} such that for any $x \in U$ we have

$$|f(x) - f_0| \leq C |\nabla f(x)|^2,$$

where f_0 is the value of f on \mathfrak{C} .

Proof. This lemma can be proved using an analogous result which holds for the Chern-Simons functional of connections on closed 3-manifolds. Define the configuration space of connections $\mathcal{B}(\tilde{Y}, \tilde{E})$ using the L_1^2 norm. Fix a connection \tilde{B}_0 satisfying (5.9) and for a connection $\tilde{B} = \tilde{B}_0 + \tilde{b}$ on the bundle \tilde{E} over \tilde{Y} , representing an element of $\mathcal{B}(\tilde{Y}, \tilde{E})$, define

$$CS_h(\tilde{B}) := \frac{1}{8\pi^2} \int_{\tilde{Y}} \text{tr}(\tilde{b} \wedge d_{\tilde{B}_0} \tilde{b} + \frac{2}{3} \tilde{b} \wedge \tilde{b} \wedge \tilde{b}) + \frac{1}{4\pi^2} \int_0^1 dt \int_{\tilde{Y}} \text{tr} \left(\tilde{b} \wedge (*_3(\nabla_{\tilde{B}_0 + t\tilde{b}} \tilde{h} - \nabla_{\tilde{B}_0} \tilde{h})) \right). \quad (5.18)$$

Note that this definition agrees with the left hand side of the expression in (5.13). The critical points of the Chern-Simons functional CS_h satisfy (5.9). In fact, with the same argument as in Proposition 5.14,

$2CS_h$ induces a map $\mathcal{B}(\tilde{Y}, \tilde{E})/\iota \rightarrow \mathbf{R}/\mathbf{Z}$, and the critical locus of this functional, denoted by $\mathfrak{C}(\tilde{Y}, \tilde{E})$, can be identified with $L(Y, E)$. A similar argument as in Lemma 2.47 shows that for any \tilde{B}_0 representing an element of $\mathfrak{C}(\tilde{Y}, \tilde{E}) \cong L(Y, E)$, the vector space

$$\mathcal{H}_h^1(\tilde{Y}; \tilde{B}_0) := \{\tilde{b} \in \Omega^1(\tilde{Y}, \tilde{E}) \mid d_{\tilde{B}_0}^* \tilde{b} = 0, *d_{\tilde{B}_0} \tilde{b} + \text{Hess}_{\tilde{B}_0} \tilde{h}(\tilde{b}) = 0\} \quad (5.19)$$

is isomorphic to the tangent space of $\mathfrak{C}(\tilde{Y}, \tilde{E})$ at \tilde{B}_0 . That is, the Chern-Simons functional CS_h of (\tilde{Y}, \tilde{E}) is Morse-Bott.

Suppose \tilde{U} is the subspace of $\mathcal{B}(\tilde{Y}, \tilde{E})/\iota$ represented by connections \tilde{B} of the form $\tilde{B}_0 + \tilde{b}$ where \tilde{B}_0 represents an element of $\mathfrak{C}(\tilde{Y}, \tilde{E})$, $\|\tilde{b}\|_{L_1^2} < \varepsilon'$, $d_{\tilde{B}_0}^* \tilde{b} = 0$ and \tilde{b} is L^2 orthogonal to $\mathcal{H}_h^1(\tilde{Y}; \tilde{B}_0)$. A straightforward application of implicit function theorem shows for ε' small enough, \tilde{U} determines an open neighborhood of $\mathfrak{C}(\tilde{Y}, \tilde{E})$, and the representation of an element of \tilde{U} as $\tilde{B}_0 + \tilde{b}$ is unique up to the action of the gauge group. In particular, (5.18) gives a well-defined real valued function on \tilde{U} . Moreover, there is a universal constant δ_1 such that

$$\delta_1 \|\tilde{b}\|_{L_1^2(\tilde{Y})} \leq \|d_{\tilde{B}_0} \tilde{b} + *_3 \text{Hess}_{\tilde{B}_0} \tilde{h}(\tilde{b})\|_{L^2(\tilde{Y})}. \quad (5.20)$$

This follows from the fact that the Chern-Simons functional CS_h of (\tilde{Y}, \tilde{E}) is Morse-Bott. The constant δ_1 can be made independent of \tilde{B}_0 because $\mathfrak{C}(\tilde{Y}, \tilde{E})$ is compact.

The inequality in (5.20) allows us to control the L_1^2 norm of \tilde{b} by the norm of the gradient of the perturbed Chern-Simons functional CS_h . We have

$$\begin{aligned} \|F_{\tilde{B}} + *_3 \nabla_{\tilde{B}} \tilde{h}\|_{L^2} &= \left\| \left(F_{\tilde{B}} + *_3 \nabla_{\tilde{B}} \tilde{h} \right) - \left(F_{\tilde{B}_0} + *_3 \nabla_{\tilde{B}_0} \tilde{h} \right) \right\|_{L^2} \\ &\geq \|d_{\tilde{B}_0} \tilde{b} + *_3 \text{Hess}_{\tilde{B}_0} \tilde{h}(\tilde{b})\|_{L^2} - \|\tilde{b} \wedge \tilde{b}\|_{L^2} - \|\nabla_{\tilde{B}} \tilde{h} - \nabla_{\tilde{B}_0} \tilde{h} - \text{Hess}_{\tilde{B}_0} \tilde{h}(\tilde{b})\|_{L^2}. \end{aligned}$$

Since we have

$$\begin{aligned} \|\nabla_{\tilde{B}} \tilde{h} - \nabla_{\tilde{B}_0} \tilde{h} - \text{Hess}_{\tilde{B}_0} \tilde{h}(\tilde{b})\|_{L^2} &\leq \left\| \left(\int_0^1 \text{Hess}_{\tilde{B}_0 + t\tilde{b}} \tilde{h}(\tilde{b}) dt \right) - \text{Hess}_{\tilde{B}_0} \tilde{h}(\tilde{b}) \right\|_{L^2}, \\ &\leq \left(\int_0^1 \|\text{Hess}_{\tilde{B}_0 + t\tilde{b}} \tilde{h} - \text{Hess}_{\tilde{B}_0} \tilde{h}\|_{L^2} dt \right) \|\tilde{b}\|_{L^2}, \end{aligned}$$

Corollary 6.2 implies that after decreasing the value of δ_1 and shrinking \tilde{U} we have

$$\begin{aligned} \|F_{\tilde{B}} + *_3 \nabla_{\tilde{B}} \tilde{h}\|_{L^2} &\geq \delta_1 \|\tilde{b}\|_{L_1^2} - \|\tilde{b}\|_{L^4}^2 \\ &\geq (\delta_1 - \|\tilde{b}\|_{L^4}) \|\tilde{b}\|_{L_1^2}. \end{aligned}$$

Thus, if ε' is small enough, then there is a constant κ_0 such that

$$\|\tilde{b}\|_{L_1^2(\tilde{Y})} \leq \kappa_0 \|F_{\tilde{B}} + *_3 \nabla_{\tilde{B}} \tilde{h}\|_{L^2(\tilde{Y})}. \quad (5.21)$$

The Chern-Simons functional of \tilde{B} in (5.18) can be bounded in the following way:

$$\begin{aligned}
CS_h(\tilde{B}) &\leq \frac{1}{8\pi^2} \left| \int_{\tilde{Y}} \text{tr}(\tilde{b} \wedge d_{\tilde{B}_0} \tilde{b} + \frac{2}{3} \tilde{b} \wedge \tilde{b} \wedge \tilde{b}) \right| + \frac{1}{4\pi^2} \int_0^1 dt \left| \int_{\tilde{Y}} \text{tr}(\tilde{b} \wedge (*_3(\nabla_{\tilde{B}_0+t\tilde{b}} \tilde{h} - \nabla_{\tilde{B}_0} \tilde{h}))) \right| \\
&\leq C \left(\|\tilde{b}\|_{L^2(\tilde{Y})} \|d_{\tilde{B}_0} \tilde{b}\|_{L^2(\tilde{Y})} + \|\tilde{b}\|_{L^3(\tilde{Y})}^3 + \|\tilde{b}\|_{L^2(\tilde{Y})}^2 \right) \\
&\leq \kappa \|F_{\tilde{B}} + *_3 \nabla_{\tilde{B}} \tilde{h}\|_{L^2(\tilde{Y})}^2,
\end{aligned} \tag{5.22}$$

where the second inequality follows from the general property of cylinder functions which is stated in part (ii) of Proposition 6.1:

$$\|\nabla_{\tilde{B}} \tilde{h} - \nabla_{\tilde{B}'} \tilde{h}\|_{L^2} \leq C \|B - B'\|_{L^2}.$$

The last inequality in (5.22) is a consequence of (5.21) and the assumption that ε' is small enough.

The upper bound on $CS_h(\tilde{B})$ in (5.22) for a connection \tilde{B} on \tilde{E} allows us to verify our main claim. There is a neighborhood U of flat mixed pairs such that for any $[B, q]$, the associated element $[\tilde{B}]$ belongs to \tilde{U} . Since we have $CS_h([B, q]) = CS_h(\tilde{B})$ and

$$\|F_{\tilde{B}} + *_3 \nabla_{\tilde{B}} \tilde{h}\|_{L^2(\tilde{Y})}^2 = \|F_B + *_3 \nabla_B h\|_{L^2(Y)}^2 + \|dq\|_{L^2([0,1])}^2,$$

(5.22) gives us the desired inequality in (5.17). \square

5.3 Exponential decay

Our next goal is to show that the solutions of the mixed equation for the special quintuples have exponential decay on the mixed ends. This subsection follows a similar scheme as the proofs of the corresponding results in the context of Yang-Mills gauge theory in [Don02, Section 4]. To obtain the desired exponential decay results and complete the proof of Proposition 3.6, we may focus on solutions of the mixed equation on the cylinder quintuple, which is introduced in Subsection 4.2.1.

Lemma 5.23. *Suppose c_0 is a connected component of $L(Y, E)$. For any open neighborhood U of the space of flat mixed pairs in $\mathbf{B}_{c_0}(Y, E)$, there is ε such that the following holds. If (A, u) is a solution of the mixed equation on the mixed cylinder $\mathfrak{c}_{(-1,1)}$ with $\mathcal{E}(A, u) < \varepsilon$, then for any $\theta \in (-\frac{1}{2}, \frac{1}{2})$, the pair $(B_\theta, q_\theta) := (A|_{\{\theta\} \times Y}, u|_{[0,2] \times \{\theta\}})$ belongs to U .*

Proof. If the claim does not hold, there is a sequence (A_i, u_i) of solutions of the mixed equation on $\mathfrak{c}_{(-\frac{1}{2}, \frac{1}{2})}$ such that $\mathcal{E}(A_i, u_i) \rightarrow 0$ as $i \rightarrow \infty$, and the class in $\mathbf{B}_{c_0}(Y, E)$ represented by the restriction of (A_i, u_i) to $(\{0\} \times Y, [0, 2] \times \{0\})$ does not belong to U . On the other hand, Theorem 5.2 implies that there is a subsequence $\{(A_i^\pi, u_i^\pi)\}$ and gauge transformations g_i^π such that $((g_i^\pi)^* A_i^\pi, u_i^\pi)$ is C^∞ convergent to the solution (A_0, u_0) of the mixed equation on $\mathfrak{c}_{(-\frac{1}{2}, \frac{1}{2})}$. In particular, $\mathcal{E}(A_0, u_0) = 0$, and after applying a gauge transformation (A_0, u_0) is the pullback of a flat mixed pair to the quintuple $\mathfrak{c}_{(-\frac{1}{2}, \frac{1}{2})}$, which is a contradiction. \square

Proposition 5.24. *There are positive constants ε , δ_0 and C such that the following holds. Suppose (A, u) is a solution of the mixed equation on the mixed cylinder $\mathfrak{c}_{(0,\infty)}$ such that $\mathcal{E}(A, u) < \varepsilon$. Then for any $\theta \in (1, \infty)$, we have*

$$\|F_A + *_3 \nabla_{A_t} h\|_{L^2((\theta-\frac{1}{2}, \theta+\frac{1}{2}) \times Y)} + \|du\|_{L^2([0,2] \times (\theta-\frac{1}{2}, \theta+\frac{1}{2}))} \leq C e^{-\delta_0 \theta}, \quad (5.25)$$

where for any $t \in (0, \infty)$, as usual A_t denotes the connection $A|_{\{t\} \times Y_0}$.

Proof. Let c_0 be the connected component of $L(Y, E)$ determined by $u(2, \theta)$ for any $\theta \in (0, \infty)$. Suppose (B_θ, q_θ) is the pair obtained by restricting A and u to $\{\theta\} \times Y$ and $[0, 2] \times \{\theta\}$. Then (B_θ, q_θ) are elements of $\mathbf{A}_{c_0}(Y, E)$. Suppose U and κ are given as in Lemma 5.16. Using Lemma 5.23, we pick $\varepsilon < \frac{1}{2}$ such that for any $\theta \in [\frac{1}{2}, \infty)$, (B_θ, q_θ) represents an element of U . Let also $\delta_0 = \frac{1}{4\pi^2 \kappa}$. Define $P : (0, \infty) \rightarrow \mathbf{R}$ by

$$P(\theta) = \frac{1}{8\pi^2} \int_{(\theta, \infty) \times Y} \text{tr}((F_A + *_3 \nabla_{A_t} h) \wedge (F_A + *_3 \nabla_{A_t} h)) + \frac{1}{4\pi^2} \int_{(\theta, \infty) \times [-1, 1]} u^* \Omega.$$

The assumption on $\mathcal{E}(A, u)$ implies that $P(\theta)$ is the lift of $CS_h(B_\theta, q_\theta)$ in $[0, \frac{1}{2})$. Since (A, u) is a solution of the mixed equation, we have

$$\frac{dP}{d\theta}(\theta) = -\frac{1}{4\pi^2} (\|F_{B_\theta} + *_3 \nabla_{B_\theta} h\|_{L^2(Y)}^2 + \|dq_\theta\|_{L^2([0,1])}^2).$$

Combining these observations and Lemma 5.16, we conclude that

$$P \leq -4\pi^2 \kappa \frac{dP}{d\theta} \quad \text{for } \theta \geq \frac{1}{2}.$$

In particular, for any $\theta \geq \frac{1}{2}$, we have the following inequality which gives the desired claim:

$$P(\theta) \leq C e^{-\delta_0 \theta}. \quad (5.26)$$

Here we can take $C = e^{\delta_0/2} \varepsilon$, because it is greater than $e^{\delta_0/2} P(\frac{1}{2})$. \square

To improve the exponential decay of Proposition 5.24, we need a variation of Theorem 4.17.

Proposition 5.27. *Suppose I and J are respectively the intervals $(-1/2, 1/2)$ and $(-1/4, 1/4)$. For any $k \geq 1$, there are constants ε_k and C_k such that the following holds. Suppose (A, u) is a solution of the mixed equation associated to the cylinder quintuple \mathfrak{c}_I with $\mathcal{E}(A, u) < \varepsilon_k$. Then for any $(\zeta, \nu) \in E_{(A,u)}^k(I)$ we have*

$$\|(\zeta, \nu)\|_{L_{k,A}^2(J)} \leq C_k \left(\|\mathcal{D}_{(A,u)}(\zeta, \nu)\|_{L_{k-1,A}^2(I)} + \|(\zeta, \nu)\|_{L^2(I)} \right). \quad (5.28)$$

where $L_{k,A}^2$ denotes the Sobolev norm defined using the connection A . In particular, only the contribution of ζ to this norm depends on A and the contribution of ν is independent of A .

Proof. Theorem 4.17 implies that for a fixed (A, u) we may find a constant C_k such that (5.28) holds for any $(\zeta, \nu) \in E_{(A,u)}^k(I)$. Using Remark 4.19 and compactness of the space of flat mixed pairs, we may in fact find a constant C'_k which works for any (A, u) which is the pullback of a flat mixed pair.

Let $C_k = 2C'_k$. If the claim does not hold for k , then there is a sequence $\{(A_i, u_i)\}$ of solutions of the mixed equation associated to the quintuple c_I such that $\mathcal{E}(A_i, u_i) < 1/i$ and there is an element of $E^k_{(A_i, u_i)}(I)$ for which the inequality in (5.28) fails for the pair (A_i, u_i) and the constant C_k . Because of our assumption about the Sobolev norms in (5.28), changing each mixed pair (A_i, u_i) by applying a gauge transformation gives us another sequence satisfying the same property. Theorem 5.2 implies that (A_i, u_i) , after passing to a subsequence and applying gauge transformations, is C^∞ convergent to a mixed pair (A_0, u_0) on compact subspaces of $I \times Y$ and $[0, 2] \times I$. The topological energy of (A_0, u_0) vanishes and hence it is the pullback of a flat mixed pair. Now, we may use Remark 4.19 to conclude that if i is large enough then the inequality in (5.28) holds for (A_i, u_i) , which is a contradiction. \square

Proposition 5.29. *For any non-negative integer l , there are positive constants ε , δ_0 and C such that the following holds. Suppose (A, u) is a solution of the mixed equation on the mixed cylinder $c_{(0, \infty)}$ such that $\mathcal{E}(A, u) < \varepsilon$ and A is in temporal gauge. Then there is a flat mixed pair (B, q) such that for any $\theta \in (1, \infty)$ and any $k \leq l$ we have*

$$|\nabla^k(A - \pi^*B)|_{\{\theta\} \times Y} \leq Ce^{-\delta_0\theta}, \quad (5.30)$$

q_θ is C^0 -convergent to the constant map to q , and

$$|\nabla^{k-1}(du)|_{\{\theta\} \times [0, 1]} \leq Ce^{-\delta_0\theta} \quad \text{for } 1 \leq k \leq l. \quad (5.31)$$

Proof. Let δ_0 be given by Proposition 5.24, and decrease the value of ε in this proposition so that it becomes smaller than the constant ε_{l+3} provided by Proposition 5.27. Let also C be the constant given by Proposition 5.24. In the following we might increase the value of C from each line to the next while keeping it independent of (A, u) . Suppose (B_θ, q_θ) is given as before, and for any $\theta \in (1, \infty)$, let $A(\theta)$ and $u(\theta)$ denote the restriction of A and u to $(\theta - \frac{1}{2}, \theta + \frac{1}{2}) \times Y$ and $[0, 2] \times (\theta - \frac{1}{2}, \theta + \frac{1}{2})$. By induction on k , we show that for any $0 \leq k \leq l + 3$ we have

$$\left\| \frac{dA}{d\theta} \right\|_{L^2_k((\theta - \frac{1}{2}, \theta + \frac{1}{2}) \times Y)} + \left\| \frac{du}{d\theta} \right\|_{L^2_k([0, 2] \times (\theta - \frac{1}{2}, \theta + \frac{1}{2}))} \leq Ce^{-\delta_1\theta}. \quad (5.32)$$

This claim in the case that $k = 0$ is proved in Proposition 5.24. Assuming (5.32), we may integrate this inequality from θ to ∞ in the case that $k = l + 3$ and show see that there is a flat mixed pair (B, q) such that the claim of this proposition holds.

The derivatives $\frac{dA}{d\theta}$ and $\frac{du}{d\theta}$ over $(\theta - \frac{1}{2}, \theta + \frac{1}{2}) \times Y$ and $[0, 2] \times (\theta - \frac{1}{2}, \theta + \frac{1}{2})$ define an element $(\zeta(\theta), \nu(\theta))$ of $E^k_{(A(\theta), u(\theta))}(I)$, where $A(\theta)$ and $u(\theta)$ are the restrictions of A and u to $(\theta - \frac{1}{2}, \theta + \frac{1}{2}) \times Y$ and $[0, 2] \times (\theta - \frac{1}{2}, \theta + \frac{1}{2})$ and $I = (-\frac{1}{2}, \frac{1}{2})$. The translation invariance of the mixed equation for cylinder quintuples implies that $(\zeta(\theta), \nu(\theta))$ is in the kernel of $\mathcal{D}_{(A(\theta), u(\theta))}$. In particular, Proposition 5.27 and (5.32) for $k = 0$ imply that

$$\|(\zeta(\theta), \nu(\theta))\|_{L^2_{k, A(\theta)}(J)} \leq Ce^{-\delta_0\theta}. \quad (5.33)$$

Using the induction assumption we may replace the left hand side of (5.33) with $\|(\zeta(\theta), \nu(\theta))\|_{L^2_k(J)}$. This allows us to prove (5.32) for the given k . \square

proof of Proposition 3.6. Most steps of Proposition 3.6 are already addressed in Subsection 5.1. The only missing part is the exponential decay of solutions of the mixed equation on the mixed ends of special quintuples, which follows from Proposition 5.29. \square

5.4 Compactness

We shall consider the compactness aspects of Proposition 3.26 in this section. Suppose $\{(A_i, u_i)\}$ is a sequence in $\mathbf{A}(\alpha, \beta)$ representing elements of $\mathbf{M}_\eta(\alpha, \beta)_d$ with $d \leq 1$, which does not have any subsequence convergent to an element of $\mathbf{M}_\eta(\alpha, \beta)_d$. Here η gives a standard perturbation of the mixed equation as in (3.22), which is trivial in a neighborhood of the matching line and is provided by Proposition 3.23. In particular, $\mathbf{M}_\eta(\alpha, \beta)_d$ is empty for negative values of d . Since the indices of the mixed operators $\mathcal{D}_{(A_i, u_i)}$ are d , Proposition 3.20 implies that $\mathcal{E}(A_i, u_i)$ is constant and hence bounded. Therefore, the analytical energy terms $\mathfrak{E}(A_i, u_i)$ are also bounded, and we may apply Theorem 5.2.

Theorem 5.2 implies that after passing to a subsequence and applying gauge transformations there is a sequence, still denoted by $\{(A_i, u_i)\}$, which is convergent in a weak sense. To be more detailed, there is a solution of the mixed equation (A_0, u_0) associated to the special quintuples q_s and finite sets $\sigma_- \subset \text{int}(X)$, $\sigma_\partial \subset \gamma$ and $\sigma_+ \subset S \setminus \gamma$ such that $\mathfrak{E}(A_0, u_0)$ is bounded, A_i is C^∞ -convergent to A_0 on compact subspaces of $X \setminus (\sigma_\partial \times \Sigma \cup \sigma_-)$ and u_i is convergent to u_0 on compact subspaces of $S \setminus (\sigma_+ \cup \sigma_\partial)$. Proposition 3.6 implies that $[A_0, u_0] \in \mathbf{M}_\eta(\alpha', \beta')_{d'}$ for some choices of $\alpha' \in \mathfrak{C}_G$, $\beta' \in \mathfrak{C}_S$ and $d' \geq 0$. To be more precise, we use adaptation of Proposition 3.6 to the perturbed mixed equation in (3.22). Since the secondary perturbations provided by Proposition 3.23 are compactly supported, we may use the exponential decay results of the previous section to prove part (iii) of Proposition 3.6 in this more general setup. As it is mentioned in Subsection 5.1, the remaining part of this generalization of Proposition 3.6 can be proved as in the original case.

Fix T_0 such that the intersections of the the finite sets σ_- , σ_∂ and σ_+ with the subspaces

$$(-\infty, -T_0] \times Y_\# \sqcup [T_0, \infty) \times Y \sqcup (-\infty, -T_0] \times Y', \quad (5.34)$$

and

$$[T_0, \infty) \times [-1, 1] \sqcup [0, 2] \times [T_0, \infty) \sqcup [0, 2] \times (-\infty, -T_0], \quad (5.35)$$

of the ends of q_s are empty and the secondary perturbations of Proposition 3.23 on these sets are empty. For any $T \geq 2$, let $(A_i(T), u_i(T))$ denote the restrictions of (A_i, u_i) to the mixed end $((T, \infty) \times Y, [0, 2] \times (T, \infty))$. Then $(A_i(T_0), u_i(T_0))$ gives a solution of the mixed equation associated to the cylinder quintuple $\mathfrak{c}_{(T_0, \infty)}$. There are gauge transformations g_i over $(T_0, \infty) \times Y$ such that $A_i(T_0)$ is in temporal gauge. Proposition 5.29 implies that $(g_i^* A_i(T_0), u_i(T_0))$ is exponentially asymptotic to a flat mixed pair (B_i, q_i) . Similarly, there is g_0 such that $(g_0^* A_0(T), u_0(T))$ is exponentially asymptotic to a flat mixed pair (B_0, q_0) . Using patching arguments of [DK90, Section 4.4.2], we may assume that the gauge transformations g_i are trivial after changing the gauge equivalence classes of connections A_i .

After passing to a further subsequence, we may assume that (B_i, q_i) is C^∞ convergent to a flat mixed pair (B, q) because the gauge equivalence classes of flat mixed pairs is a compact set. A priori, (B, q) might not be necessarily equal to (B_0, q_0) because we only know that $A_i(T) - \pi^*(B_i)$ is C^∞ -convergent to $A_0(T) - \pi^*(B)$ on compact subspaces of $((T, \infty) \times Y, [0, 2] \times (T, \infty))$. However, if there is $T \geq T_0$ such that the topological energies of $(A_i(T), u_i(T))$ for large enough values of i is less than the constant ε given by Proposition 5.29, then Proposition 5.29 and the dominated convergence theorem imply that

$$\lim_{i \rightarrow \infty} \int_{(T, \infty) \times Y} |\nabla^k ((A_0(T) - \pi^* B) - (A_i(T) - \pi^*(B_i)))|^2 e^{\delta_0 \theta} = 0, \quad 0 \leq k \leq l. \quad (5.36)$$

In particular, $B = B_0$ and hence $q = q_0$. Moreover, by working in a smooth chart about $q \in \mathcal{M}(\Sigma, F)$, we obtain a similar exponential convergence of $u_i(T)$ to $u_0(T)$.

Now, suppose there is no T satisfying the above properties. After passing to a subsequence, we may assume that there is a sequence $\{T_i\}_i$ converging to ∞ such that $\mathcal{E}(A_i(T_i), u_i(T_i))$ is equal to ε . In particular, we have

$$\mathcal{E}(A_0(T_0), u_0(T_0)) \leq \limsup_{i \rightarrow \infty} \mathcal{E}(A_i(T_0), u_i(T_0)) - \varepsilon. \quad (5.37)$$

(In fact, we may replace ε with $\frac{1}{2}$ using the fact that the minimal topological energy of a solution of the mixed equation on $\mathfrak{c}_{\mathbf{R}}$ with finite energy is $\frac{1}{2}$. However, we do not need this stronger upper bound.) We obtain a similar dichotomy for the mixed end associated to Y' : either the solutions (A_i, u_i) restricted to the mixed end $((-\infty, -T_0) \times Y', [0, 2] \times (-\infty, -T_0))$ is convergent to (A_0, u_0) as in (5.36), or there is a loss of topological energy by at least ε on the mixed end analogous to (5.37).

A similar analysis can be applied to study the behavior of the sequence $\{(A_i, u_i)\}$ on the gauge theoretical and symplectic ends of \mathfrak{q}_s , and one can even obtain more efficient results. For instance, we may apply the results of [Don02, Section 5] to the sequence of (perturbed) ASD connections $A_i|_{(-\infty, -T_0) \times Y_{\#}}$ and obtain a sequence

$$A_1^G \in \check{M}_G(\alpha, \alpha_1)_{p_1}, A_2^G \in \check{M}_G(\alpha_1, \alpha_2)_{p_2}, \dots, A_n^G \in \check{M}_G(\alpha_{n-1}, \alpha')_{p_n}$$

such that $[A_i|_{(-\infty, -T_0) \times Y_{\#}}]$ is *chain convergent* to $[A_1^G, A_2^G, \dots, A_n^G, A_0|_{(-\infty, -T_0) \times Y_{\#}}]$ on the complement of a finite set of bubbling points in the sense of [Don02, Section 5]. This means that in addition to the convergent of A_i to A_0 on compact subspaces of $(-\infty, -T_0) \times Y_{\#}$, there are finite subsets $\sigma_i \subset \mathbf{R} \times Y_{\#}$ and a sequence of real numbers

$$s_0^i = 0 < s_1^i < s_2^i < \dots < s_n^i$$

such that for any $1 \leq j \leq n$, we have $s_j^i - s_{j-1}^i \rightarrow \infty$ as $i \rightarrow \infty$ and $\tau_{s_j^i}^*(A_i)$ is C^∞ convergent to A_j^G on the compact subspaces of $\mathbf{R} \times Y_{\#} \setminus \sigma_i$. Here $\tau_{s_j^i}^*(A_i)$ denotes the translate of $A_i|_{(-\infty, -T_0) \times Y_{\#}}$ by s_j^i , which is a connection on $(-\infty, -T_0 + s_j^i) \times Y_{\#}$. If k_G is the sum of the size of the bubbling sets σ_i , the chain convergence implies that we have

$$\mathcal{E}(A_0|_{(-\infty, -T_0) \times Y_{\#}}) + \sum_{i=1}^n \mathcal{E}(A_i^G) \leq \limsup_{i \rightarrow \infty} \mathcal{E}(A_i|_{(-\infty, -T_0) \times Y_{\#}}) - k_G. \quad (5.38)$$

Similar argument shows that on the gauge theory side there is an integer k_S and a sequence of holomorphic strips

$$u_1^S \in \check{M}_S(\beta', \beta_1)_{p'_1}, u_2^S \in \check{M}_S(\beta_1, \beta_2)_{p'_2}, \dots, u_m^S \in \check{M}_S(\beta_{m-1}, \beta)_{p'_m}$$

such that the sequence of pseudo-holomorphic maps $u_i|_{(T_0, \infty) \times [-1, 1]}$ are chain convergent to the broken pseudo-holomorphic map $(u_0|_{(T_0, \infty) \times [-1, 1]}, u_1^S, u_2^S, \dots, u_m^S)$ on the complement of a set of k_S bubbling points. Consequently, we have

$$\mathcal{E}(u_0|_{(T_0, \infty) \times [-1, 1]}) + \sum_{i=1}^m \mathcal{E}(u_i^S) \leq \limsup_{i \rightarrow \infty} \mathcal{E}(u_i|_{(T_0, \infty) \times [-1, 1]}) - \frac{k_S}{2}, \quad (5.39)$$

where for a map u from an oriented Riemann surface S to $\mathcal{M}(\Sigma, F)$, we define $\mathcal{E}(u)$ as the integral of $u^*\Omega$ over S divided by $4\pi^2$. To get the term $k_s/2$ in (5.39), we use the fact that the pairing of Ω with an element of $\pi_2(\mathcal{M}(\Sigma, F))$ is a positive integer multiple of $2\pi^2$ (see Subsection 2.2).

Theorem 5.2 implies that the topological energy of (A_0, u_0) over the complement of the ends of q_s in (5.34) and (5.35) is bounded by the lim sup of the topological energies of (A_i, u_i) over the same set, and this inequality can be improved by \hbar unless $\sigma_-, \sigma_\partial$ and σ_+ are empty. Combining all of the above inequalities we conclude that

$$\mathcal{E}(A_0, u_0) + \sum_i \mathcal{E}(A_i^G) + \sum_j \mathcal{E}(u_j^S) \leq \limsup_i \mathcal{E}(A_i, u_i). \quad (5.40)$$

Since $\mathcal{E}(A_i, u_i)$ is constant, we may replace the right hand side with $\mathcal{E}(A_i, u_i)$ for any $i \geq 1$. The inequality in (5.40) is strict unless $\sigma_-, \sigma_\partial$ and σ_+ are empty, (A_i, u_i) is convergent to (A_0, u_0) with respect to the $L_{i,\delta}^2$ on the mixed ends as in (5.36), and $k_s = k_G = 0$. If the inequality is strict, then Lemma 3.15 implies that in fact the difference between the two sides of (5.40) is at least $\frac{1}{2}$. To see the latter claim note that we can glue (A_0, u_0) , the connections A_i^G and the maps u_j^S to obtain an element of $\mathbf{B}(\alpha, \beta)$, and the topological energy is additive with respect to gluing. Using Proposition 3.20 we also have

$$\text{index}(\mathcal{D}_{(A_0, u_0)}) + \sum_i \text{index}(\mathcal{D}_{A_i^G}) + \sum_j \text{index}(\mathcal{D}_{u_j^S}) \leq d, \quad (5.41)$$

and if the inequality in (5.40) is strict, then the left hand side of (5.41) is at most $d - 4$, which is impossible because $d \leq 1$ and all the indices on the left hand side of (5.41) are non-negative. In fact, $\text{index}(\mathcal{D}_{A_i^G})$ and $\text{index}(\mathcal{D}_{u_j^S})$ are at least one. Thus, in the case that $d = 0, m = n = 0$, which shows that $[A_0, u_0] \in \mathbf{M}_\eta(\alpha, \beta)_0$, and $[A_i, u_i]$ is a sequence in $\mathbf{M}_\eta(\alpha, \beta)_0$ convergent to $[A_0, u_0]$. This is a contradiction, and hence $\mathbf{M}_\eta(\alpha, \beta)_0$ is compact. In the case that $d = 1$, we conclude that either $n = 0$ or $m = 0$. In the first case, $\beta = \beta', [A_0, u_0] \in \mathbf{M}_\eta(\alpha', \beta)_0, A_1^G$ is in the 0-dimensional moduli space $\check{\mathbf{M}}_G(\alpha, \alpha')_{p_1}$, and $[A_i, u_i]$ is chain convergent to $[A_1^G, (A_0, u_0)]$. In the case that $m = 0, \alpha = \alpha', [A_0, u_0] \in \mathbf{M}_\eta(\alpha, \beta')_0, u_1^S$ is in the 0-dimensional moduli space $\mathbf{M}_S(\beta', \beta)_{p'_1}$, and $[A_i, u_i]$ is chain convergent to $[(A_0, u_0), u_1^S]$. Therefore, we can compactify $\mathbf{M}_\eta(\alpha, \beta)_1$ by adding the points

$$\check{\mathbf{M}}_G(\alpha, \alpha')_p \times \mathbf{M}_\eta(\alpha', \beta)_0, \quad \mathbf{M}_\eta(\alpha, \beta')_0 \times \check{\mathbf{M}}_S(\beta', \beta)_p, \quad (5.42)$$

as it is stated in Proposition 3.26.

To complete the proof of part (ii) of Proposition 3.26, we need a gluing theorem as an inverse to the above compactness theorem, which shows that the non-compact ends of $\mathbf{M}_\eta(\alpha, \beta)_1$ can be parametrized by gluing the *broken solutions* of the mixed equation in (5.42). Such gluing theory are standard in the context of instanton Floer homology [Flo88a], [Don02, Section 4] and Lagrangian Floer homology [Flo88b]. In the present setup, we need to either glue solutions of (perturbed) ASD equation on $\mathbf{R} \times Y_\#$ to solutions of the mixed equation on the gauge theoretical end of q_s or glue pseudo-holomorphic strips in $\mathcal{M}(\Sigma, F)$ to solutions of the mixed equation on the symplectic end of q_s . In these two cases the proofs in the gluing theory of instanton Floer homology and Lagrangian Floer homology can be adapted without any essential change to prove the desired result. Using similar arguments as in instanton Floer homology and Lagrangian Floer homology, one can also see easily that the conventions in Subsection 4.7 determine orientations of the moduli spaces that satisfy the claim in Proposition 3.26.

6 Perturbations

In several stages in the definition of instanton Floer homology group $I_*(Y_\#, E_\#)$, its symplectic counterpart $SI_*(Y_\#, E_\#)$ and the isomorphism between them, we had to use perturbations of the relevant equations. Up until this point we treat such perturbations as blackboxes, and only exploited their properties to obtain the required results. In this section, we recall the definition of these perturbations [Don87, Flo88a, Tau90, Her94, KM11] and collect the properties which are used in the earlier sections. The first subsection reviews the definition of cylinder functions. We used the formal gradients of cylinder functions to perturb the defining equation of 3-manifold Lagrangians (Subsection 2.3) and the ASD equation in the definition of $I_*(Y_\#, E_\#)$ (Subsection 2.4). The primary version of the mixed equation in (3.4) is also defined using cylinder functions h and h' . Then we use secondary perturbations to deform the mixed equation. These secondary perturbations are the main subject of the second subsection of the present section.

In this section, we slightly change our viewpoint on the configuration spaces of connections. As it is mentioned in the proof of Lemma 2.43, the $SO(3)$ -bundle $E_\#$ over $Y_\#$ is the adjoint bundle associated to a $U(2)$ -bundle $\tilde{E}_\#$ over $Y_\#$. Suppose B_0 is a fixed connection on the determinant bundle $\Lambda^2 \tilde{E}_\#$ of $\tilde{E}_\#$. Then the configuration space of $SO(3)$ -connections on $E_\#$ modulo the determinant one gauge group $\mathcal{G}(E_\#)$ can be identified with the quotient by $\mathcal{G}(E_\#)$ of the space of $U(2)$ -connections on $\tilde{E}_\#$ with B_0 being the induced connection on the determinant bundle. There is a similar description for the configuration spaces of connections on Y , Y' and X in terms of $U(2)$ -connections and we use this viewpoint through this section. (We assume that the $U(1)$ -connections fixed on these four spaces are compatible with each other in the obvious way.) In particular, holonomy of connections will be regraded as a section of a fiber bundle with fiber $U(2)$. This will give us some flexibility in the construction of holonomy perturbations.

6.1 Cylinder functions and holonomy perturbations

In this subsection we prove Proposition 2.16 and Lemma 2.56. We first recall the definition of cylinder functions and then collect some of their properties from the literature which are used in the earlier subsections. As before (Y, E) and (Y', E') denote pairs which are introduced in Subsection 2.1.

Let $\gamma : S^1 \times D^2 \rightarrow Y$ be a smooth immersion, supported in the interior of $Y_0 \subset Y$. Given a connection $B \in \mathcal{A}(Y, E)$ and $z \in D^2$, let $\tau_z(B)$ denote the trace of the holonomy of B along the loop $\gamma(S^1 \times \{z\})$. Fix a compactly supported 2-form μ on D^2 with integral 1. Define

$$\tau_\gamma(B) = \int_{D^2} \tau_z(B) \mu.$$

Now suppose λ is the data of a positive integer n , smooth immersions $\gamma_i : S^1 \times D^2 \rightarrow Y_0$ for $1 \leq i \leq n$, and a smooth function $G : [-3, 3]^n \rightarrow \mathbf{R}$. The *cylinder function* h_λ associated to λ is defined as

$$h_\lambda(B) := G(\tau_{\gamma_1}(B), \dots, \tau_{\gamma_n}(B)).$$

The formal gradient of the cylinder function h_λ can be regarded as a section ∇h_λ of the tangent bundle \mathcal{T}_l of $\mathcal{A}(Y, E)$, which is the Banach space bundle $\mathcal{A}(Y, E) \times L_l^2(Y, \Lambda^1 \otimes E)$. Clearly, h_λ is invariant with

respect to the action of $\mathcal{G}(E)$ and induces a map $\mathcal{B}(Y, E) \rightarrow \mathbf{R}$, still denoted by h_λ . The formal gradient ∇h_λ is also $\mathcal{G}(E)$ -invariant and its value at any $B \in \mathcal{A}(Y, E)$ belongs to $X_B \subset L^2_l(Y, \Lambda^1 \otimes E)$.

Following [KM11], we may form a Banach space \mathcal{P} of perturbations from cylinder functions. Fix once and for all a sequence $\{\lambda_i\}_{i \in \mathbf{N}}$ where λ_i is the information of a positive integer n_i , immersions $\gamma_{i,j} : S^1 \times D^2 \rightarrow Y_0$ for $1 \leq j \leq n_i$ and a smooth function $G_i : \mathbf{R}^{n_i} \rightarrow \mathbf{R}$. We require this sequence to be dense in the following sense: given any $(n, \{\gamma_i\}, G)$ as above there is a subsequence $\{\phi_{i_k}\}_{k \in \mathbf{N}}$ of $\{\lambda_i\}$ such that $n_{i_k} = n$, $\gamma_{i_k,j}$, for any j , is convergent to γ_j in the C^∞ topology and G_{i_k} is C^∞ convergent to G . For a sequence of real numbers $\rho = \{a_i\}_{i \in \mathbf{N}}$, define $h_\rho : \mathcal{A}(Y, E) \rightarrow \mathbf{R}$ as

$$h_\rho = \sum_{i=1}^{\infty} a_i h_{\lambda_i}.$$

To have convergent cylinder functions h_ρ belonging to appropriate function spaces, we need to control the growth of the sequence $\{a_i\}$ by a condition of the form

$$|\rho| := \sum_{i=1}^{\infty} C_i |a_i| < \infty$$

for a carefully chosen sequence of positive real numbers C_i .

Proposition 6.1. [KM11, Proposition 3.7] *The constants C_i can be chosen such that the cylinder functions h_ρ and their formal gradients ∇h_ρ satisfy the following properties.*

(i) *The association*

$$(B, \rho) \rightarrow \nabla_B h_\rho$$

determines a smooth map $H : \mathcal{A}(Y, E) \times \mathcal{P} \rightarrow \mathcal{T}_l$.

(ii) *There is a constant C such that for any $B \in \mathcal{A}(Y, E)$*

$$|h_\rho(B)| \leq C|\rho|, \quad \|\nabla_B h_\rho\|_{L^\infty} \leq C|\rho|,$$

and for any $B, B' \in \mathcal{A}(Y, E)$ and $1 \leq p \leq \infty$

$$\|\nabla_B h_\rho - \nabla_{B'} h_\rho\|_{L^p} \leq C|\rho| \|B - B'\|_{L^p}.$$

(iii) *The derivative DH of H with respect to the component in $\mathcal{A}(Y, E)$ defines a smooth section of $\mathcal{A}(Y, E) \times \mathcal{P} \rightarrow \text{Hom}(\mathcal{T}_l, \mathcal{T}_l)$ which extends smoothly to a smooth section of $\mathcal{A}(Y, E) \times \mathcal{P} \rightarrow \text{Hom}(\mathcal{T}_k, \mathcal{T}_k)$ for any $0 \leq k \leq l$. Analogous to \mathcal{T}_l , the Banach space bundle \mathcal{T}_k over $\mathcal{A}(Y, E)$ is given by $\mathcal{A}(Y, E) \times L^2_k(Y, \Lambda^1 \otimes E)$.*

The following corollary is used in the proof of Proposition 5.16.

Corollary 6.2. *For any $B \in \mathcal{A}(Y, E)$, $\rho \in \mathcal{P}$ and any constant ε , there is a neighborhood U of B such that for any $B' \in U$ we have*

$$\|\text{Hess}_B h_\rho - \text{Hess}_{B'} h_\rho\|_{L^2} < \varepsilon.$$

Proof. This follows immediately from the special case of part (iii) of Proposition 6.1 that $k = 0$. \square

Recall that $\mathbf{L}(Y, E)$ is the subspace of $\mathcal{B}(Y, E) \times \mathcal{P}$ consisting of pairs $([B], \rho)$ such that $\Phi(B, \rho) = 0$ where Φ is defined in (2.15). Let $\pi : \mathbf{L}(Y, E) \rightarrow \mathcal{P}$ be the projection map sending $([B], \rho) \in \mathbf{L}(Y, E)$ to ρ . Given $([B_0], \rho_0) \in \mathbf{L}(Y, E)$, the space $\mathbf{L}(Y, E)$ in a neighborhood of a point $([B_0], \rho_0)$ can be identified with the solutions of the equation

$$\Pi_{B_0} \circ \Phi : \mathcal{B}(Y, E) \times \mathcal{P} \rightarrow \ker(d_{B_0}^*),$$

with Π_{B_0} being projection into $\ker(d_{B_0}^*)$. The linearization of this equation acts as

$$(b, \sigma) \rightarrow *d_{B_0} b + \text{Hess}_{B_0} h_{\rho_0}(b) + \nabla_{B_0} h_\sigma \quad (6.3)$$

on the elements of $X_{B_0} \times \mathcal{P}$. Restricting to pairs $(b, 0)$ in (6.3) gives the Fredholm operator L_{B_0} in (2.9) which governs the local behavior of the space $L_{h_{\rho_0}}(Y, E) = \pi^{-1}(\rho_0) \subset \mathcal{B}(Y, E)$. According to Proposition 2.10, the kernel and the cokernel of the operator L_{B_0} are given by $\mathcal{H}_{h_{\rho_0}}^1(Y; B_0)$ and $\mathcal{H}_{h_{\rho_0}}^1(Y, \Sigma; B_0)$, and it has Fredholm index $-\frac{3}{2}\chi(\Sigma)$. We sketch a slight modification of an argument of [Her94] showing that the operator in (6.3) is surjective. Therefore, the implicit function theorem implies that $\mathbf{L}(Y, E)$ is a smooth Banach manifold and $\pi : \mathbf{L}(Y, E) \rightarrow \mathcal{P}$ is a Fredholm map with index $-\frac{3}{2}\chi(\Sigma)$.

To verify surjectivity of (6.3), it suffices to show that for any non-zero $b \in \mathcal{H}_{h_{\rho_0}}^1(Y, \Sigma; B_0)$, there is $\sigma \in \mathcal{P}$ such that $\langle \nabla_{B_0} h_\sigma, b \rangle$, which is equal to the derivative $D_{B_0} h_\sigma(b)$ of h_σ at the connection B_0 evaluated at b , is non-zero. Our assumption on the density of the sequence $\{\rho_i\}$ implies that this claim holds unless the derivative $D_{B_0} \tau_\gamma$ vanishes at b for any γ as above. The latter condition holds only if the derivative $D_{B_0} \text{Hol}_\ell$ evaluates to zero at b for any closed loop ℓ in Y_0 [Don02, Section 5.5]. Here $\text{Hol}_\ell : \mathcal{A}(Y, E) \rightarrow \text{U}(2)$ is the holonomy of a connection on Y along ℓ . The same claim holds for loops in the collar neighborhood of the boundary $Y \setminus Y_0$ because $d_{B_0} b$ vanishes on $Y \setminus Y_0$ and B_0 restricts to a flat connection on $Y \setminus Y_0$. Finally vanishing of the derivative of Hol_ℓ for closed loops ℓ in Y_0 and $Y \setminus Y_0$ can be used to conclude the vanishing of these derivatives for any closed loop in Y .

We claim that if $D_{B_0} \text{Hol}_\ell$ vanishes at b , then there is a section ζ of E on Y such that $d_{B_0} \zeta = b$. Fix a basepoint $p_0 \in \Sigma$, and for any $p \in Y$, take a path $\ell : [0, 1] \rightarrow Y$ with $\ell(0) = p_0$, $\ell(1) = p$. Trivialize $E|_\ell$ by parallel transport and define

$$\zeta(p) := \int_0^1 \ell^*(b). \quad (6.4)$$

The assumption on b implies that (6.4) is independent of the choice of ℓ and hence is well-defined. From the definition, it is straightforward to check that $d_{B_0} \zeta = b$. In particular, ζ restricts to zero on Σ because the same property holds for b and the restriction of B_0 is irreducible. From this we obtain

$$\begin{aligned} \int_Y \langle b, b \rangle &= - \int_Y \text{tr}(d_{B_0} \zeta \wedge *b) \\ &= - \int_\Sigma \text{tr}(\zeta \wedge *b) - \int_Y \text{tr}(\zeta \wedge *d_{B_0}^* b) = 0, \end{aligned}$$

where in the second identity we use $d_{B_0}^* b = 0$ and the vanishing of ζ on Σ . This shows that $b = 0$, which is a contradiction and hence the operator in (6.3) is surjective.

Proposition 6.5. *Suppose (B_0, ρ_0) is as above. Suppose a is an L^2_{l-1} section of $\Lambda^1 \otimes E$ with $d^*_{B_0} a = 0$. Then there are $\sigma \in \mathcal{P}$ and $b \in L^2_l(Y, \Lambda^1 \otimes E)$ with $d^*_B b = 0$ and $*b|_\Sigma = 0$ such that*

$$a = *d_{B_0} b + \text{Hess}_{B_0} h_{\rho_0}(b) + \nabla_{B_0} h_\sigma \quad (6.6)$$

*Similarly, if a is an L^2_{l-1} section of $\Lambda^1 \otimes E$ with $d^*_{B_0} a = 0$ and $*a|_\Sigma = 0$, then there are $\sigma \in \mathcal{P}$ and $b \in L^2_l(Y, \Lambda^1 \otimes E)$ with $d^*_B b = 0$ and $b|_\Sigma = 0$ such that (6.6) holds.*

Proof. We already addressed the first part of this lemma. To prove the second part, we consider the map in (6.3) as an operator from $X'_{B_0} \times \mathcal{P}$ with

$$X'_{B_0} := \{b \in L^2_l(Y, \Lambda^1 \otimes E) \mid d^*_B b = 0, b|_\Sigma = 0\},$$

to the subspace of $\ker(d^*_{B_0})$ given by a with $*a|_\Sigma = 0$. Restricting to pairs $(b, 0)$ determines an operator L'_{B_0} with the kernel and the cokernel $\mathcal{H}^1_{h_{\rho_0}}(Y, \Sigma; B_0)$ and $\mathcal{H}^1_{h_{\rho_0}}(Y; B_0)$. Now a similar argument as above can be used to complete the proof. \square

Proposition 6.7 (cf. [Her94]). *The map $r : \mathbf{L}(Y, E) \rightarrow \mathcal{M}(\Sigma, F)$ which maps any element $([B_0], \rho_0)$ to the restriction α_0 of B_0 to the boundary is a submersion.*

This proposition together with Proposition 6.5 completes the proof of Proposition 2.16.

Proof. Suppose c is a 1-form with values in F over Σ which represents an element of $T_{\alpha_0} \mathcal{M}(\Sigma, F)$. That is to say, $d_{\alpha_0} c = 0$. Extend c to a smooth section b_0 of $\Lambda^1 \otimes E$ over Y . Then $a := *d_{B_0}(b_0) + \text{Hess}_{B_0} h_{\rho_0}(b_0)$ is in the kernel of $d^*_{B_0}$ and hence Proposition 6.5 applied to a implies that there are $\sigma \in \mathcal{P}$ and $b_1 \in L^2_l(Y, \Lambda^1 \otimes E)$ with $d^*_B b_1 = 0$ and $b_1|_\Sigma = 0$ such that

$$*d_{B_0}(b_1 - b_0) + \text{Hess}_{B_0} h_{\rho_0}(b_1 - b_0) + \nabla_{B_0} h_\sigma = 0.$$

In particular, the restriction of $b_0 - b_1$ to Σ is equal to c . There is an L^2_{l+1} section ζ of E such that $b := b_1 - b_0 - d_{B_0} \zeta$ is in the Coulomb gauge

$$d^*_{B_0} b = 0, \quad *b|_\Sigma = 0.$$

Thus (b, σ) gives a vector tangent to $\mathbf{L}(Y, E)$ whose restriction to the boundary represents the same element as c . \square

Next, we turn into the proof of Lemma 2.56. As before, suppose $(Y_\#, E_\#)$ is an admissible pair with an admissible splitting

$$(Y, E) \cup_{(\Sigma, F)} (Y', E').$$

Associated to the pairs (Y, E) , (Y', E') we may form Banach spaces \mathcal{P} and \mathcal{P}' parametrizing perturbations. Since the map $\pi : \mathbf{L}(Y, E) \rightarrow \mathcal{P}$ is Fredholm, the Sard-Smale theorem implies that there is a residual subset \mathcal{P}_{reg} of \mathcal{P} such that for any $\sigma \in \mathcal{P}_{\text{reg}}$, the space $L_{h_\sigma}(Y, E)$ is a smooth Lagrangian in $\mathcal{M}(\Sigma, F)$. We fix one such element σ of \mathcal{P}_{reg} . Similarly we may form a residual subset $\mathcal{P}'_{\text{reg}}$ of \mathcal{P}' . The restriction map $r' : \mathbf{L}(Y', E') \rightarrow \mathcal{M}(\Sigma, F)$ is transversal to $L_{h_0}(Y, E)$ because the former map is a submersion.

Therefore, by passing to a smaller residual subset, we may assume that any $\sigma' \in \mathcal{P}'_{\text{reg}}$ has the property that $L_{h_0}(Y, E)$ and $L_{h_{\sigma'}}(Y', E')$ are transversal to each other. We fix one such σ' .

The functions h_σ and $h_{\sigma'}$ induce a perturbation of the flat equation for connections on $E_\#$ as in (2.44). We follow the same notation as before to denote the solutions of this equation with \mathfrak{C}_G . According to Lemma 2.47 elements of \mathfrak{C}_G are regular. We form (2.48) which is a perturbation of the ASD equation on configuration spaces $\mathcal{B}(\alpha, \beta)_p$ with $\alpha, \beta \in \mathfrak{C}_G$. Regularity of elements of \mathfrak{C}_G implies that these equations are Fredholm (see, for example, [Don02, Chapter 4]). Note that the perturbation of the ASD equation in (2.48) are supported in the subset $\mathbf{R} \times (Y_0 \sqcup Y'_0)$ of the cylinder $\mathbf{R} \times Y$. Lemma 2.56 is a consequence of the following proposition.

Proposition 6.8. *The Riemannian metrics on Y, Y' and cylinder functions h and h' supported in the interior of $Y_0 \subset Y$ and $Y'_0 \subset Y'$ can be chosen such that the following conditions are satisfied.*

- (i) *Solutions of the perturbation of the flat equation (2.44) for the functions h and h' agree with \mathfrak{C}_G .*
- (ii) *Solutions of the perturbation of the ASD equation (2.48) for the functions h and h' with index at most seven are regular.*

Here the assumption on the index of the solutions of (2.48) is not essential and we are making this assumption to avoid the bubbling phenomena.

Proof. The proof is just a slight modification of a similar result in [Don02, Section 5.5]. The main difference is that we want to guarantee our perturbation vanishes in the complement of $Y_0 \sqcup Y'_0$. Following the notation of Subsection 2.1, let Y_1 (resp. Y'_1) be the union of Y_0 and $[-1, 0] \times \Sigma$ (resp. Y'_0 and $[0, 1] \times \Sigma$). Then the intersection of Y_1 and Y'_1 is a copy of Σ with the collar neighborhood $[-1, 1] \times \Sigma$. Fix Riemannian metrics on Y and Y' in the same way as before. We show inductively that for any $k \leq 7$, there is a pair (h_k, h'_k) of cylinder functions such that h_k and h'_k agree with h_σ and $h'_{\sigma'}$ in a neighborhood of \mathfrak{C}_G , $\nabla_B h_k$ (resp. $\nabla_{B'} h'_k$) is compactly supported in the interior of Y_1 (resp. Y'_1) for any B (resp. B'), solutions of the perturbation of the flat equation (2.44) for the pair (h_k, h'_k) agree with \mathfrak{C}_G , and the moduli space $M_G(\alpha, \beta)_p$ of solutions to the perturbed ASD equation associated to (h_k, h'_k) is regular for any path p of index at most k . In fact, the pair $h_k - h_\sigma, h'_k - h'_{\sigma'}$ are sums of finitely many cylinder functions where our assumption on cylinder functions is slightly relaxed and we allow cylinder functions associated to immersion into the interior of Y_1 and Y'_1 . In particular, h_k and h'_k vanish in the regular neighborhood $[-\varepsilon, \varepsilon] \times \Sigma$ of $Y_1 \cap Y'_1$ if ε is small enough. Note that if k is small enough, then solutions of the perturbed ASD equation with index at most k should have negative topological energy and hence these moduli spaces are empty. Thus the claim for such values of k holds if we set $(h_k, h'_k) = (h_\sigma, h'_{\sigma'})$.

Next, we show that the claim holds for k assuming that it already holds for $k - 1$. We find a collection of cylinder functions (h_ρ, h'_ρ) where ρ belongs to a finite dimensional vector space $\mathcal{P}_\#$, (h_ρ, h'_ρ) depends linearly on ρ , (h_ρ, h'_ρ) is a cylinder functions associated to an immersion into the interior of Y_1 and Y'_1 , and for small values of ρ , solutions of the perturbation of the flat equation (2.44) for the pair $(h_{k-1} + h_\rho, h'_{k-1} + h'_\rho)$ agree with \mathfrak{C}_G . Moreover, for any path p of index k , the family moduli space $\mathbb{M}_G(\alpha, \beta)_p \subset \mathcal{B}_G(\alpha, \beta)_p \times \mathcal{P}_\#$, which is the union of the moduli spaces $M_G(\alpha, \beta)_p$ of solutions to the perturbed ASD equation associated to $(h_{k-1} + h_\rho, h'_{k-1} + h'_\rho)$ for all $\rho \in \mathcal{P}_\#$, is cut down transversely at elements of the form $([A], 0)$.

Suppose $([A], 0)$ is an element of a moduli space $\mathbb{M}_G(\alpha, \beta)_p$ with index k which is not cut down transversely. Our induction assumption and the assumption $k \leq 7$ imply that the space of all such non-regular elements of $\mathbb{M}_G(\alpha, \beta)_p$ is compact. Assume that A is given in the temporal gauge, and let A_t denote the restriction of A to $\{t\} \times Y_\#$. A non-trivial element in the cokernel of the linearized operator for $\mathbb{M}_G(\alpha, \beta)_p$ is given by a smooth family of 1-forms $\{\phi_t\}_{t \in \mathbb{R}}$ on $Y_\#$ with values in $E_\#$ such that

$$\frac{d}{dt}\phi_t = -L_t\phi_t, \quad (6.9)$$

$$d_{A_t}^*\phi_t = 0, \quad \int_{-\infty}^{\infty} \langle \nabla_{A_t} h_\rho + \nabla_{A_t'} h'_\rho, \phi_t \rangle = 0, \quad \forall \rho \in \mathcal{P}_\#, \quad (6.10)$$

and the L^2 norm of ϕ_t converges to zero as $|t| \rightarrow \infty$. Here L_t is a self-adjoint operator (with respect to the L^2 norm) defined on the sections of $\Lambda^1(Y_\#) \otimes E_\#$. The operator L_t depends on A_t , $(h_0 + h_\rho, h'_0 + h'_\rho)$ and $L_t(b)$ is equal to $*_3 d_{A_t} b$ outside the supports of $h_0 + h_\rho$ and $h'_0 + h'_\rho$. Unique continuation of the solutions of the equations of the form (6.9) [KM07, Lemma 7.1.3] implies that ϕ_t is non-zero for all values of t . For $\dot{A}_t := \frac{d}{dt} A_t$, we have

$$\frac{d}{dt} \dot{A}_t = L_t \dot{A}_t. \quad (6.11)$$

Using (6.9), (6.11) and the decay of ϕ_t and $\frac{d}{dt} \dot{A}_t$, we can see that ϕ_t is L^2 -orthogonal to $\frac{d}{dt} \dot{A}_t$ for all t . This claim can be proved by differentiating the inner product of ϕ_t and \dot{A}_t with respect to t .

Since the restrictions of ϕ_t and \dot{A}_t to $Y_\# \setminus \Sigma$ are linearly independent, we may use the same argument as in [Don02, Proposition 5.17] to find cylinder functions h_ρ, h'_ρ supported in the interior of Y, Y' such that h_ρ, h'_ρ vanish in a neighborhood of \mathfrak{C}_G and

$$\int_{-\infty}^{\infty} \langle \nabla_{A_t} h_\rho + \nabla_{A_t'} h'_\rho, \phi_t \rangle > 0.$$

In particular, if we enlarge $\mathcal{P}_\#$ using (h_ρ, h'_ρ) , then the dimension of the cokernel of the linearization of the moduli space $\mathbb{M}_G(\alpha, \beta)_p$ at $([A], 0)$ decreases by one. Because of the compactness of the space of the non-regular elements of $\mathbb{M}_G(\alpha, \beta)_p$ with index k , we may iterate this process and modify $\mathcal{P}_\#$ such that $\mathbb{M}_G(\alpha, \beta)_p$ for any p with index k is regular. Now, a standard application of Sard's theorem shows that for a generic small $\rho \in \mathcal{P}_\#$, the pair $(h_k, h'_k) = (h_{k-1} + h_\rho, h'_{k-1} + h'_\rho)$ verifies the claim for k .

Let (h_7, h'_7) be the cylinder functions given for $k = 7$. Then (h_7, h'_7) is supported in the complement of $\Sigma \subset Y_\#$. In fact, these functions are supported in the complement of a regular neighborhood of Σ because h_σ, h'_σ are already supported in Y_0, Y'_0 , and $h_7 - h_\sigma$ and $h'_7 - h'_\sigma$ are defined using finitely many immersions into the interior of Y_1 and Y'_1 . If we rescale the metric on $Y_\#$ by a constant, the same assumption on the regularity of the moduli spaces hold. Moreover, we may assume that the rescaling constant is large enough so that $(h_0 + h_1, h'_0 + h'_1)$ is supported in the complement of a copy of $(-1, 1) \times \Sigma$ equipped with the product metric of the standard metric on $(-1, 1)$ and some Riemannian metric on Σ . \square

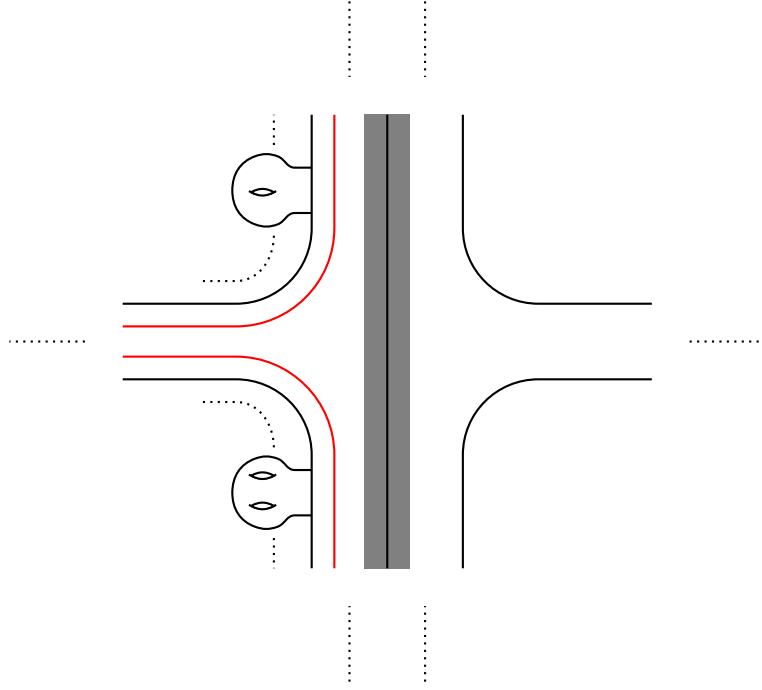


Figure 5: Support of the secondary perturbation terms is in the complement of the gray region.

6.2 Secondary perturbations

The purpose of this subsection is to prove Proposition 3.23 using a secondary perturbation of the mixed equation. The perturbed equation has the form in (3.22), copied below again for the reader's convenience:

$$\begin{cases} F^+(A) + (*_3 \nabla_{A_t} h)^+ + (*_3 \nabla_{A'_t} h')^+ + \eta(A) = 0, \\ \bar{\partial}_J u = 0. \end{cases} \quad (6.12)$$

The perturbation of the ASD equation is given by the holonomy perturbation term $\eta(A)$, and the perturbation of the CR equation is provided by perturbing the complex structure J . Let X_c and U_c be the sunspaces of X and U_+ obtained by the complement of the gray region sketched in Figure 5. Since we established the analysis of mixed equation in a neighborhood of the matching line only in the unperturbed case, we limit ourselves to holonomy perturbations which are supported in X_c and complex structures J which differ from the standard complex structure J_* only in U_c .

The definition of the holonomy perturbation term $\eta(A)$ is analogous to the definition of cylinder functions. Given a Riemannian 4-manifold M , let $\gamma : S^1 \times D^4 \rightarrow M$ be a smooth submersion such that γ restricted to $\{1\} \times D^4$ is an embedding of D^4 into M . Let also ω be a self-dual 2-form on the image of $\gamma(\{1\} \times D^4)$. For any connection A on a $U(2)$ bundle E over M , the holonomy of A along the loop $\gamma(S^1 \times \{z\})$ is an element of the fiber of $\text{End}(E)$ over the point $\gamma(1, z)$. Let $H_z(A)$ be the image of this holonomy with respect to the homomorphism $\text{End}(E) \rightarrow \mathfrak{su}(E)$ induced by the projection of

$\text{End}(\mathbf{C}^2) \rightarrow \mathfrak{su}(2)$. Then $H_z(A)$ determines a section of $\mathfrak{su}(E)$ over $\gamma(\{1\} \times D^4)$. Define

$$P_{\gamma,\omega}(A) := H_z(A) \otimes \omega \quad (6.13)$$

which gives a gauge invariant map from the space of connections on E to the space of sections of $\Lambda^+ \otimes \mathfrak{su}(E)$. Note that the $|P_{\gamma,\omega}(A)|$ is bounded by the C^0 norm of ω .

Now, we turn into the proof of Proposition 3.23. We shall define the perturbation term η as a linear combination

$$\eta = \sum_{i=1}^N P_{\gamma_i,\omega_i} \quad (6.14)$$

for a finite collection of $\gamma_i : S^1 \times D^4 \rightarrow X_c$ and self-dual 2-forms ω_i . For any solution (A, u) of (6.12), we have

$$\mathcal{E}(A, u) = \frac{1}{8\pi^2} \mathfrak{E}(A, u) - \frac{1}{4\pi^2} \|\eta(A)\|_{L^2(X)}^2 \geq -C \sum_{i=1}^N \|\omega_i\|_{L^2}^2$$

where C is a fixed constant, independent of (γ_i, ω_i) . Thus, if the sum of $\|\omega_i\|_{L^2}^2$ is less than ε for a small enough ε , then the topological energy of any solution (A, u) of (6.12) is greater than $-\delta$ for a given positive constant δ . Since the set of possible values for the topological energy $\mathcal{E}(A, u)$ for elements of the configuration spaces $\mathbf{B}(\alpha, \beta)$ is a discrete subset of \mathbf{R} , we can pick ε such that the topological energy of any solution of (6.12) is non-negative. To prove Proposition 3.23, we follow a similar strategy as in Proposition 6.8. By induction on the expected dimension of the moduli spaces of mixed equation, we show that for any $k \leq 3$, there are η_k and a family complex structures $\mathcal{J}_k = \{J_{(s,\theta)}\}_{(s,\theta) \in U_+}$ as in Subsection 3.1 such that all moduli spaces $\mathbf{M}_{\eta_k}(\alpha, \beta)_d$, with $d \leq k$ and defined with respect to \mathcal{J}_k , are cut down transversely, and the moduli spaces $\mathbf{M}_{\eta_k}(\alpha, \alpha)_0$ consists of a single regular element. Moreover, the perturbation term η has the form in (6.14) with the sum of $\|\omega_i\|_{L^2}^2$ being less than $\varepsilon - 1/2^k$. If k is small enough, then the trivial perturbation of the ASD equation and a family of almost complex structures as in Subsection 3.1 satisfy this claim.

Next, we show that the claim holds for k assuming that it already holds for $k - 1$. Suppose η_{k-1} and \mathcal{J}_{k-1} are chosen satisfying the above properties. We show that there is a collection $\{(\gamma_i, \omega_i)\}_{i=1}^N$, an open neighborhood \mathcal{U} of the origin in a Euclidean space, and for each $x \in \mathcal{U}$, a family of compatible almost complex structures $\{J_{(s,\theta)}^x\}_{(s,\theta) \in U_+}$ on $\mathcal{M}(\Sigma, F)$ with $\{J_{(s,\theta)}^0\}_{(s,\theta) \in U_+} = \mathcal{J}_{k-1}$ such that the following claim holds. Suppose $\mathfrak{M}(\alpha, \beta)$ is the subspace of $([A, u], r_1, \dots, r_N, x) \in \mathbf{B}(\alpha, \beta) \times \mathbf{R}^N \times \mathcal{U}$ where (A, u) is a solution of (6.12) defined using almost complex structures $\{J_{(s,\theta)}^x\}_{(s,\theta) \in U_+}$ and the perturbation

$$\eta = \eta_{k-1} + \sum_{i=1}^N r_i P_{\gamma_i,\omega_i}.$$

Then any solution $(A, u, 0, 0, \dots, 0, \vec{0}) \in \mathfrak{M}(\alpha, \beta)$ with $\text{index}(\mathcal{D}_{(A,u)}) = k$ is cutdown transversely. Here $J_{(s,\theta)}^x$ depends smoothly on (x, s, θ) , is equal to the standard complex structure J_* for $s \leq 1$ and is equal to J_θ , the complex structure given by Lemma 2.35 for $s \geq 3$. Moreover, $J_{(s,\theta)}^x$ is constant in the θ direction if $|\theta| > 2$.

Assuming the claim of the previous paragraph, a standard application of Sard's theorem shows that there are $\mathbf{r} = (r_1, \dots, r_N) \in \mathbf{R}^N$ and $x \in \mathcal{U}$ with arbitrary small norms such that if we define $\mathbf{M}_\eta(\alpha, \beta)_d$ using $\eta = \eta_{k-1} + \sum_{i=1}^N r_i P_{\gamma_i, \omega_i}$ and the family of compatible almost complex structures $\{J_{(s, \theta)}^x\}_{(s, \theta) \in U_+}$, then any such moduli space is regular if $d \leq k$. Since any moduli space of the form $\mathbf{M}_{\eta_{k-1}}(\alpha, \alpha)_0$ already contains a unique regular element, if \mathbf{r} is small enough, then the moduli space $\mathbf{M}_{\eta_{k-1}}(\alpha, \alpha)_0$ contains a unique regular element, too. Moreover, a small enough \mathbf{r} allows us to guarantee that η has the desired form in (6.14) with the sum of $\|\omega_i\|_{L^2}^2$ being less than $\varepsilon - 1/2^k$. This completes the proof of Proposition 3.23.

Now, we turn to the construction of the family of perturbations of the ASD equation and the compatible almost complex structures on $\mathcal{M}(\Sigma, F)$. Suppose $(A, u, 0, 0, \dots, 0, \vec{0}) \in \mathfrak{M}(\alpha, \beta)$ is a non-regular solution of $\mathfrak{M}(\alpha, \beta)$ with the index of $\mathcal{D}_{(A, u)}$ being k . The induction assumption and the compactness results of Subsection 5.4 imply that the set of all such non-regular solutions is compact. Fix a non-zero element (μ, ξ, z) in the cokernel of the linearization of $\mathfrak{M}(\alpha, \beta)$ at $(A, u, 0, 0, \dots, 0, \vec{0})$, where as before μ and ξ respectively denote a 0-form and a self-dual 2-form on X , and z denotes a section of $u^*T\mathcal{M}(\Sigma, F)$ over U_+ .

If (μ, ξ) is non-trivial, then unique continuation implies that the restriction of μ to X_c is nontrivial [Aro57] because the restriction of A to the complement of X_c satisfies the (unperturbed) ASD equation. Since A over X_c is irreducible, we may find (ω, γ) as above such that $P_{\gamma, \omega}(A) \neq 0$ (See, for example, the proof of Lemma 13 in [Kro05]). Therefore, if we add (ω, γ) to the collection $\{(\gamma_i, \omega_i)\}_{i=1}^N$, then the dimension of the cokernel of $\mathfrak{M}(\alpha, \beta)$ at $(A, u, 0, 0, \dots, 0, \vec{0})$ decreases by 1. In the case that (μ, ξ) is trivial, the restriction of z to U_∂ needs to be trivial. If u is a constant map, then one can use the arguments of Subsection 4.3 to see that the vanishing of z on U_∂ implies that z vanishes globally. This is in contradiction with non-triviality of (μ, ξ, z) , and hence u is a non-constant map. In particular, there are points in U_c where the derivative of u does not vanish. This implies that we may enlarge \mathcal{U} by adding another direction to deform $\{J_{(s, \theta)}^x\}_{(s, \theta) \in U_+}$ such that the deformation is compactly supported in U_c and the dimension of the cokernel of $\mathfrak{M}(\alpha, \beta)$ at $(A, u, 0, 0, \dots, 0, \vec{0})$ decreases by 1. (See, for example, the proof of Proposition 6.7.7. in [MS04]. Note also that the domain U_+ is rigid and does not have any non-trivial automorphism mapping U_∂, η_+ and η'_+ to themselves.) By iterating this process finitely many times, we may guarantee the regularity of $(A, u, 0, 0, \dots, 0, \vec{0})$. Using the compactness of the space of all such elements of $\mathfrak{M}(\alpha, \beta)$, we can more generally achieve regularity at all points $(A, u, 0, 0, \dots, 0, \vec{0}) \in \mathfrak{M}(\alpha, \beta)$ with the same topological energy. This completes the proof of our claim.

7 Extensions of the main theorem

The goal of this section is to show that the isomorphism \mathbf{N} of Theorem 2 is compatible with certain additional structures on instanton Floer homology and its symplectic variant. In the first subsection, we define the structure relevant for Theorem 3 in the introduction and then prove this theorem. A more precise version of Theorem 3 is stated as Theorem 7.7. In the second subsection, we review the definition of the operators m_σ^G, m_σ^S and then prove Theorem 4.

7.1 Filtered framed Floer homology groups

Topological energy of solutions to the mixed equation plays a key role in the proof of our main theorem in Section 3. In fact, we can use the notion of topological energy to define an additional structure on instanton Floer homology and its symplectic version. We call this additional structure the *Chern–Simons filtration*. In the discussion below, we follow similar conventions as in [DS20].

To define the Chern–Simons filtration, it is convenient to introduce $\overline{\mathfrak{C}}_G$, as a variation of \mathfrak{C}_G . Fix an element α_0 of \mathfrak{C}_G , and let $\overline{\mathfrak{C}}_G$ consist of pairs $\overline{\alpha} = (\alpha, p)$ where $\alpha \in \mathfrak{C}_G$ and p is a path from α to α_0 . We call $\overline{\alpha}$ a *lift* of α . Define $\deg_I(\overline{\alpha})$, the I-grading of $\overline{\alpha}$, to be the topological energy of the path p defined in (2.51). Any $\overline{\alpha} = (\alpha, p)$ in $\overline{\mathfrak{C}}_G$ is determined by α and $\deg_I(\overline{\alpha})$. Moreover, for any two different lifts $\overline{\alpha}$ and $\overline{\alpha}'$ of α , the expression $2(\deg_I(\overline{\alpha}) - \deg_I(\overline{\alpha}'))$ is an integer. We define a bijection $U : \overline{\mathfrak{C}}_G \rightarrow \overline{\mathfrak{C}}_G$ by requiring that for a lift $\overline{\alpha}$ of α we have

$$\deg_I(U(\overline{\alpha})) - \deg_I(\overline{\alpha}) = \frac{1}{2}.$$

Similarly, define $\overline{C}_G(Y_{\#}, E_{\#})$ to be the variation of $C_G(Y_{\#}, E_{\#})$ which is the free abelian group generated by the elements of $\overline{\mathfrak{C}}_G$. The bijection U defines the structure of a $\mathbf{Z}[U^{\pm 1}]$ -module on $\overline{C}_G(Y_{\#}, E_{\#})$. We also modify the differential on $C_G(Y_{\#}, E_{\#})$ in the following way to get a differential $d : \overline{C}_G(Y_{\#}, E_{\#}) \rightarrow \overline{C}_G(Y_{\#}, E_{\#})$:

$$d(\overline{\alpha}) := \sum_{p: \alpha \rightarrow \beta} \#\check{M}_G(\alpha, \beta)_p \cdot \overline{\beta}. \quad (7.1)$$

Here $\overline{\alpha}, \overline{\beta} \in \overline{\mathfrak{C}}_G$ are lifts of $\alpha, \beta \in \mathfrak{C}_G$ which are related by the path p . That is to say, the path from α to α_0 given by $\overline{\alpha}$ is equal to the composition of p and the path provided by $\overline{\beta}$. The differential map in (7.1) is a $\mathbf{Z}[U^{\pm 1}]$ -module homomorphism. The following lemma implies that the I-grading defines a filtration on $\overline{C}_G(Y_{\#}, E_{\#})$, which we call the Chern–Simons filtration.

Lemma 7.2. *Suppose $\overline{\alpha}, \overline{\beta} \in \overline{\mathfrak{C}}_G$ are lifts of $\alpha, \beta \in \mathfrak{C}_G$ which are related by the path p and the moduli space $M_G(\alpha, \beta)_p$ is non-empty. Then*

$$\deg_I(\overline{\alpha}) \geq \deg_I(\overline{\beta}),$$

and the equality holds if and only if $\overline{\alpha} = \overline{\beta}$ and p is the constant path.

Proof. This is a consequence of the fact that for $[A] \in M_G(\alpha, \beta)_p$ the topological energy $\mathcal{E}(A)$ is non-negative because it is equal to $\frac{1}{8\pi^2} \mathfrak{E}(A)$ where

$$\mathfrak{E}(A) := \int_{\mathbf{R} \times Y_{\#}} |F_A + *_3 \nabla_{A_t} h + *_3 \nabla_{A_t'} h'|^2 dt \, \text{dvol}_{Y_{\#}}.$$

□

We may define a Chern–Simons filtration on symplectic instanton Floer homology in a similar way. The sets \mathfrak{C}_G and \mathfrak{C}_S are naturally identified with each other and we define the \mathbf{Z} -covering $\overline{\mathfrak{C}}_S$ of \mathfrak{C}_S to be the

same as $\overline{\mathfrak{C}}_G$ with the same I-grading as above. We define $\overline{C}_S(Y_\#, E_\#)$ to be the abelian group generated by $\overline{\mathfrak{C}}_S$. Suppose $\alpha, \beta \in \mathfrak{C}_S$ and p is a path from α to β represented by a strip $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$. By gluing u to the constant mixed pair (A_α, u_α) as in Subsection 4.5 and then applying mixed shifting we obtain a connection A on $\mathbf{R} \times Y_\#$ from α to β which is glued to the constant mixed pair (A_β, u_β) . This construction allows us to associate to p a well-defined gauge theoretical path p' from $\alpha \in \mathfrak{C}_G$ to $\beta \in \mathfrak{C}_G$. In particular, p' can be used to assign a lift $\overline{\beta}$ of β in $\overline{\mathfrak{C}}_S$ to any lift $\overline{\alpha} \in \overline{\mathfrak{C}}_S$ of α , which is characterized by

$$\deg_I(\overline{\alpha}) = \mathcal{E}(u) + \deg_I(\overline{\beta}).$$

Analogous to (7.1), we define a differential $d : \overline{C}_S(Y_\#, E_\#) \rightarrow \overline{C}_S(Y_\#, E_\#)$ which is a $\mathbf{Z}[U^{\pm 1}]$ -module homomorphism and the following lemma implies that it is filtered with respect to the I-grading.

Lemma 7.3. *Suppose $\overline{\alpha}, \overline{\beta} \in \overline{\mathfrak{C}}_S$ are lifts of $\alpha, \beta \in \mathfrak{C}_S$ which are related by the path p and the moduli space $M_S(\alpha, \beta)_p$ is non-empty. Then*

$$\deg_I(\overline{\alpha}) \geq \deg_I(\overline{\beta}),$$

and the equality holds if and only if $\overline{\alpha} = \overline{\beta}$ and p is the constant path.

We can more generally use any mixed pair $(A, u) \in \mathbf{A}(\alpha, \beta)$, or rather its connected component, to associate a lift $\overline{\beta} \in \overline{\mathfrak{C}}_S$ of $\beta \in \mathfrak{C}_S$ to any lift $\overline{\alpha} \in \overline{\mathfrak{C}}_G$ of $\alpha \in \mathfrak{C}_G$. In fact, one can see that $\deg_I(\overline{\alpha}) - \mathcal{E}(A, u)$ is the I-grading of a lift of β by using Lemma 4.62 and reducing this claim to the case that (A, u) is a symplectically constant pair. Therefore, we can define $\overline{\beta}$ by requiring

$$\deg_I(\overline{\alpha}) = \mathcal{E}(A, u) + \deg_I(\overline{\beta}).$$

This allows us to lift the isomorphism $\mathbf{N} : C_G(Y_\#, E_\#) \rightarrow C_G(Y_\#, E_\#)$ of Subsection 3.2 to a $\mathbf{Z}[U^{\pm 1}]$ -module homomorphism from $\overline{C}_G(Y_\#, E_\#)$ to $\overline{C}_S(Y_\#, E_\#)$. The third part of Lemma 3.23 implies that if the perturbation term in the definition of \mathbf{N} is small enough, then for any $\overline{\alpha} \in \overline{\mathfrak{C}}_G$

$$\mathbf{N}(\overline{\alpha}) = \overline{\alpha} + \sum_{\overline{\beta}} n_{\alpha, \beta} \overline{\beta},$$

where $\overline{\beta}$ appears in the sum above only if $\deg_I(\overline{\beta}) < \deg_I(\overline{\alpha})$. In another word, \mathbf{N} is filtered with respect to the Chern-Simons filtration and its leading term is equal to the identity map.

Definition 7.4. An *I-graded complex* is a chain complex (C, d) which is freely and finitely generated over the ring $\mathbf{Q}[U^{\pm 1}]$ and has a $\mathbf{Z} \times \mathbf{R}$ -bigrading. If $C_{i,j}$ is the subgroup of C consisting of elements with bigrading (i, j) , then we have

- (i) $U C_{i,j} \subset C_{i+4, j+\frac{1}{2}}$,
- (ii) $d C_{i,j} \subset \bigcup_{j' \leq j} C_{i-1, j'}$.

Here i and j are respectively called the Floer grading and the I-grading of $C_{i,j}$. A chain map $f : (C, d) \rightarrow (C', d')$ of I-graded complexes is of level $\lambda > 0$, if it is a module homomorphism satisfying

$$f C_{i,j} \subset \bigcup_{j' \leq j+\lambda} C_{i, j'}$$

A level λ chain homotopy $h : (C, d) \rightarrow (C', d')$ between two chain maps $f, g : (C, d) \rightarrow (C', d')$ of level λ is a module homomorphism satisfying

$$f - g = d'h + hd$$

and

$$h C_{i,j} \subset \bigcup_{j' \leq j+\lambda} C_{i+1,j'}.$$

Our discussion above shows that the instanton Floer complex $\overline{C}_G(Y_\#, E_\#)$ and its symplectic version $\overline{C}_S(Y_\#, E_\#)$ are I-graded complexes, and \mathbf{N} defines an I-graded chain map of level 0 which is an isomorphism. However, this is not completely satisfactory for two reasons. First the I-gradings on $\overline{C}_G(Y_\#, E_\#)$ and $\overline{C}_S(Y_\#, E_\#)$ depend on the auxiliary choices of perturbations, the Riemann metrics and almost complex structures. (It turns out the dependence on perturbation terms h and h' are more serious than the other two items.) Second we need to fix a distinguished element α_0 of \mathfrak{C}_G to define the I-grading. Since the set \mathfrak{C}_G changes by varying the auxiliary choices, we need to slightly modify this choice to resolve the second issue. Mostly for the ease of exposition, from now on we focus on the case of framed Floer homology discussed in the introduction.

Recall that for any closed 3-manifold M , framed Floer homology is defined by introducing $Y_\# := M \# T^3$. We also fix an $SO(3)$ bundle $E_\#$ which is induced by the trivial bundle on M and the pullback to T^3 of the non-trivial bundle F_1 on T^2 . There is a unique flat connection on F_1 up to the action of the gauge group $\mathcal{G}(F_1)$ and the pullback of this connection to T^3 and the trivial connection on M induce an $SO(3)$ family of flat connections on $E_\#$. Let α_0 be an arbitrary element of this family. As it is explained in the introduction, any Heegaard splitting of M induces an admissible splitting of $(Y_\#, E_\#)$ as $(Y \cup Y', E \cup E')$. For any such admissible splitting, there is a sequence $\{(\sigma_i, \sigma'_i)\}$ in the perturbation space $\mathcal{P} \times \mathcal{P}'$ which converges to zero and $(h_i, h'_i) := (h_{\sigma_i}, h'_{\sigma'_i})$ satisfies the following properties.

- (i) The spaces $L_{h_i}(Y, E), L_{h'_i}(Y', E')$ are smooth embedded Lagrangians which intersect transversely. The intersection of these two Lagrangians is denoted by \mathfrak{C}_S^i and it can be identified with its gauge theoretical counterpart \mathfrak{C}_G^i . The set \mathfrak{C}_G^i includes the flat connection α_0 .
- (ii) The claim of Lemma 2.56 holds. In particular, we can use (h_i, h'_i) to define I-graded Floer complexes $(\overline{C}_G^i(Y_\#, E_\#), d^i)$ and $(\overline{C}_S^i(Y_\#, E_\#), d^i)$, where the I-grading is defined using α_0 .

Standard continuation maps in Floer theory provide chain maps $f_i^j : \overline{C}_G^i(Y_\#, E_\#) \rightarrow \overline{C}_G^j(Y_\#, E_\#)$ for any i, j such that $f_i^i = \text{Id}$ and $f_j^k \circ f_i^j$ is chain homotopic to f_i^k using a chain homotopy $l_{i,j,k}$. In fact, the chain maps f_i^j and the chain homotopy $l_{i,j,k}$ for $i, j, k \geq n$ are of level λ_n where $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. (See [Dae20, Subsection 2.2] for the proof of a similar claim in a closely related context.) In particular, the chain complexes $\overline{C}_G^i(Y_\#, E_\#)$ form an *enriched complex* in the sense of the following definition. (This is a slight variation of [DS20, Definition 7.16], which is adapted to the case of instanton Floer homology for admissible bundles.)

Definition 7.5. An *enriched complex* $\mathfrak{E} = \{(C^i, d^i), f_i^j, \lambda_n\}_{i,j,n}$ is a sequence of I-graded complexes $\{(C^i, d^i)\}$ and a family of chain maps $f_i^j : C^i \rightarrow C^j$ such that

- (i) f_i^j is of level λ_n for any $i, j \geq n$,
- (ii) $f_i^i = \text{Id}$,
- (iii) $f_j^k \circ f_i^j$ is chain homotopic to f_i^k using a chain homotopy $l_{i,j,k}$ of level λ_n whenever $i, j, k \geq n$,
- (iv) $\lim_{n \rightarrow \infty} \lambda_n = 0$.

Definition 7.6. If $\mathfrak{C}_1 = \{(C_1^i, d_1^i), f_i^j, \lambda_n\}_{i,j,n}$ and $\mathfrak{C}_2 = \{(C_2^i, d_2^i), g_i^j, \mu_n\}_{i,j,n}$ are two enriched complexes, then an enriched morphism $\mathfrak{N} : \mathfrak{C}_1 \rightarrow \mathfrak{C}_2$ consists of level κ_i chain maps $\mathbf{N}_i : C_1^i \rightarrow C_2^i$ for any i such that $\mathbf{N}_j f_i^j$ and $g_i^j \mathbf{N}_i$ are chain homotopic using a chain homotopy of level κ_n whenever $i, j \geq n$, and $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$. Enriched morphisms $\mathfrak{N} = \{\mathbf{N}_i : C_1^i \rightarrow C_2^i\}$ and $\mathfrak{M} = \{\mathbf{M}_i : C_1^i \rightarrow C_2^i\}$ are chain homotopic to each other, if there is a sequence $\{\mathbf{K}_i : C_1^i \rightarrow C_2^i\}$ where \mathbf{K}_i is a chain homotopy of level κ_i between \mathbf{N}_i and \mathbf{M}_i with $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$. The enriched complexes \mathfrak{C}_1 and \mathfrak{C}_2 are chain homotopy equivalent to each other if there are enriched morphisms $\mathfrak{N} = \{\mathbf{N}_i : C_1^i \rightarrow C_2^i\}$ and $\mathfrak{M} = \{\mathbf{M}_i : C_2^i \rightarrow C_1^i\}$ such that the enriched morphism $\mathfrak{M} \circ \mathfrak{N} := \{\mathbf{M}_i \circ \mathbf{N}_i : C_1^i \rightarrow C_1^i\}$ (resp. $\mathfrak{N} \circ \mathfrak{M} := \{\mathbf{N}_i \circ \mathbf{M}_i : C_2^i \rightarrow C_2^i\}$) is chain homotopy equivalent to an isomorphism of \mathfrak{C}_1 (resp. \mathfrak{C}_2).

The following theorem is an extension of our main theorem, which in particular shows that framed Floer homology and symplectic framed Floer homology together with their Chern-Simons filtrations are isomorphic to each other. In the statement of the theorem, we write $\mathfrak{E}_G(M)$ for the enriched framed Floer complex $\{(\overline{C}_G^i(Y_\#, E_\#), d^i), f_i^j, \lambda_n\}$ of M .

Theorem 7.7. *The 1-graded complexes $(\overline{C}_S^i(Y_\#, E_\#), d^i)$ can be completed into an enriched complex*

$$\mathfrak{E}_S(M) = \{(\overline{C}_S^i(Y_\#, E_\#), d^i), g_i^j, \mu_n\},$$

and the enriched framed Floer theories $\mathfrak{E}_G(M)$ and $\mathfrak{E}_S(M)$ are chain homotopy equivalent to each other as enriched complexes.

Proof. The construction of Subsection 3.2 gives a level 0 chain map $\mathbf{N}_i : \overline{C}_G^i(Y_\#, E_\#) \rightarrow \overline{C}_S^i(Y_\#, E_\#)$ for each i , which is an isomorphism. Then

$$g_i^j := \mathbf{N}_j \circ f_i^j \circ \mathbf{N}_i^{-1} : (\overline{C}_S^i(Y_\#, E_\#), d_i) \rightarrow (\overline{C}_S^j(Y_\#, E_\#), d^j)$$

is a chain map, and $g_j^k \circ g_i^j$ is chain homotopic to g_i^k using the chain homotopy $\mathbf{N}_k \circ l_{i,j,k} \circ \mathbf{N}_i^{-1}$. By picking $\mu_n = \lambda_n$, we may easily see that $\mathfrak{E}_S(M) = \{(\overline{C}_S^i(Y_\#, E_\#), d^i), g_i^j, \mu_n\}$ is an enriched complex. The maps \mathbf{N}_i and $\mathbf{M}_i := \mathbf{N}_i^{-1}$ give the desired chain homotopy equivalence between $\mathfrak{E}_G(M)$ and $\mathfrak{E}_S(M)$. \square

Remark 7.8. In this paper we have been concerned with the instanton Floer homology for admissible bundles on 3-manifolds. The key feature of these bundles is that they do not admit reducible flat connections. However, there are versions of instanton Floer homology for 3-manifolds [Flo88a, Don02, Frø02] and knots [CS99, DS20, Ech19] where one works with bundles which admit flat reducible connections. In these cases, one still has the Chern-Simons filtration which can be used to produce numerical invariants of

3-manifolds [Dae20, NST19] and knots [DS20]. Although this has not been investigated in the literature, it is reasonable to expect that the Chern-Simons filtration on framed Floer homology, in the form of the enriched complex $\mathfrak{E}_G(M)$ (or equivalently $\mathfrak{E}_S(M)$), could be useful in the study low of dimensional manifolds.

7.2 Quantum cohomology and the μ -operator

Associated to any configuration space of connections on a principal bundle, there is a universal bundle, which can be used to produce cohomology classes of the configuration space. As the first example, let $\mathcal{A}^*(\Sigma, F)$ be the open subspace of $\mathcal{A}(\Sigma, F)$ given by irreducible connections, and define $\mathcal{B}^*(\Sigma, F) \subset \mathcal{B}(\Sigma, F)$ as the quotient of $\mathcal{A}^*(\Sigma, F)$ by $\mathcal{G}(F)$. The gauge group $\mathcal{G}(F)$ acts in the obvious way on the product space $\mathcal{A}^*(\Sigma, F) \times \Sigma$, and this action can be lifted to the pullback of F to $\mathcal{A}^*(\Sigma, F) \times \Sigma$. The stabilizer of the action of $\mathcal{G}(F)$ at any point of $\mathcal{A}^*(\Sigma, F) \times \Sigma$ is $\{\pm 1\}$ which act trivially at any point on the bundle. In particular, taking quotient with respect to $\mathcal{G}(F)$ defines an $\text{SO}(3)$ -bundle on $\mathcal{B}^*(\Sigma, F) \times \Sigma$ which is called the universal bundle associated to F . We write \mathbb{F} for the restriction of this bundle to the subspace $\mathcal{M}(\Sigma, F) \times \Sigma$ of $\mathcal{B}^*(\Sigma, F) \times \Sigma$. Similar constructions give rise to the universal bundles $\mathbb{E}_\#$ over $\mathcal{B}^*(Y_\#, E_\#) \times Y_\#$ and \mathbb{V} over $\mathbf{B}(\alpha, \beta) \times X$ for any $\alpha, \beta \in \mathfrak{C}_G$. Here $\mathcal{B}^*(Y_\#, E_\#)$ is again the subspace of $\mathcal{B}(Y_\#, E_\#)$ given by irreducible connections.

The universal bundles \mathbb{F} , $\mathbb{E}_\#$ and \mathbb{V} are related to each other. For any $(A, u) \in \mathbf{A}(\alpha, \beta)$, the restriction of A to $\{(0, \theta)\} \times \Sigma \subset U_\partial \times \Sigma$ gives a flat connection in $\mathcal{A}^*(\Sigma, F)$ and this association is equivariant. This induces a map $\rho : \mathbf{B}(\alpha, \beta) \times U_\partial \times \Sigma \rightarrow \mathcal{M}(\Sigma, F) \times \Sigma$, and the restriction of \mathbb{V} to $\mathbf{B}(\alpha, \beta) \times U_\partial \times \Sigma$ is given by the pullback of \mathbb{F} via the map ρ . Let \tilde{X} be the manifold obtained as the union of X and $U_+ \times \Sigma$ where $U_\partial \times \Sigma \subset X \times \Sigma$ is identified with $U_\partial \times \Sigma \subset U_+ \times \Sigma$ in the obvious way. The above discussion shows that the bundles \mathbb{V} over $\mathbf{B}(\alpha, \beta) \times X$ and the pullback of \mathbb{F} to $\mathbf{B}(\alpha, \beta) \times U_+ \times \Sigma$ via the map

$$([A, u], s, \theta, x) \in \mathbf{B}(\alpha, \beta) \times U_+ \times \Sigma \rightarrow (u(s, \theta), x) \in \mathcal{M}(\Sigma, F) \times \Sigma, \quad (7.9)$$

can be naturally identified with each other over $U_\partial \times \Sigma$, and hence they induce a bundle over $\mathbf{B}(\alpha, \beta) \times \tilde{X}$, which we denote by $\tilde{\mathbb{V}}$. Fix $T \geq 3$ and let $\mathbf{B}^*(\alpha, \beta)$ denote the subspace of $\mathbf{B}(\alpha, \beta)$ given by mixed pairs $[A, u]$ that the restriction of A to $\{t\} \times Y_\#$, for any $t \in (-\infty, -T]$, is irreducible. Then the restriction of \mathbb{V} to $\mathbf{B}^*(\alpha, \beta) \times (-\infty, -T] \times Y_\#$ is the pullback of the bundle $\mathbb{E}_\#$ with respect to the map

$$([A, u], t, x) \in \mathbf{B}^*(\alpha, \beta) \times (-\infty, -T] \times Y_\# \rightarrow ([A]_{\{t\} \times Y_\#}, x) \in \mathcal{B}^*(Y_\#, E_\#) \times Y_\#. \quad (7.10)$$

Later in this subsection we shall need a further constraint on T .

Universal bundles can be used to produce cohomology classes in configuration spaces of connections. For instance, in the case of $\mathcal{M}(\Sigma, F)$, the first Pontryagin class of \mathbb{F} has the Künneth decomposition

$$p_1(\mathbb{F}) = \alpha_0 \otimes \omega_0 + \alpha_1 \otimes \omega_1 + \beta_0 \otimes x_0 + \beta_1 \otimes x_1 + \sum_i \psi_j \otimes \gamma_j \in H^4(\mathcal{M}(\Sigma, F) \times \Sigma)$$

where the cohomology classes $\omega_j \in H^2(\Sigma)$, $x_j \in H^0(\Sigma)$ and $\gamma_j \in H^1(\Sigma)$ give a basis for the corresponding cohomology groups of Σ , and $\alpha_j \in H^2(\mathcal{M}(\Sigma, F))$, $\beta_j \in H^4(\mathcal{M}(\Sigma, F))$ and $\psi_j \in$

$H^3(\mathcal{M}(\Sigma, F))$. These cohomology classes of $\mathcal{M}(\Sigma, F)$ provide a multiplicative generating set for the cohomology ring $H^*(\mathcal{M}(\Sigma, F))$ [AB83].

Explicit representatives for these cohomology classes can be constructed in the following way. Let $\mathbb{F}_{\mathbf{C}}$ denote the complexification $\mathbb{F} \otimes \mathbf{C}$. Fix two sections s_1^S and s_2^S of $\mathbb{F}_{\mathbf{C}}$ and define

$$Z := \{(\alpha, x) \in \mathcal{M}(\Sigma, F) \times \Sigma \mid s_1^S(\alpha, x), s_2^S(\alpha, x) \text{ are linearly dependent}\}.$$

If the sections s_1^S and s_2^S are chosen generically, then Z is a codimension four compact stratified subspace of $\mathcal{M}(\Sigma, F) \times \Sigma$ where the top stratum Z_0 is a smooth submanifold and $Z \setminus Z_0$ given by the common zeros of s_1^S and s_2^S has codimension twelve. In particular, if we co-orient Z using the product orientation on $\mathcal{M}(\Sigma, F) \times \Sigma$ and the complex orientation on the fibers of $\mathbb{F}_{\mathbf{C}}$, then it has a well-defined fundamental class which gives the Poincaré dual for $c_2(\mathbb{F}_{\mathbf{C}}) = -p_1(\mathbb{F})$. For this space Z and the analogous ones defined in the following, we use the non-standard product orientation convention to get a representative for $p_1(\mathbb{F})$. Thus Z can be used to produce representatives for the cohomology classes α_j, β_j and ψ_j . For instance, the projection of $Z \cap (\mathcal{M}(\Sigma, F) \times \Sigma_j)$ to $\mathcal{M}(\Sigma, F)$ gives a cycle representing α_j and the intersection of Z with $\mathcal{M}(\Sigma, F) \times \{x\}$ for a generic $x \in \Sigma_j$ gives a representative for β_j . Similarly, if ℓ is a closed oriented loop representing γ_j , then the intersection $Z \cap (\mathcal{M}(\Sigma, F) \times \ell)$, after possibly a perturbation of ℓ , is transversal and the projection of this intersection to $\mathcal{M}(\Sigma, F)$ gives a cycle representing γ_j .

As in Subsection 2.4, for $\alpha, \beta \in \mathfrak{C}_S$, let $M_S(\alpha, \beta)_p$ be the moduli space of pseudo-holomorphic maps $u : \mathbf{R} \times [-1, 1] \rightarrow \mathcal{M}(\Sigma, F)$ satisfying Lagrangian boundary conditions corresponding to the path p from α to β . Let $\text{Ev} : M_S(\alpha, \beta)_p \times \Sigma \rightarrow \mathcal{M}(\Sigma, F) \times \Sigma$ be the evaluation map that sends (u, x) in $M_S(\alpha, \beta)_p \times \Sigma$ to $(u(0, 0), x)$. We also fix an element of $H^2(\Sigma)$ represented by one of the connected components Σ_σ of Σ . A generic choice of s_1^S and s_2^S allows us to assume that Ev is transversal to Z . We form the cutdown moduli space

$$L_S(\alpha, \beta)_p := \{(u, x) \in M_S(\alpha, \beta)_p \times \Sigma_\sigma \mid \text{Ev}(u, x) \in Z\}.$$

Our transversality assumption implies that $\dim(L_S(\alpha, \beta)_p) = \dim(M_S(\alpha, \beta)_p) - 2$.

Now, we are ready to review the definition of the operator m_σ^S . If the index of p is at most 1, the moduli space $L_S(\alpha, \beta)_p$ is empty and if the index of p is 2, then $L_S(\alpha, \beta)_p$ is a compact 0-dimensional manifold which we may orient using the orientation of $M_S(\alpha, \beta)_p$ and the co-orientation of Z that realizes $p_1(\mathbb{F})$. Orientation of these moduli spaces allows us to define a homomorphism $m_\sigma^S : C_S((Y, E), (Y', E')) \rightarrow C_S((Y, E), (Y', E'))$ as

$$m_\sigma^S(\alpha) := \sum_{p: \alpha \rightarrow \beta} \#L_S(\alpha, \beta)_p \cdot \beta$$

where the sum is over all paths p from $\alpha \in \mathfrak{C}_S$ to $\beta \in \mathfrak{C}_S$ of index two and $\#L_S(\alpha, \beta)_p$ denotes the signed count of the elements of $L_S(\alpha, \beta)_p$. An analysis of the ends of 1-dimensional cutdown moduli spaces $L_S(\alpha, \beta)_p$ shows that the homomorphism m_σ^S is a chain map, and we use the same notation to denote the induced map $m_\sigma^S : \text{SI}_*(Y_\#, E_\#) \rightarrow \text{SI}_*(Y_\#, E_\#)$ at the level of homology.

The above construction has a counterpart in the case of instanton Floer homology for admissible pairs, as we review the construction now for the pair $(Y_\#, E_\#)$. For any $\alpha, \beta \in \mathfrak{C}_G$, any path p from α to β and any $[A] \in M_G(\alpha, \beta)_p$, the restriction A_t of A to $\{t\} \times Y_\#$ for any $t \in \mathbf{R}$ is irreducible. Otherwise, if

A_t has a non-trivial stabilizer u , then A and the pullback u^*A are two solutions of the perturbed ASD equation that agree on $\{t\} \times Y_\#$. Unique continuation implies that these two connections are equal to each other which contradicts with the irreducibility of A . In particular, we obtain a well-defined map $M_G(\alpha, \beta)_p \rightarrow \mathcal{B}^*(Y_\#, E_\#)$ by restricting $[A]$ to $\{0\} \times Y_\#$.

Next, we fix sections s_1^G and s_2^G of the complexified universal bundle $\mathbb{E}_\# \otimes \mathbf{C}$ over $\mathcal{B}^*(Y_\#, E_\#) \times Y_\#$ and define

$$L_G(\alpha, \beta)_p := \{([A], x) \in M_G(\alpha, \beta)_p \times \Sigma_\sigma \mid s_1^G([A|_{\{0\} \times Y_\#}], x) \text{ and } s_2^G([A|_{\{0\} \times Y_\#}], x) \text{ are linearly dependent.}\}$$

Again, we may assume that the space $L_G(\alpha, \beta)_p$ is cut down transversely. In particular, it is empty, if the index of p is at most one, and it is a compact 0-dimensional manifold if the index of p is two. In the latter case, we use the product orientation of $M_G(\alpha, \beta)_p \times \Sigma_\sigma$ to orient $L_G(\alpha, \beta)_p$. Oriented 0-dimensional moduli spaces $L_G(\alpha, \beta)_p$ can be used to define the operator $m_\sigma^G : C(Y_\#, E_\#) \rightarrow C(Y_\#, E_\#)$ as

$$m_\sigma^G(\alpha) := \sum_{p: \alpha \rightarrow \beta} \#L_G(\alpha, \beta)_p \cdot \beta$$

where the sum is over all paths p from $\alpha \in \mathfrak{C}_G$ to $\beta \in \mathfrak{C}_G$ with index two. Using 1-dimensional moduli spaces $L_G(\alpha, \beta)_p$ one can see again that m_σ^G is a chain map. The induced operator acting on $I_*(Y_\#, E_\#)$ is denoted by the same notation.

To relate the operators m_σ^G and m_σ^S we need sections of the bundle $\tilde{\mathbb{V}}$ interpolating between s_1^G, s_2^G on the gauge theoretical side and s_1^S, s_2^S on the symplectic side. Fix continuous sections \mathbf{s}_1 and \mathbf{s}_2 of $\tilde{\mathbb{V}}$ defined over $\mathbf{B}^*(\alpha, \beta) \times \tilde{X}$ which satisfy the following properties:

- (i) The restriction of \mathbf{s}_i to $\mathbf{B}^*(\alpha, \beta) \times X$ is smooth.
- (ii) The restriction of \mathbf{s}_i to $\mathbf{B}^*(\alpha, \beta) \times (-\infty, -T] \times Y_\#$ is given by pulling back s_i^G using the map (7.10).
- (iii) As in Subsection 4.2.2, let X_T denote the compact subspace of X given as the complement of $(T, \infty) \times T$, $(-\infty, -T) \times Y'$ and $(-\infty, -T) \times Y_\#$ in X . Then $\mathbf{s}_i([A, u], x)$ for $([A, u], x) \in \mathbf{B}^*(\alpha, \beta) \times X_T$ depends on the restriction of $[A]$ to X_T . To be more precise, the bundle \mathbb{V} over $\mathbf{B}^*(\alpha, \beta) \times X_T$ is the pullback of the universal bundle over $\mathcal{B}^*(X_T) \times X_T$ and we demand that \mathbf{s}_i is the pullback of a section of this universal bundle. Here $\mathcal{B}^*(X_c)$ denotes the configuration space of irreducible L_T^2 connections on X_T and the universal bundle over this space is defined analogous to the previous instances of universal bundles.
- (iv) The restriction of \mathbf{s}_i to $\mathbf{B}^*(\alpha, \beta) \times U_- \times \Sigma$ is given by pulling back s_i^S using the map (7.9).

Suppose the constant T in the definition of $\mathbf{B}^*(\alpha, \beta)$ is chosen such that the secondary perturbation term η vanishes on $(-\infty, -T] \times Y_\#$. Then unique continuation again implies that the moduli space $\mathbf{M}_\eta(\alpha, \beta)$ is contained in $\mathbf{B}^*(\alpha, \beta)$. We may arrange the sections s_i^G, s_i^S and \mathbf{s}_i so that the following subspace of

$\mathbf{M}_\eta(\alpha, \beta) \times \mathbf{R} \times \Sigma_\sigma$ for $d \leq 2$ is cut down transversely:

$$\mathbf{L}(\alpha, \beta)_d := \{([A, u], t, x) \in \mathbf{M}_\eta(\alpha, \beta)_{d+1} \times \mathbf{R} \times \Sigma_\sigma \mid \\ \mathbf{s}_1([A, u], t, x) \text{ and } \mathbf{s}_2([A, u], t, x) \text{ are linearly dependent}\}.$$

Here we use the embedding of $\mathbf{R} \times \Sigma_\sigma$ into \tilde{X} where $(-\infty, 0] \times \Sigma_\sigma$ is mapped to $(-\infty, 0] \times \{0\} \times \Sigma_\sigma \subset U_- \times \Sigma$ and $[0, \infty) \times \Sigma_\sigma$ is mapped to $[0, \infty) \times \{0\} \times \Sigma_\sigma \subset U_+ \times \Sigma$. By assumption, the restrictions of \mathbf{s}_i to $\mathbf{M}(\alpha, \beta)_d \times \Sigma_\sigma \times (-\infty, 0]$, $\mathbf{M}(\alpha, \beta)_d \times \Sigma_\sigma \times [0, \infty)$ and $\mathbf{M}(\alpha, \beta)_d \times \Sigma_\sigma \times \{0\}$ are smooth. The transversality assumption above means that the loci that \mathbf{s}_1 and \mathbf{s}_2 are linearly dependent over each of these subspaces is cut down transversely. This transversality assumption implies that $\mathbf{L}(\alpha, \beta)_d$ is empty for $d < 0$ and it is a 0-dimensional manifold for $d = 0$.

Lemma 7.11. *The moduli space $\mathbf{L}(\alpha, \beta)_0$ is compact.*

Proof. Suppose $\{([A_i, u_i], t_i, x_i)\}$ is a sequence of elements in $\mathbf{L}(\alpha, \beta)_0$. After passing to a subsequence, we may assume that x_i converges to $x_0 \in \Sigma_\sigma$ and t_i converges to t_0 which is either a finite real number or $\pm\infty$. The argument of Subsection 5.4 implies that there is a solution of the mixed equation $[A_0, u_0] \in \mathbf{M}_\eta(\alpha', \beta')_{d+1}$ with $d \leq 0$, perturbed ASD connections

$$A_1^G \in \check{\mathbf{M}}_G(\alpha, \alpha_1)_{p_1}, A_2^G \in \check{\mathbf{M}}_G(\alpha_1, \alpha_2)_{p_2}, \dots, A_n^G \in \check{\mathbf{M}}_G(\alpha_{n-1}, \alpha')_{p_n}$$

and pseudo-holomorphic strips

$$u_1^S \in \check{\mathbf{M}}_S(\beta', \beta_1)_{p'_1}, u_2^S \in \check{\mathbf{M}}_S(\beta_1, \beta_2)_{p'_2}, \dots, u_m^S \in \check{\mathbf{M}}_S(\beta_{m-1}, \beta)_{p'_m}$$

such that $[A_i, u_i]$ is chain convergent to $([A_1^G], \dots, [A_n^G], [A_0, u_0], u_1^S, \dots, u_m^S)$ on the complement of a set of bubble points. Since the index of the mixed pairs $[A_i, u_i]$ is 1, we may show that the set of bubble points is empty by arguing as in Subsection 5.4. If t_0 is a finite number, then the continuity of the sections \mathbf{s}_1 and \mathbf{s}_2 (together with the properties (ii) and (iii) of \mathbf{s}_i if $t_0 < 0$) implies that $\mathbf{s}_1([A_0, u_0], t_0, x_0)$ and $\mathbf{s}_2([A_0, u_0], t_0, x_0)$ are linearly dependent. In particular, the moduli space $\mathbf{L}(\alpha', \beta')_d$ is non-empty which shows that $d = 0$. This in turn implies that $m = n = 0$ and $\alpha = \alpha'$, $\beta = \beta'$, and $\{([A_i, u_i], t_i, x_i)\}$ is convergent to $([A_0, u_0], t_0, x_0) \in \mathbf{L}(\alpha, \beta)_0$ with respect to the topology of $\mathbf{L}(\alpha, \beta)_0$. Thus in this case $\{([A_i, u_i], t_i, x_i)\}$ is convergent after passing to a subsequence.

Next, we consider the case that t_0 is not a finite number. First let $t_0 = -\infty$. After passing to a subsequence, we may assume that $t_i \leq -T$ for any i . Translating the restriction of A_i to $(-\infty, -T]$ by t_i gives a connection A'_i on $(-\infty, -T - t_i]$ such that $\mathbf{s}_1^G([A'_i|_{\{0\} \times Y_\#}], x_i)$ and $\mathbf{s}_2^G([A'_i|_{\{0\} \times Y_\#}], x_i)$ are linearly independent. Since $t_i \rightarrow -\infty$, our assumption on the chain convergence of $\{([A_i, u_i], t_i, x_i)\}$ implies that the connections A'_i modulo the action of the gauge group are convergent to B which is one of the connections A_i^G or the pullback of one of the connections $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_{n-1}, \alpha_n = \alpha'$. Moreover, property (ii) of the sections \mathbf{s}_i implies that $\mathbf{s}_1^G([B|_{\{0\} \times Y_\#}], x_0)$ and $\mathbf{s}_2^G([B|_{\{0\} \times Y_\#}], x_0)$ are linearly dependent. In particular, B equals one of the connections A_i^G , and this connections represents an element of $L_G(\alpha_{i-1}, \alpha_i)_p$. This implies that the index of A_i^G is at least 2. On the other hand, the sum of the indices of the connections A_i^G , the mixed pair $[A_0, u_0]$ and u_j^S is 1, which is a contradiction. This shows that t_0 cannot be $-\infty$. A similar proof rules out the case $t_0 = \infty$. \square

We orient the compact 0-dimensional manifold $\mathbf{L}(\alpha, \beta)_0$ using the orientation of $\mathbf{M}_\eta(\alpha, \beta)_1$ and the induced product orientation on $\mathbf{M}_\eta(\alpha, \beta)_1 \times \mathbf{R} \times \Sigma_\sigma$. These oriented moduli spaces allow us to define a map $K : C(Y_\#, E_\#) \rightarrow C_S((Y, E), (Y', E'))$ as

$$K(\alpha) := \sum_{\beta} \#\mathbf{L}(\alpha, \beta)_0 \cdot \beta.$$

To prove Theorem 4 for the operators m_σ^G and m_σ^S , it suffices to show that $\mathbf{N} \circ m_\sigma^G - m_\sigma^S \circ \mathbf{N} = dK + Kd$. The following proposition, which is the counterpart of part (ii) of Proposition 3.26, shows that this relation follows from analyzing the ends of the 1-dimensional moduli spaces $\mathbf{L}(\alpha, \beta)_1$.

Proposition 7.12. *The moduli spaces of the form $\mathbf{L}(\alpha, \beta)_1$ can be compactified into compact 1-manifolds by adding points in correspondence to the 0-dimensional spaces*

$$\mathbf{L}(\alpha, \gamma)_0 \times \check{\mathbf{M}}_S(\gamma, \beta)_p, \quad \check{\mathbf{M}}_G(\alpha, \gamma)_p \times \mathbf{L}(\gamma, \beta)_0, \quad (7.13)$$

and

$$\mathbf{M}_\eta(\alpha, \gamma)_0 \times \mathbf{L}_S(\gamma, \beta)_p, \quad \mathbf{L}_G(\alpha, \gamma)_p \times \mathbf{M}_\eta(\gamma, \beta)_0. \quad (7.14)$$

where $\gamma \in \mathfrak{C}_G \cong \mathfrak{C}_S$, in (7.13) p denotes a path of index 1, and in (7.14) p denotes a path of index 2. Moreover, the induced orientation on the boundary components of the compactified moduli space $\mathbf{L}(\alpha, \beta)_1$ agree with the product orientation on the two terms in (7.13) and the first term in (7.14) and disagrees with the induced orientation on the second term in (7.14).

Proof. A straightforward adaptation of the proof of Lemma 7.11 shows that any sequence of elements in $\mathbf{L}(\alpha, \beta)_1$ without any subsequence convergent to an element of this moduli space has a subsequence chain convergent to an element in one of the spaces in (7.13) or (7.14). We need a gluing theory as a converse to this compactness result to show all elements of (7.13) and (7.14) appear as the ends of the moduli space $\mathbf{L}(\alpha, \beta)_1$. As in the case of Proposition 3.26, the desired gluing theory results concern gluing mixed pairs to ASD connections or pseudo-holomorphic strips on the gauge theoretical or symplectic ends. In particular, they can be proved by a straightforward adaptation of the corresponding gluing results in the context of instanton Floer theory and Lagrangian Floer theory. The discussion of the induced orientations of the moduli spaces on the boundary components is also similar to the standard corresponding results in the context of instanton Floer theory and Lagrangian Floer theory. \square

We may follow a similar discussion to prove the variation of the above result in the case that $\sigma \in H^1(\Sigma)$ and is represented by a loop ℓ_σ in Σ . The main modifications applied to the proof are replacing Σ_σ with ℓ_σ and working with the moduli spaces of instantons, pseudo-holomorphic strips and mixed pairs of one dimension higher. The above argument does not immediately generalize to the case that $\sigma \in H^0(\Sigma)$ because we need to work with the 4-dimensional moduli spaces of solutions to the mixed equation. The main obstacle in this case is that we may have bubbling along the matching line in the compactification of the moduli spaces of mixed pairs, and we have not studied the behavior of the compactified moduli spaces in a neighborhood of such bubbles.

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