3-MANIFOLDS WITHOUT ANY EMBEDDING IN SYMPLECTIC 4-MANIFOLDS

ALIAKBAR DAEMI, TYE LIDMAN, AND MIKE MILLER EISMEIER

Abstract. We show that there exist infinitely many closed 3-manifolds that do not embed in closed symplectic 4-manifolds, disproving a conjecture of Etnyre-Min-Mukherjee. To do this, we construct L-spaces that cannot bound positive or negative definite manifolds. The arguments use Heegaard Floer correction terms and instanton moduli spaces.

Theorem 1. *There exist infinitely many rational homology spheres which cannot embed in a closed symplectic 4-manifold.*

The family of manifolds we use are particular connected sums of elliptic manifolds. Let P denote the Poincaré homology sphere oriented as the boundary of the negative definite E_8 plumbing. Let O denote the "first" octahedral manifold with Seifert invariants $(-2; 1/2, 2/3, 3/4)$, oriented as the boundary of the negative definite E_7 plumbing. The manifolds in the theorem are those of the form $mP\# - kO$ with $m \geq 1$ and $k > 8m$. This answers the conjecture of Etnyre-Min-Mukherjee from [EMM19, p.6] in the negative. (Note that this is stronger than saying that the manifolds are not symplectically fillable, since a separating 3-manifold may sit in a symplectic 4-manifold in a way which is not compatible with any contact structure on the 3-manifold.)

The rational homology spheres above are L-spaces, since they are connected sums of elliptic manifolds [OS05, Section 2]. It is shown in [Muk20] that if an L-space embeds in a symplectic 4-manifold, then it must bound a definite 4-manifold. Hence, we are able to prove Theorem 1 by proving:

Theorem 2. *For any pair of integers k and m with $m \geq 1$ and $k > 8m$, the manifolds $mP\# - kO$ are L-spaces which cannot bound positive- or negative-definite 4-manifolds.*

The argument has two steps. First, to obstruct the negative-definite manifolds, we use the Heegaard Floer correction terms, which is carried out in Section 1. To obstruct the positive-definite manifolds, we use Chern–Simons invariants and ASD moduli spaces. This is done in Section 2.

Examples of 3-manifolds that do not bound any definite 4-manifold were previously given in [NST19, GL20]. In [NST19], a filtration of instanton Floer homology given by

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the Chern–Simons functional is used to construct integer homology spheres without any positive- or negative-definite 4-manifold filling. In [GL20], the Heegaard Floer correction terms are used to construct examples of rational homology spheres that bound no definite 4-manifold.

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1. The d-invariant argument

In this section, we use Heegaard Floer d-invariants [OS03] to obstruct the manifolds $mP\#-kO$ from bounding negative definite manifolds for suitable positive values of k, m.

Proposition 1.1. *Let $k, m > 0$. The manifold $mP\#-kO$ cannot bound a negative-definite 4-manifold for $k > 8m$.*

Before proving the proposition, we need to compute the Heegaard Floer d-invariants of O.

Lemma 1.2. *For a choice of labelling, the two Spin^c structures on O, $\mathfrak{s}_0, \mathfrak{s}_1$, satisfy*

$$d(-O, \mathfrak{s}_0) = -7/4, \; d(-O, \mathfrak{s}_1) = -1/4.$$

Proof. There are several ways to compute the d-invariants of O. We opt for a surgery approach for simplicity. We have that $-O = S^3_2(T_{2,3})$ (see for example [Doi15, Theorem 2, Equation 2])). By [OS12, Theorem 6.1] (and the formulas following), there is a labeling of the Spin^c structures such that

$$d(S^3_2(T_{2,3}), \mathfrak{s}_0) = \frac{1}{4} - 2t_0(T_{2,3}), \quad d(S^3_2(T_{2,3}), \mathfrak{s}_1) = -\frac{1}{4} - 2t_1(T_{2,3}),$$

where $t_i(K)$ denotes the ith torsion coefficient, $\sum_{j \geq 1} j a_{|i|+j}$, and a_k is the kth coefficient of the symmetrized Alexander polynomial. The result now follows, since $t_0(T_{2,3}) = 1$ and $t_1(T_{2,3}) = 0$ (see [OS12, Equations 2 and 3]). □

Proof of Proposition 1.1. Suppose that $mP\#-kO$ bounds a negative-definite cobordism. Then [OS12, Proposition 5.2] implies that

$$\max_{\mathfrak{s}} d(mP\#-kO, \mathfrak{s}) \geq 0. \tag{1}$$

Recall that d-invariants are additive under connected sum and that $d(P) = 2$. We thus have from Lemma 1.2

$$\max_{\mathfrak{s}} d(mP\#-kO, \mathfrak{s}) = 2m - \frac{k}{4}.$$

It follows that for $k > 8m$, (1) is violated. □

Remark 1.3. It seems likely that Proposition 1.1 can also be proved using Donaldson's diagonalizability theorem and lattice techniques. We anticipate that the assumption $k > 8m$ can be relaxed somewhat using refinements of Frøyshov's instanton h-invariant for rational homology spheres.

2. The Chern–Simons argument

In this section, we use the instanton moduli spaces to obstruct $mP\#-kO$ from bounding a positive-definite cobordism, complementary to the results in Proposition 1.1.

Proposition 2.1. *Let $m>0$ and k be integers. Then the manifold $mP\#-kO$ cannot bound a positive-definite 4-manifold.*

To prove this claim, we consider moduli spaces of $SU(2)$-instantons. We first provide a sketch of the argument for non-experts, with further details below.

Sketch of proof. Suppose there exists such a positive-definite manifold W_0. By reversing the orientation, adding 3-handles, and surgering out a set of loops giving a generating set of $H_1(W_0;\mathbb{R})$, we obtain a negative-definite cobordism $W : P \to \sqcup_k O \sqcup_{m-1} -P$ with $b_1 = 0$. We will obtain a contradiction by considering an orientable 1-dimensional moduli space M of instantons on this cobordism W (more precisely we attach cylindrical ends to W and consider a perturbation of the ASD equation). Our contradiction will come from showing that the number of ends, counted with sign, is non-zero. The ends of this moduli space M correspond to gluing instantons on W to instantons on the incoming end $\mathbb{R}\times P$ or the outgoing ends $\mathbb{R}\times O$ and $\mathbb{R}\times -P$.

We first construct a family of ends of M by gluing a particular instanton on $\mathbb{R}\times P$ to the reducible flat connections over W. These reducibles are determined by $H_1(W;\mathbb{Z})$, and are isolated and well-behaved with respect to gluing because $b_1(W)=0$ and $b^+(W)=0$, respectively. This construction produces as many ends of M as there are elements of $H_1(W;\mathbb{Z})/H_1(\partial W;\mathbb{Z})$, and they are all oriented in the same direction.

We use *topological energy* $\kappa(A)$ of instantons to establish that these are the only ends. The moduli space M is the moduli space of instantons with topological energy equal to $\frac{1}{120}$. In general, topological energy is non-negative, additive under gluing of instantons, and multiplicative under passing to covering spaces. An instanton A on W determines flat connections α and α' on the incoming and outgoing boundary components of W, and the topological energy $\kappa(A)$ is equal modulo $\mathbb{Z}$ to the difference of *Chern–Simons invariants* $\mathrm{CS}(\alpha)-\mathrm{CS}(\alpha')\in\mathbb{R}/\mathbb{Z}$.

There are two key points.

- The instanton on $\mathbb{R}\times P$ used above has $\kappa(A)=\frac{1}{120}$, and the reducible flat connections on W have $\kappa(A)=0$. By additivity of energy, the instantons we constructed on W above have energy $\kappa=\frac{1}{120}$, as do all other instantons in the moduli space M. All other instantons on $\mathbb{R}\times P$ have larger energy; when glued to instantons on W they produce instantons of energy larger than $\frac{1}{120}$, which do not lie in M.
- All instantons on $\mathbb{R}\times\pm O$ have $\kappa(A)\geq\frac{1}{48}$, so also do not contribute to ends of M. Here we use the relation to the Chern–Simons invariant: for any flat connection α on O, we have $48\mathrm{CS}(\alpha)\equiv 0\in\mathbb{R}/\mathbb{Z}$. This follows because the universal cover of O is the 3-sphere, where the Chern–Simons invariant is zero; the Chern–Simons invariant is multiplicative under covers and $|\pi_1(O)|=48$.

Because every end of M is constructed by the gluing procedure (gluing an instanton on a cylindrical end to one on W), the only ends are those initially described. This gives the desired contradiction: the signed count of ends is $\pm|H_1(W;\mathbb{Z})/H_1(\partial W;\mathbb{Z})| \neq 0$, but the signed count of ends of an oriented 1-manifold without boundary must be zero. □

Remark 2.2. Proposition 2.1 holds more generally for any closed oriented 3-manifold Y with $|\pi_1(Y)| < 120$ in place of O, with the same proof.

Remark 2.3. Using moduli spaces of $SU(2)$-instantons to study negative definite smooth closed 4-manifolds goes back to Donaldson's groundbreaking work [Don83]. Here we use an energy argument to analyze boundary components of a 1-dimensional moduli space of $SU(2)$-instantons. Similar strategies appear, for example, in [FS85, Fur90, FS90, HK12, PC17]. Another key tool in the study of negative definite 4-manifolds with integer homology sphere boundary is Frøyshov's invariant h of [Frø02]. Frøyshov's invariant is the instanton counterpart of Heegaard Floer d-invariant used in the previous section. Topological energy is employed in [Dae20, NST19] to construct refinements of Frøyshov's invariant. We expect that the invariants of [Dae20, NST19] generalize to invariants of rational homology spheres using the results of [Mil19], and that the argument above can be recast in that language.

2.1. **Detailed argument.** Choose a metric on W which is cylindrical (identical to the product metric) in a collar neighborhood of the boundary. We will consider instantons on the complete Riemannian manifold $W \cup_{\partial W} [0,\infty) \times \partial W$, where we have attached infinite cylindrical ends. By an abuse of notation, we ignore this subtlety and refer by the same name W to both the compact manifold W and the version with cylindrical ends, as each may be recovered from the other. By partitioning ∂W into a set of *incoming ends* and *outgoing ends*, we may write that $W : Y \to Y'$ is a cobordism, where $\partial W = Y' \sqcup -Y$.

We are interested in $SU(2)$-connections A on the trivial bundle over W which are asymptotic to a flat connection α over Y and a flat connection α' over Y', and for which the *topological energy*, defined as

$$\kappa(A) := \frac{1}{8\pi^2}\int_W \operatorname{tr}(F_A \wedge F_A),$$

is finite. This quantity is constant with respect to continuous deformations of A, and its mod $\mathbb{Z}$ value is equal to $\mathrm{CS}(\alpha) - \mathrm{CS}(\alpha')$. (Taking Y' to be empty, this serves as a definition of $\mathrm{CS}(\alpha)$.)

Define the *gauge group* to be the space of all maps $u : W \to SU(2)$, regarded as automorphisms of the trivial $SU(2)$-bundle, that are asymptotic to a map $v : Y \to SU(2)$ on the incoming end and a map $v' : Y' \to SU(2)$ on the outgoing end. Then we may pull back any connection A as above with respect to u to obtain the connection u^*A that has the same topological energy as A and is asymptotic to $v^*\alpha$ and $(v')^*\alpha'$ on the incoming and the outgoing ends of W. The automorphisms $u = \pm I$ act trivially on A, and A is called *irreducible* if these are the only elements of the stabilizer Γ_A of A under the action of the gauge group. The other possibilities for the isomorphism type of Γ_A are $U(1)$ and $SU(2)$ where A is called respectively an *abelian* and a *central* connection. For instance, the trivial connection is a central connection. We use similar terminology to define the three types of connections on 3-manifolds.

Suppose $W : Y \to Y'$ is a cobordism. For any flat connections α, α' on Y and any real number κ, let $M_\kappa(W; \alpha, \alpha')$ denote the moduli spaces of connections A on W with topological energy κ that are asymptotic to α along the incoming ends and α' along the outgoing ends, and which satisfy the ASD equation

$$F^+(A) = 0$$

with respect to the metric on W. Any solution A of the ASD equation satisfies

$$\kappa(A) = 8\pi^2 \|F_A\|^2_{L^2}.$$

In particular, $\kappa(A) \geq 0$, and $\kappa(A) = 0$ if and only if A is flat.

A special case of interest is when $W = \mathbb{R} \times N$ for a connected 3-manifold N. For any flat connections α, β on N and any non-negative κ with

$$\kappa \equiv \mathrm{CS}(\alpha) - \mathrm{CS}(\beta) \mod \mathbb{Z},$$

we have a moduli space $M_\kappa(\mathbb{R} \times N; \alpha, \beta)$. Translation along the $\mathbb{R}$ factor determines an action of $\mathbb{R}$ on this moduli space. This action is free if κ is positive, and the quotient in this case is denoted by $\breve{M}_\kappa(N; \alpha, \beta)$.

Next, we review some of the properties of the moduli spaces $\breve{M}_\kappa(N; \alpha, \beta)$ in the case that N is one of the 3-manifolds $\pm P$, $\pm O$ with spherical metric. These manifolds lie in the more general family of *binary polyhedral spaces*; their instanton moduli spaces are studied in [Aus95], and the following lemma can be deduced from the general calculation of [Aus95, Section 4.3]. See also the more recent work [Ola22] in this direction.

Lemma 2.4 ([Aus95])**.** *If Y is a spherical 3-manifold, then $\breve{M}_\kappa(Y; \alpha, \beta)$ are smooth manifolds. We also have the following facts about flat connections and instantons over $\mathbb{R} \times Y$ for $Y = \pm O, \pm P$.*

- *(i) The manifold P supports three flat connections: the trivial connection θ and two irreducible flat connections α_1, α_2 with* CS*-values $\frac{1}{120}$, $\frac{49}{120}$ respectively. The moduli space $\breve{M}_{\frac{1}{120}}(P; \alpha_1, \theta)$ is a singleton.*
- *(ii) Every other nonempty moduli space $\breve{M}_\kappa(P; \alpha, \beta)$ has $\kappa \geq 2/5$.*
- *(iii) Every nonempty moduli space $\breve{M}_\kappa(-P; \alpha, \theta)$ has $\kappa \geq \frac{71}{120}$.*
- *(iv) Every nonempty moduli space $\breve{M}_\kappa(\pm O; \alpha, \beta)$ has $\kappa \geq \frac{1}{48}$.*

Proof. The claim about smoothness is proved in [Aus95, Section 4.5]; it follows from the corresponding fact for instantons over S^4. The flat connections and Chern–Simons values on P are computed as in [FS90, Proposition 2.8] and [FS90, Theorem 3.7(3)].[1] Item (ii) follows from inspection of these Chern–Simons invariants, as does item (iii) because $\mathrm{CS}_{-Y}(\alpha) = -\mathrm{CS}_Y(\alpha)$ and $\kappa > 0$ has $\kappa \equiv \mathrm{CS}(\alpha) - \mathrm{CS}(\beta)$.

[1] In comparing, Fintushel–Stern's normalization of the Chern–Simons functional is four times ours; though their function appears to be defined using the same formula, the factor of 4 arises because they use $SO(3)$-connections, whereas we use $SU(2)$-connections.

That the 0-dimensional component of $\breve{M}(P;\alpha_1,\theta)$ is a singleton follows from the computation of [Aus95, Section 4.3],[2] which also shows that this singleton descends from an instanton on S^4 over a bundle with $c_2 = 1$, hence topological energy 1. Because topological energy is multiplicative under covers and $|\pi_1(P)| = 120$, item (i) follows.

Item (iv) was explained in the sketch of the proof above, and follows because S^3 is a 48-fold cover of O. □

If $mP\#-kO$ bounds a positive-definite manifold for some $m > 0$, then as in the sketch of the proof we can construct a cobordism $W : P \to \sqcup_k O \sqcup_{m-1} -P$ with $b_1(W) = b^+(W) = 0$; glue cylindrical ends to W and fix a Riemannian metric on W compatible with the spherical metrics on the ends. The proof of Proposition 2.1 will follow from an analysis of the instanton moduli spaces on W. A standard reference for the study of the moduli spaces of $SU(2)$-instantons on 4-manifolds with cylindrical ends is [Don02], which mostly focuses on the case of 4-manifolds whose boundary components are integer homology spheres. Since the flat connections on the boundary components of our 4-manifold W are either irreducible or central, the results of [Don02] can be readily adapted to the present setup, and we assume the reader has some basic familiarity with them.

The moduli spaces $M_\kappa(W;\alpha,\alpha')$ are not necessarily smooth anymore. However, the local behavior of the moduli space around any instanton A is governed by the *ASD operator* $D_A := d_A^+ \oplus d_A^* : \Omega^1 \to \Omega^+ \oplus \Omega^0$ which is a Fredholm operator obtained as a combination of the *Coulomb gauge* condition and linearizing the instanton equation at A [Don02, Chapter 3]. Moreover, the instanton equation can be perturbed by a *holonomy perturbation* π so that the moduli space $M^\pi_\kappa(W;\alpha,\alpha')$ of solutions of the perturbed ASD equation

$$F^+(A) + \pi(A) = 0 \tag{2}$$

is a smooth manifold away from reducible elements of the moduli space [Mil19, Theorem 4.37]. In fact, we can pick π so that the L^2 norm of $\pi(A)$ is less than a given positive constant ϵ, the perturbation $\pi(A)$ vanishes for any reducible connection A, and $\pi(A)$ depends on the restriction of A to a compact subspace of W. (See [Kro05, Section 3] for a review of holonomy perturbations on 4-manifolds.) The dimension of the irreducible locus equals the Fredholm index of D_A where A is any connection on W that has topological energy κ and is asymptotic to α and α' on the cylindrical ends. Since $b_1(W) = b^+(W) = 0$, the moduli space has a canonical orientation. (In general, one needs an orientation of the vector space $H^1(W;\mathbb{R}) \oplus H^+(W;\mathbb{R})^*$ to orient the moduli spaces $M^\pi_\kappa(W;\alpha,\alpha')$.)

Lemma 2.5. *There is a holonomy perturbation π for W such that the following holds.*

- *(i) All irreducible π-instantons have surjective ASD operator, so that the irreducible part of $M^\pi_\kappa(W;\alpha,\alpha')$ is a smooth manifold of dimension equal to the index of D_A.*
- *(ii) The moduli space $M^\pi_0(W;\theta,\theta')$ of π-instantons with vanishing topological energy, which are asymptotic to the trivial connections θ and θ' on the incoming and the outgoing ends of W, does not contain any irreducible.*

[2] In comparing, our α_1 is Austin's Q, and our $\breve{M}(Y;\alpha,\beta)$ is Austin's $\widetilde{\mathcal{M}}(\beta,\alpha)$.

(iii) *There is a one to one correspondence between the central elements of $M_0^\pi(W;\theta,\theta')$ and the homomorphisms $H_1(W;\mathbb{Z}) \to \mathbb{Z}/2$ which are trivial on ∂W. For any such central connection, the perturbed ASD operator is injective.*

(iv) *There is a one to one correspondence between the abelian elements of $M_0^\pi(W;\theta,\theta')$ and the free orbits of complex-conjugation on the space of homomorphisms $H_1(W;\mathbb{Z}) \to U(1)$ which are trivial on ∂W. For any such abelian connection, the perturbed ASD operator is injective.*

(v) *All π-instantons on W have non-negative topological energy.*

In particular, if a and b denote the number of central and abelian elements of $M_0^\pi(W;\theta,\theta')$, then $a+2b$ is equal to the cardinality of $H_1(W;\mathbb{Z})/H_1(\partial W;\mathbb{Z})$.

Proof. A perturbation satisfying (i) is given in [Mil19, Theorem 4.37]. Because reducible connections on a cobordism with $b^+(W)=0$ are cut out transversely in the reducible locus [Mil19, Lemma 4.20], the perturbation $\pi(A)$ can be assumed to vanish when A is reducible. It follows that the reducible elements of $M_0(W;\theta,\theta')$ coincide with the reducible elements of $M_0^\pi(W;\theta,\theta')$.

The index of the trivial connection is -3, so that the expected dimension of the irreducible part of $M_0^\pi(W;\theta,\theta')$ is -3, and hence the irreducible part of this moduli space is empty, establishing (ii).

Because the reducibles in $M_0^\pi(W;\theta,\theta')$ agree with those in $M_0(W;\theta,\theta')$, they correspond to flat connections on W which are trivial on ∂W, hence conjugacy classes of homomorphisms $\pi_1(W) \to SU(2)$ that restrict to the trivial homomorphism on each boundary component. Those homomorphisms with image in $\{\pm I\}$ correspond to central connections, for which the conjugation action is trivial; those homomorphisms with image conjugate to a subgroup of $U(1)$ correspond to abelian connections, for which the conjugation action is the action of complex conjugation. Because homomorphisms from $\pi_1(W)$ to an abelian group factor through $H_1(W;\mathbb{Z})$, this gives the enumeration of items (iii) and (iv). The concluding enumeration follows because $a+2b$ coincides with the number of homomorphisms $H_1(W;\mathbb{Z})/H_1(\partial W;\mathbb{Z}) \to S^1$. Because $H_1(W;\mathbb{Z})$ is a finite abelian group, Pontryagin duality shows that this coincides with the number of elements of $H_1(W;\mathbb{Z})/H_1(\partial W;\mathbb{Z})$.

The central connection has injective ASD operator for the trivial perturbation, and hence the same holds for a small perturbation π. That we may choose π so that this is also true for abelian connections follows from the argument of [Mil19, Theorem 4.37] and the fact that the *normal index* $\mathrm{Ind}(D_A)+1+b^+(W)-b_1(W)$ is nonpositive (here, it is -2); see also [CDX20, Section 7.3] for a similar discussion and conclusion.

It remains to verify the claim in (v), which follows from the assumption that π is small. For any connection A we have

$$\kappa(A) = \frac{1}{8\pi^2}\int_W \mathrm{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2}\left(\|F_A\|_{L^2}^2 - 2\|F_A^+\|_{L^2}^2\right)$$

Thus for any solution of (2) we have $\kappa(A) = \frac{1}{8\pi^2}\left(\|F_A\|_{L^2}^2 - 2\|\pi(A)\|_{L^2}^2\right)$.

By taking π small enough, we can guarantee that $\kappa(A)$ is greater than $-\epsilon$ for any fixed positive constant ϵ. Since the mod $\mathbb{Z}$ value of $\kappa(A)$ belongs to a fixed finite set, we obtain (v) for a small enough value of ϵ. (In fact, $\epsilon = \frac{1}{240}$ will do.) □

Proof of Proposition 2.1. For the duration of this argument we write $\kappa = \frac{1}{120}$. For the perturbation π constructed in Lemma 2.5, we consider the moduli space $M^\pi_\kappa(W; \alpha_1, \theta')$. Since α_1 is irreducible, all elements of this moduli space are irreducible, and $M^\pi_\kappa(W; \alpha_1, \theta')$ is a smooth manifold. Gluing the connection $B \in \breve{M}_\kappa(P; \alpha_1, \theta)$ to the trivial connection Θ on W determines a connection A on W asymptotic to α_1 at $-\infty$ and the trivial connection θ' at $+\infty$; by additivity of topological energy, $\kappa(A) = \frac{1}{120}$. By additivity of ASD index [Don02, Chapter 3], we have

$$\begin{aligned}\operatorname{ind}(D^+_A) &= \operatorname{ind}(D^+_B) + \operatorname{ind}(D^+_\Theta) + \dim(\Gamma_\theta)\\ &= 1 + (-3) + 3 = 1.\end{aligned}$$

Thus $M^\pi_\kappa(W; \alpha_1, \theta')$ is an oriented 1-dimensional smooth manifold.

Next, we study the ends of the 1-dimensional manifold $M^\pi_\kappa(W; \alpha_1, \theta')$ using the standard compactification and gluing theory results in Yang–Mills gauge theory. Since $\frac{1}{120} < 1$, there is no room for bubbling. Thus the only source of non-compactness is the possibility of energy sliding off the ends of W, in two possible ways:

(i) Energy could slide off the incoming end, corresponding to gluing an instanton $\alpha \to \beta$ over $\mathbb{R} \times P$ to a π-instanton $\beta \to \theta'$ over W.
(ii) Energy could slide off one of the outgoing ends, corresponding to gluing a π-instanton $\alpha \to \beta'$ over W to an instanton $\beta' \to \theta$ over $\mathbb{R} \times -P$ or $\mathbb{R} \times \pm O$.

However, Lemma 2.4 and additivity of topological energy imply that the only possibility is case (i), where $\beta = \theta$. It follows that if $[A_i]$ is a sequence in $M^\pi_\kappa(W; \alpha_1, \theta')$ without any convergent subsequence, then this sequence is chain convergent to $[B, A]$ in the sense of [Don02, Chapter 5] where $[B]$ is the element of $\breve{M}_\kappa(P; \alpha_1, \theta)$ and $[A] \in M^\pi_0(W; \theta, \theta')$. Now the same argument as in [Don87] shows that for any abelian (resp. central) $[A] \in M^\pi_0(W; \theta, \theta')$, the pair $[A, B]$ contributes two ends (resp. one end) to the moduli space $M^\pi_\kappa(W; \alpha_1, \theta')$, and all of these ends have the same orientation. (See also [Dae20, Section 4.2] for a similar discussion.) By the conclusion of Lemma 2.5, we see that this gives as many ends of $M^\pi_\kappa(W; \alpha_1, \theta')$ as there are elements of $H_1(W, \mathbb{Z})/H_1(\partial W; \mathbb{Z})$, all of which are oriented in the same direction. As every oriented 1-manifold with finitely many ends has zero ends when counted with sign, this is a contradiction; there is no positive-definite 4-manifold with boundary $mP\# -kO$ where $m > 0$. □

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Department of Mathematics and Statistics, Washington University in St. Louis
Email address: adaemi@wustl.edu

Department of Mathematics, North Carolina State University
Email address: tlid@math.ncsu.edu

Department of Mathematics, Columbia University
Email address: smm2344@columbia.edu