MONOTONE LAGRANGIAN FLOER THEORY IN SMOOTH DIVISOR COMPLEMENTS: II

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Abstract. In [DF2], the first part of the present paper, we study the moduli spaces of holomorphic discs and strips into an open symplectic, isomorphic to the complement of a smooth divisor in a closed symplectic manifold. In particular, we introduce a compactification of this moduli space, which is called the relative Gromov-Witten compactification. The goal of this paper is to show that the RGW compactifications admit Kuranishi structures which are compatible with each other. This result provides the remaining ingredient for the main construction of [DF2]: Floer homology for monotone Lagrangians in a smooth divisor complement.

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1. Introduction

In [DF2], the authors studied Lagrangian intersection Floer homology of a pair of monotone Lagrangians in an open symplectic manifold, which is isomorphic to a divisor complement. At the heart of the construction of [DF2], there is a compactification of the moduli spaces of holomorphic discs and strips satisfying Lagrangian boundary condition. The main purpose of this sequel to [DF2] is to show that this compactification, called the RGW compactification, admits a Kuranishi structure. The virtual count of the elements of these Kuranishi spaces is used in [DF2] to define the desired Lagrangian Floer homology.

To be more detailed, let \((X, \omega)\) be a symplectic manifold and \(D\) be a symplectic submanifold of \(X\) with codimension 2. Another probably non-essential\(^1\) assumption is that \(D\) and \(N_D(X)\), the normal bundle of \(D\) in \(X\), admit integrable complex structures compatible with the symplectic structure. The complex structure on \(N_D(X)\) induces an integrable complex structure in a neighborhood of \(D\) and we extend this complex structure to an almost complex structure \(J\) which is tamed by \(\omega\). (See [DF2, Subsection 3.3] for more details on the choice of almost complex structures.)

Let \(L_0\) and \(L_1\) be compact orientable and transversal Lagrangians in \(X \setminus D\). For any \(\beta \in \Pi_2(X; L_i)\) with \(\beta \cap D = 0\), let \(M^\text{reg}_{k+1}(L_i; \beta)\) be the moduli space of \(J\)-holomorphic disks of homology class \(\beta\) with \(k + 1\) boundary marked points and Lagrangian boundary condition associated to \(L_i\). In [DF2], we introduced the RGW compactification \(M^\text{RGW}_{k+1}(L_i; \beta)\) of \(M^\text{reg}_{k+1}(L_i; \beta)\). (See [DF2, Section 3] for the definition of this moduli space. Note that this compactification is different from the stable map compactification.)

\(^1\)In [DF2], this assumption is made to simplify the arguments. However, the authors believe that this assumption can be removed at the expense of a more complicated analysis of holomorphic curves in a neighborhood of \(D\) in \(X\).
The RGW compactification $\mathcal{M}_{k_1, k_0}^{reg}(L_1, L_0; p, q; \beta)$ of the moduli space $\mathcal{M}_{k_1, k_0}^{reg}(L_1, L_0; p, q; \beta)$ in a similar way in [DF2, Section 3].

**Theorem 1.** The moduli spaces $\mathcal{M}_{k+1}^{RGW}(L_i; \beta)$ and $\mathcal{M}_{k_1, k_0}^{RGW}(L_1, L_0; p, q; \beta)$ admit Kuranishi structures.

A topological space with a Kuranishi structure is locally modeled by the vanishing locus of an equation defined on a manifold, or more generally an orbifold. These orbifolds and equations for different points need to satisfy some compatibility conditions. (See [FOOO2, Definition A.1.1] for a more precise definition of Kuranishi spaces.) Given a point of a space with Kuranishi structure, the zeros of the corresponding equation might be cut down transversally. In that case, our space looks like an orbifold in a neighborhood of such point. The main point is that such equations might not be transversal to zero and we might end up with a space which is not as regular as an orbifold. Nevertheless, a Kuranishi structure would be sufficient to have some of the interesting properties of smooth orbifolds. For example, it makes sense to talk about a space with Kuranishi structure which has boundary and corners. In fact, the Kuranishi structures of Theorem 1 have boundary and corners. Any corner of this moduli space is given by appropriate fiber products of the spaces of the form $\mathcal{M}_{k+1}^{RGW}(L_i; \beta)$ and $\mathcal{M}_{k_1, k_0}^{RGW}(L_1, L_0; p, q; \beta)$.

**Theorem 2.** The Kuranishi structures on the moduli spaces $\mathcal{M}_{k+1}^{RGW}(L_i; \beta)$ and $\mathcal{M}_{k_1, k_0}^{RGW}(L_1, L_0; p, q; \beta)$ can be chosen such that they are compatible over the boundary and corners. These Kuranishi structures are also compatible with the forgetful maps of boundary marked points.

Suppose Lagrangians $L_1, L_2$ are monotone in $X \setminus \mathcal{D}$, namely, there is a positive constant $c$ such that for any $\beta \in \Pi_2(X; L_i) := \text{Im}(\pi_2(X, L_i) \to H_2(X, L_i; \mathbb{Z}))$ with $\beta \cap \mathcal{D} = 0$, we have:

$$\omega(\beta) = c\mu(\beta).$$

Here $\mu : H_2(X, L_i; \mathbb{Z}) \to \mathbb{Z}$ is the Maslov index associated to $L_i$. The first part of Theorem 2 is the essential ingredient in the definition of Lagrangian Floer homology for the monotone pairs $L_1, L_0$ and verifying its independence from auxiliary choices in the construction.

Given any of the boundary marked points, we may define a forgetful map from $\mathcal{M}_{k+1}^{RGW}(L_i; \beta)$ to $\mathcal{M}_k^{RGW}(L_i; \beta)$. Similarly, we can define forgetful maps for $\mathcal{M}_{k_1, k_0}^{RGW}(L_1, L_0; p, q; \beta)$. The second part of Theorem 2 concerns compatibility with these maps which is necessary in the definition of Floer homology when the minimal Maslov index of one of the Lagrangians is 2. For a more precise version of Theorem 2, see Theorem 10.1, Theorem 11.7 [DF2, Lemma 3.70] and [DF2, Lemma 3.75].

One of the novel features of the RGW compactification is that it has some strata which consist of obstructed objects by default. To make this point more clear, we make a comparison with the stable map compactification. In the stable compactification of the moduli space of holomorphic
discs, each stratum is described by a fiber product of the moduli spaces of holomorphic discs and strips. If each of the moduli spaces appearing in such a fiber product consists of Fredholm regular elements and the fiber product is transversal, then the moduli space in a neighborhood of this stratum consists of regular objects and hence it is a smooth orbifold in this neighborhood. However, the situation in the case of the RGW compactification looks significantly different. There are strata of the compactification which belong to the singular locus of the moduli space, even if each element of the associated fiber product is Fredholm regular and the fiber product is cut down transversely. This subtlety is the main point that our treatment diverges from the proof of the analogues of Theorems 1 and 2 for the stable map compactification of the moduli spaces of holomorphic discs and strips. (See [FOOO2, FOOO4, FOOO6, FOOO7] for such results in the context of the stable map compactification.)

We resolve the above issue by introducing the notion of inconsistent solutions to the Cauchy-Riemann equation. Under the assumption of the previous paragraph, the space of inconsistent solutions forms a smooth manifold. Moreover, the elements of the moduli spaces $\mathcal{M}_{k+1}^{RGW}(L_i; \beta)$ and $\mathcal{M}_{k_1,k_0}^{RGW}(L_1, L_0; p, q; \beta)$ can be regarded as the zero sets of appropriate equations on the moduli space of inconsistent solutions. We treat these equations as extra terms for Kuranishi maps. We believe this approach could be also useful for the analysis of the relative Gromov-Witten theory in symplectic category. We also believe that this idea as well as some of the arguments provided in this paper can be generalized to study various conjectures proposed in [DF2, Section 6].

The main steps of the construction of the Kuranishi structures required for the proof of Theorems 1 and 2 are parallel to the ones for the stable map compactification. Throughout the paper, we point out relevant references for the corresponding results in the context of the stable map compactification. At the same time, we try to make our exposition as self-contained as possible. One of the exceptions is the exponential decay result of [FOOO5] where the same arguments can be used to deal with the exponential decay result which we need for this paper.

**Outline of Contents.** In order to make the main ideas of the construction more clear, we devote the first part of the paper to the construction of a Kuranishi chart around a special point in the RGW compactification of moduli spaces of discs. This special point, described in Section 2, belongs to a stratum of the moduli space which is always obstructed. Motivated by this example, we introduce the notion of inconsistent solutions in Section 5. The stratum of this special example is given by the fiber product of a moduli space of discs and two moduli spaces of spheres. In Sections 3 and 4, we study the deformation theory of the elements of the moduli space within this stratum. A Kuranishi chart for each element of this stratum is constructed
in Section 8. The main analytical results required for the construction of the Kuranishi chart is verified in Section 7.

In Section 8, we explain how the method of the first part of the paper can be used to construct a Kuranishi chart around any point of the RGW moduli space. Section 9 is devoted to showing that these Kuranishi charts are compatible with each other using appropriate coordinate changes. This completes the proof of Theorem 1. We study compatibility of Kuranishi structures at boundary components and corners in Section 10.1. In order to verify the second part of Theorem 2, compatibility of Kuranishi structures with forgetful maps is studied in Section 11. Since the case of strips are only notationally heavier, we focus on the moduli spaces of discs up to this point in the paper.

In the final section, we turn our attention back to the moduli spaces of strips. Using the general theory of Kuranishi structures and the system of Kuranishi structures provided by Theorems 1 and 2, we can construct a system of multi-valued perturbations on such moduli spaces. These perturbations allow us to make a virtual count of the elements of the moduli spaces of the RGW strips and show that such counts are independent of auxiliary choices. Thus they provide the crucial ingredient for the construction of [DF2, Section 4].

2. A Special Point of the Moduli Spaces of Discs

In the first half of the paper, we focus on the analysis of a special case. We hope that this allows the main features of our construction stand out. The special case can be described as follows. Let $\Sigma$ be a surface with nodal singularities, which has three irreducible components $\Sigma_d$, $\Sigma_s$ and $\Sigma_D$. The irreducible component $\Sigma_d$ is a disc and the remaining ones are spheres. The components $\Sigma_d$, $\Sigma_s$ and $\Sigma_D$ are respectively called the disc component, the sphere component and the divisor component. The divisor component $\Sigma_D$ intersects $\Sigma_d$ and $\Sigma_s$ at the points, $z_d$ and $z_s$, respectively. When we want to emphasize that we consider these points as elements of $\Sigma_D$, we denote them by $z_{D,d}$ and $z_{D,s}$. There is no intersection between $\Sigma_d$ and $\Sigma_s$.

We are given a $J$-holomorphic map $u : (\Sigma, \partial \Sigma) \to (X, L)$. The restriction of this map to $\Sigma_d$, $\Sigma_s$, $\Sigma_D$ are denoted by $u_d$, $u_s$, $u_D$. We assume that the image of $u_D$ is contained in the divisor $D$. The images of $\Sigma_d$, $\Sigma_s$ intersect $D$ only at the points $z_d$ and $z_s$ with multiplicities 2 and 3, respectively. Following [DF2, Section 3], we also associate a level function $\lambda$ that evaluates to 0 at the components $\Sigma_d$ and $\Sigma_s$ and to 1 at $\Sigma_D$. We assume that there is one boundary marked point $z_0$ on $\Sigma_d$.

We also assume that the homology class $(u_D)_*([\Sigma_D])$ satisfies the following identity: (Compare to [DF2, Condition (3.12)].)

$$2 + 3 + c_1(N_D(X)) \cap (u_D)_*([\Sigma_D]) = 0.$$ 

This condition implies that there exists a meromorphic section $s$ of $(u_D)^*N_D X$ such that $s$ has a pole of order 2 (resp. 3) at $z_d$, (resp. $z_s$), and $s$ has no other
pole or zero. The choice of this section \( s \) is unique up to a multiplicative constant in \( \mathbb{C}_* \). We fix one such section \( s \) and define:

\[
U_D : \Sigma_D \setminus \{z_d, z_s\} \to N_D(X) \setminus D = \mathbb{R} \times SN_D(X)
\]

where \( U_D(z) \), for \( z \in \Sigma_D \setminus \{z_d, z_s\} \), is defined to be \((u_D(z), s(z))\).

The Riemann surface \( \Sigma \) and the detailed ribbon tree corresponding to \( u \) are sketched in Figures 1, 2. These data define an element of \( \mathcal{M}^{RGW}_1(L; \beta) \) for an appropriate choice of \( \beta \in H_2(X, L; \mathbb{Z}) \). See [DF2, Section 3] for the definitions of detailed ribbon trees and moduli spaces \( \mathcal{M}^{RGW}_{k+1}(L; \beta) \). Constructing a Kuranishi neighborhood for this element of \( \mathcal{M}^{RGW}_1(L; \beta) \) is the main goal of the first half of the paper.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Detailed tree of the element we study.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The source curves of \((u_d, u_D, u_s)\)}
\end{figure}

3. Fredholm Theory of the Irreducible Components

In this subsection, we shall be concerned with the deformations of the restrictions of the map \( u \) to the irreducible components \( \Sigma_d, \Sigma_s \) and \( \Sigma_D \). We will see that the deformation theory of each irreducible component is governed by a Fredholm operator.

Throughout this section, we use cylindrical coordinates both for the target and the source. There is a neighborhood \( \mathcal{U} \) of the divisor \( D \) with a partial \( \mathbb{C}_* \)-action such that the complex structure \( J \) on \( \mathcal{U} \) is integrable. (See [DF2, Subsection 3.2] for the definition of partial \( \mathbb{C}_* \)-actions and [DF2, Subsection 3.3] for the existence of \( \mathcal{U} \).) We may assume that the open set \( \mathcal{U} \) is chosen such that its closure is diffeomorphic to:

\[
[0, \infty)_\tau \times SN_D(X)
\]
where \( SN_D(X) \) is the unit \( S^1 \)-bundle associated to the normal bundle \( N_D(X) \) of \( D \) in \( X \). We use \( \tau \) to denote the standard coordinate on \([0, \infty)\). The 1-form 
\[ \theta := -\frac{1}{r} \text{d}r \circ J \]
determines a connection 1-form for the \( S^1 \)-bundle \( SN_D(X) \). Let \( g' \) be a metric on \( D \) which is fixed for the rest of the paper. We also fix a metric \( g \) on \( X \setminus D \) such that its restriction to (3.1) is given by:
\begin{equation}
(3.2) \quad g|_{[0,\infty)_r \times SN_D(X)} = d\tau^2 + \theta^2 + g'.
\end{equation}
In particular, \( g \) is invariant with respect to the partial \( \mathbb{C}_r \cong (-\infty, \infty) \times S^1 \)-action on (3.1), where \((-\infty, \infty) \) acts (partially) by translation along the factor \([0, \infty)_{\tau} \) and the action of \( S^1 \) is induced by the obvious circle action on \( SN_D(X) \). We also fix another metric \( g_{NC} \) on \( X \setminus D \) whose restriction to (3.1) has the following form:
\begin{equation}
(3.3) \quad g_{NC}|_{[0,\infty)_r \times SN_D(X)} = e^{-2\tau}(d\tau^2 + \theta^2) + g'.
\end{equation}
This non-cylindrical metric extends to \( D \) to give a smooth metric on \( X \) which is also denoted by \( g_{NC} \).

**Remark 3.1.** We do not make any assumption on compatibility of the metric \( g' \) with the almost complex structure or the symplectic structure.

The metric \( g \) determines the decomposition:
\begin{equation}
(3.4) \quad TX|_U = \mathbb{R} \oplus \mathbb{R} \oplus \pi^*(TD)
\end{equation}
where the first factor is given by the action of \((-\infty, \infty) \times \{1\} \subset (-\infty, \infty) \times S^1 \), the second factor is given by the action of \( \{1\} \times S^1 \subset (-\infty, \infty) \times S^1 \), and the third factor is given by the vectors orthogonal to the first two factors. Note that the last factor and the direct sum of the first two factors determine complex subspaces of \( TX|_U \).

### 3.1. The Disk Component.

The surface \( \Sigma_d \) can be identified uniquely with the standard unit disc \( D^2 \subset \mathbb{C} \) such that \( z_0 \) and \( z_d \) are mapped to 1 and 0. The map \( u_d : D^2 \to X \) induces a map from \( D^2 \setminus \{0\} \) to \( X \setminus D \), which we also denote by \( u_d \). We identify \( D^2 \setminus \{0\} \) with \([0, \infty)_{r_1} \times S^1 \) and denote the standard coordinates on the \([0, \infty) \) and \( S^1 \) factors with \( r_1 \) and \( s_1 \). Namely, the point \((r_1, s_1) \in [0, \infty) \times S^1 \) is mapped to \( \exp( - r_1 - \sqrt{-1} s_1 ) \). (Here and in what follows, \( S^1 \) is identified with \( \mathbb{R}/2\pi \mathbb{Z} \).) Since the multiplicity of the intersection of \( u_d \) and \( D \) at \( z_d \) is 2, there exist \( R_d \in \mathbb{R} \) and \( x_d \in SN_D(X) \) such that:
\begin{equation}
(3.5) \quad d_{C^m}(u_d(r_1, s_1), (2r_1 + R_d, 2s_1 + x_d)) \leq C_m e^{-\delta_1 r_1}
\end{equation}
for some \( C_m, \delta_1 > 0 \). The constant \( \delta_1 \) is independent of \( m \) and in fact, we can pick it to be 1. Here we regard \( (2r_1 + R_d, 2s_1 + x_d) \) as an element of \([0, \infty)_{r_1} \times SN_D(X) \) using the partial action of \((-\infty, \infty) \times S^1 \) on \([0, \infty)_{\tau} \times SN_D(X) \). The expression on the left hand side of (3.5) is the \( C^m \) distance between the following two maps from \([0, \infty)_{r_1} \times S^1 \) to \( X \setminus D \):
\[
(r_1, s_1) \mapsto u_d(r_1, s_1) \quad (r_1, s_1) \mapsto (2r_1 + R_d, 2s_1 + x_d)
\]
Note that there exists $R'_d > 0$ such that $u_d(s_1, r_1)$ is an element of $(3.1)$ for $r_1 > R'_d$. The $C^m$ norm is defined with respect to the cylindrical metric on $D^2 \setminus \{0\}$ and the metric $g$ on $X \setminus D$. The inequalities in (3.5) are immediate consequences of the fact that $u_d : D^2 \setminus \{0\} \to X \setminus D$ extends to a holomorphic map from $D^2$ to $X$, which intersects $D$ with multiplicity 2.

**Definition 3.2.** We define $C^\infty(\{0, \infty\} \times S^1; (u^*_d TX, u^*_d TL))$ to be the space of all smooth sections $V$ of $u^*_d TX$ on the space $\{0, \infty\} \times S^1$ with the boundary condition:

$$V(0, s_1) \in T_{u_d(0, s_1)} L.$$ 

We extend each vector $v \in T_{u_d(z_d)} D$ to a vector field defined on a neighborhood of $u_d(z_d)$ in $D$. Let $\hat{v}$ be the horizontal lift of this vector field using the decomposition in (3.4) to a neighborhood of $u_d(z_d)$ in $X$. We may assume that the map $v \mapsto \hat{v}$ is linear. Using (3.4), we can also obtain a vector field $[t_\infty, s_\infty]$ on $X \setminus D$ for each $(t_\infty, s_\infty) \in \mathbb{R} \times \mathbb{R}$. These vector fields are also $(-\infty, \infty) \times S^1$-invariant.

**Definition 3.3.** Let $C^\infty(\{0, \infty\} \times S^1; (u^*_d TX, u^*_d TL))^+$ be the space of all triples $(V, (t_\infty, s_\infty), v)$ such that $V \in C^\infty(\{0, \infty\} \times S^1, \{0\} \times S^1; (u^*_d TX, u^*_d TL))$, $(t_\infty, s_\infty) \in \mathbb{R} \times \mathbb{R}$, $v \in T_{u_d(z_d)} D$, and

$$V - [t_\infty, s_\infty] - \hat{v}$$

has compact support. We define a weighted Sobolev norm on this vector space as follows:

$$\| (V, (t_\infty, s_\infty), v) \|^2_{W^2_{m, \delta}} = \| V \|^2_{L^2_{\delta}(\{0, R'_d\} \times S^1)}$$

$$+ \sum_{j=0}^{m} \int_{\{R'_d, \infty\} \times S^1} e^{\delta r_1} |\nabla^j (V - [t_\infty, s_\infty] - \hat{v})|^2 dr_1 ds_1$$

$$+ |(t_\infty, s_\infty)|^2 + |v|^2. \tag{3.6}$$

Later we shall be concerned with the case that $\delta > 0$ is a sufficiently small positive number and $m$ is a sufficiently large positive integer. We denote by $W^2_{m, \delta}(\Sigma_d \setminus \{z_d\}; (u^*_d TX, u^*_d TL))$ the completion of $C^\infty(\{0, \infty\} \times S^1; \{0\} \times S^1; (u^*_d TX, u^*_d TL))^+$ with respect to the norm $\| \cdot \|_{W^2_{m, \delta}}$. This completion is a Hilbert space and is independent of how we extend the vectors $v$ to $\hat{v}$.

**Definition 3.4.** Let $C^\infty_0(\{0, \infty\} \times S^1; u^*_d TX \otimes \Lambda^{0,1})$ be the space of all smooth sections with compact supports, and define a weighted Sobolev norm on it by:

$$\| V \|^2_{L^2_{m, \delta}} = \| V \|^2_{L^2_{m}(\{0, R'_d\} \times S^1)} + \sum_{j=0}^{m} \int_{\{R'_d, \infty\} \times S^1} e^{\delta r_1} |\nabla^j (V)|^2 dr_1 ds_1.$$
The completion of $C_0^\infty([0, \infty) \times S^1; u_d^* TX \otimes \Lambda^{0,1})$ with respect to the norm $\| \cdot \|_{L^2_{m,d}}$ is denoted by

$$L^2_{m,d}(\Sigma_d \setminus \{z_d\}; u_d^* TX \otimes \Lambda^{0,1}).$$

(3.7)

Linearization of the Cauchy-Riemann equation at $u_d$ gives a first order differential operator

$$D_{u_d} \bar{\partial} : C_0^\infty(([0, \infty) \times S^1, \{0\} \times S^1); (u_d^* TX, u_d^* TL))$$

$$\rightarrow C_0^\infty([0, \infty) \times S^1; u_d^* TX \otimes \Lambda^{0,1}).$$

Lemma 3.5.

(1) The operator $D_{u_d} \bar{\partial}$ induces a continuous linear map

$$D_{u_d} \bar{\partial} : W^2_{m+1,d}(\Sigma_d \setminus \{z_d\}; (u_d^* TX, u_d^* TL))$$

$$\rightarrow L^2_{m,d}(\Sigma_d \setminus \{z_d\}; u_d^* TX \otimes \Lambda^{0,1}).$$

(3.8)

In particular, for an element:

$$(V, (r_\infty, s_\infty), v) \in C_0^\infty(([0, \infty) \times S^1, \{0\} \times S^1); (u_d^* TX, u_d^* TL))^+$$

we have $D_{u_d} \bar{\partial}(V, (r_\infty, s_\infty), v) = D_{u_d} \bar{\partial}(V)$.

(2) (3.8) is a Fredholm operator.

(3) The index of the operator (3.8) is equal to the virtual dimension of the moduli space $M^\text{reg, d}_1(\beta_d; (2))^2$ which contains $u_d$.

Proof. (1) is a consequence of (3.5). (We choose $\delta$ to be smaller than the constant $\delta_1$ in (3.5).) The differential operator $D_{u_d} \bar{\partial}$ is asymptotic to an operator of the form

$$\frac{\partial}{\partial r_1} + P$$

as $r_1$ goes to infinity. Furthermore, $P = J\partial/\partial s_1$ and the kernel of this operator can be identified with $\mathbb{R} \oplus \mathbb{R} \oplus T_{u_d(z_d)} \mathcal{D}$. Part (2) is a consequence of this observation and general results about Fredholm operators on manifolds with cylindrical ends [APS]. Part (3) is also standard. \hfill \Box

3.2. The Sphere Component. In this part, we study the linearization of the problem governing the map $u_s$. This can be done similar to the case of $u_d$. We take a compact subset $K_s$ of $\Sigma_s \setminus \{z_s\}$ such that $u_s(\Sigma_s \setminus K_s)$ is contained in (3.1). We may assume that $\Sigma_s \setminus K_s$ is a disk. We take a coordinate $(r_2, s_2) \in \mathbb{R} \times S^1$ of $\Sigma_s \setminus (K_s \cup \{z_s\})$ such that $(r_2, s_2)$ is identified with $\exp(-r_2 - \sqrt{-1}s_2) \in D^2 \setminus \{0\} \cong \Sigma_s \setminus (K_s \cup \{z_s\})$. In the same way as in (3.5), we have the following inequality:

$$d_C^m(u_s(r_2, s_2), (3r_2 + R_s, 3s_2 + x_s)) \leq C_m e^{-\delta_1 r_2},$$

(3.9)

for a constant $R_s$ and $x_s \in SN_D(X)$.

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\[^2\text{See [DF2, Definition 3.35] for the definition of this moduli space. Here (2) stands for the multiplicity number 2.}\]
Definition 3.6. (Compare to [FOOO2, Lemma 7.1.5].) Let:

\[ C_0^\infty(\Sigma_s \setminus \{z_s \}; u_s^*TX) ^+ \]

be the space of all triples \((V, (r_\infty, s_\infty), v)\) such that \( V \in C^\infty(\Sigma_s \setminus \{z_s \}; u_s^*TX), (r_\infty, s_\infty) \in \mathbb{R} \times \mathbb{R}, v \in \mathcal{T}_{u_\infty(z_s)}\mathcal{D} \) and:

\[ V - [r_\infty, s_\infty] - \hat{v} \]

is compactly supported.\(^3\) Analogous to (3.6), we define a Sobolev norm on this space as follows:

\[
\| (V, (r_\infty, s_\infty), v) \|^2_{W^2_m,\delta} = \| V \|^2_{L^2_{m,\delta}(K_s)} \\
+ \sum_{j=0}^{m} \int_{[0,\infty) \times S^1} e^{\delta r_2} |\nabla^j (V - [r_\infty, s_\infty] - \hat{v})|^2 \, dr_2 \, ds_2 \\
+ |(r_\infty, s_\infty)|^2 + |v|^2.
\]

We shall be concerned with the case that \( \delta \) is a sufficiently small positive number and \( m \) is a sufficiently large positive integer. We denote by

\[ W^2_{m,\delta}(\Sigma_s \setminus \{z_s \}; u_s^*TX) \]

the completion of \( C_0^\infty(\Sigma_s \setminus \{z_s \}; u_s^*TX) ^+ \) with respect to the norm \( \| \cdot \|_{W^2_{m,\delta}} \). This completion is a Hilbert space.

We can also define the Hilbert space:

\[ L^2_{m,\delta}(\Sigma_s \setminus \{z_s \}; u_s^*TX \otimes \Lambda^{0,1}) , \]

in the same way as in (3.7).

Lemma 3.7. \( (1) \) The linearization of the Cauchy-Riemann equation at \( u_s \) defines a continuous linear map

\[
D_{u_\infty} \overline{\partial} : W^2_{m+1,\delta}(\Sigma_s \setminus \{z_s \}; u_s^*TX) \rightarrow L^2_{m,\delta}(\Sigma_s \setminus \{z_s \}; u_s^*TX \otimes \Lambda^{0,1}).
\]

\( (2) \) (3.11) is a Fredholm operator.

\( (3) \) The index of the operator (3.11) is equal to 4 plus the virtual dimension of the moduli space \( M^{\text{reg,3}}(\beta_3; (3)) \)

which contains \( u_s \).

The proof is similar to the proof of Lemma 3.5. The number 4, that appears in Item (3), is the dimension of the group of automorphisms of \((S^2, z_s)\).

\(^3\)Note that \((r_\infty, s_\infty)\) and \( v \) determine a vector fields \([r_\infty, s_\infty]\) and \( \hat{v} \) on (3.1) in the same way as in the last subsection.

\(^4\)See [DF2, Definition 3.36]. (3) stands for the multiplicity number 3.
3.3. The Divisor Component. Finally, we analyze the deformation theory of $u_D$. Note that $u_D$ is a map to the Kähler manifold $D$. So we firstly describe a Fredholm theory for the deformation of $u_D$ as a map to $D$. This is a standard task in Gromov-Witten theory. We have a Fredholm operator:

\[(3.12) \quad D_{u_D} : L^2_{m+1}(\Sigma_D; u_D^*TD) \to L^2_m(\Sigma_D; u_D^*TD \otimes \Lambda^{0,1}).\]

To perform gluing analysis, we compare this Fredholm operator with another Fredholm operator associated to the map $U_D$ in (2.1).

**Definition 3.8.** As in previous two subsections, we extend any $v_{\infty,d} \in T_{u_d(z_d)}D$, $v_{\infty,s} \in T_{u_s(z_s)}D$, to vector fields $\tilde{v}_{\infty,d}$, $\tilde{v}_{\infty,s}$ on open neighborhoods of the fibers of $\mathbb{R}_+ \times SN_D(X)$ over $u_d(z_d)$ and $u_s(z_s)$. For any $(r_{\infty,d}, s_{\infty,d}) \in \mathbb{R} \times \mathbb{R}$, we can also define a vector field $[r_{\infty}, s_{\infty}]$ in a neighborhood of any of the points $u_d(z_d)$ and $u_s(z_s)$, as in the last two subsections. Let:

$$C^0_0(\Sigma_D \setminus \{ z_d, z_s \}; U^*_D T(\mathbb{R}_+ \times SN_D(X)))^+$$

be the space of all 5-tuples $(V, (r_{\infty,d}, s_{\infty,d}), (r_{\infty,s}, s_{\infty,s}), v_d, v_s)$ such that:

(i) The restriction of $V - [r_{\infty,d}, s_{\infty,d}] - \hat{v}_d$ to a punctured neighborhood of $z_d$ in $\Sigma_D \setminus \{ z_d, z_s \}$ vanishes;

(ii) The restriction of $V - [r_{\infty,s}, s_{\infty,s}] - \hat{v}_s$ to a punctured neighborhood of $z_s$ in $\Sigma_D \setminus \{ z_d, z_s \}$ vanishes.

We define a weighted Sobolev norm on this space as follows:

$$\| (V, (r_{\infty,d}, s_{\infty,d}), (r_{\infty,s}, s_{\infty,s}), v_d, v_s) \|^2_{H^2_{m, \delta}}$$

\[
= \| V \|^2_{L^2_{m}(K_D)} + \sum_{j=0}^{m} \int_{[0, \infty) \times S^1} e^{\delta r_1} |\nabla^j (V - [r_{\infty,d}, s_{\infty,d}] - \hat{v}_d)|^2 dr_1 ds_1 \\
+ \sum_{j=0}^{m} \int_{[0, \infty) \times S^1} e^{\delta r_2} |\nabla^j (V - [r_{\infty,s}, s_{\infty,s}] - \hat{v}_s)|^2 dr_2 ds_2 \\
+ |(r_{\infty,d}, s_{\infty,d})|^2 + |(r_{\infty,s}, s_{\infty,s})|^2 + |v_d|^2 + |v_s|^2.
\]

In order to clarify the notation in (3.13), the following comments are in order. We take a compact subset $K_D \subset \Sigma_D \setminus \{ z_d, z_s \}$ such that $\Sigma_D \setminus K_D$ is the union of two discs. We fix coordinates $(r_1, s_1) \in [0, \infty) \times \mathbb{R}/2\pi \mathbb{Z}$ and $(r_2, s_2) \in [0, \infty) \times \mathbb{R}/2\pi \mathbb{Z}$ on the complement of the origins of these two discs. That is, we identify $[0, \infty) \times \mathbb{R}/2\pi \mathbb{Z}$ with $D^2 \setminus \{ 0 \}$ using $(r_1, s_1) \mapsto \exp(-r_i + \sqrt{-1}s_i))$.

We denote the completion of $C^0_0(\Sigma_D \setminus \{ z_d, z_s \}; U^*_D T(\mathbb{R}_+ \times SN_D(X)))^+$ with respect to the norm $\| \cdot \|_{H^2_{m, \delta}}$ by:

$$H_1 := W^2_{m, \delta}(\Sigma_D \setminus \{ z_d, z_s \}; U^*_D T(\mathbb{R}_+ \times SN_D(X))).$$

This completion is a Hilbert space.

We also define a Hilbert space

$$H_2 := L^2_{m, \delta}(\Sigma_D \setminus \{ z_d, z_s \}; U^*_D T(\mathbb{R}_+ \times SN_D(X)) \otimes \Lambda^{0,1}).$$
in the same way as in (3.7).

We have a short exact sequence of holomorphic bundles on $\Sigma_D \setminus \{z_d, z_s\}$ as follows:

$$0 \to \mathbb{C} \to U_D^* T(\mathbb{R} \times SN_D(X)) \to u_D^* TD \to 0.$$ 

Here the first map is defined by the $\mathbb{C}_+$-action. This short exact sequence induce a diagram of the following form:

$$
\begin{array}{cccccc}
0 & \to & A_1 & \to & H_1 & \to & B_1 & \to & 0 \\
& & f & \downarrow & g & \downarrow & h & & \\
0 & \to & A_2 & \to & H_2 & \to & B_2 & \to & 0
\end{array}
$$

where we have:

$$A_1 = W_{m+1,\delta}^2(\Sigma_D \setminus \{z_d, z_s\}; \mathbb{C}), \quad A_2 = L_{m,\delta}^2(\Sigma_D \setminus \{z_d, z_s\}; \Lambda^{0,1}),$$

and

$$B_1 = W_{m+1,\delta}^2(\Sigma_D \setminus \{z_d, z_s\}; u_D^* TD), \quad B_2 = L_{m,\delta}^2(\Sigma_D \setminus \{z_d, z_s\}; u_D^* TD \otimes \Lambda^{0,1}).$$

The spaces $A_1$ and $B_1$ are defined similar to $H_1$ in an obvious way. In the same way as in the proof of Lemma 3.5, we can show that the linearization of the Cauchy-Riemann equation at $U_D$ defines a continuous linear map:

$$D_{U_D} \overline{\partial} : W_{m+1,\delta}^2(\Sigma_D \setminus \{z_d, z_s\}; U_D^* T(\mathbb{R} \times SN_D(X))) \to L_{m,\delta}^2(\Sigma_D \setminus \{z_d, z_s\}; u_D^* TD \otimes \Lambda^{0,1}).$$

which is the map $g$ in (3.14). The map $f$ is the standard Cauchy-Riemann operator and $h$ is the linearized Cauchy-Riemann operator associated to the map $u_D$. The diagram in (3.14) commutes and each row of the diagram forms an exact sequence.

**Lemma 3.9.**

(1) The operator in (3.15) is Fredholm.

(2) The kernel and the cokernel of the operator $h$ in (3.14) can be identified with the kernel and the cokernel of $D_{U_D} \overline{\partial}$ in (3.12). Moreover, (3.14) induces a short exact sequence of the following form:

$$0 \to \mathbb{C} \to \text{Ker} D_{U_D} \overline{\partial} \to \text{Ker} D_{u_D} \overline{\partial} \to 0.$$

and an isomorphism

$$\text{CoKer} D_{U_D} \overline{\partial} \cong \text{CoKer} D_{u_D} \overline{\partial}.$$

**Proof.** The proof of the claim in (1) is similar to the proof of Lemma 3.5. Identification of the kernels and cokernels of the operators $h$ and $D_{U_D} \overline{\partial}$ is straightforward. Similarly, we can identify the kernels and cokernels of the operators $f$ and the Cauchy-Riemann operator associated to the trivial bundle on the sphere $\Sigma_D$. The latter operator is surjective and its kernel is a copy of $\mathbb{C}$, consists of constant sections of the trivial bundle. We can use this observation and properties of the diagram in (3.14) to obtain the remaining claims in part (2).
4. STABILIZATION OF THE SOURCE CURVES AND THE OBSTRUCTION BUNDLES

The operators $D_{u_4\overline{\partial}}, D_{u_3\overline{\partial}}, D_{u_2\overline{\partial}}$ are not necessarily surjective. If these operators are not surjective, then the deformation theories of $u_d, u_s, u_D$ are obstructed. Following a general idea due to Kuranishi, we introduce obstruction bundles:

**Definition 4.1.** We consider linear subspaces

$$E_d \subset C^\infty(\Sigma_d \setminus \{z_d\}; u_d^*TX \otimes \Lambda^{0,1}),$$

$$E_s \subset C^\infty(\Sigma_s \setminus \{z_s\}; u_s^*TX \otimes \Lambda^{0,1}),$$

$$E_D \subset C^\infty(\Sigma_D \setminus \{z_d, z_s\}; u_D^*TD \otimes \Lambda^{0,1})$$

of finite dimensions, which have the following properties:

1. Elements of $E_d, E_s, E_D$ have compact supports away from $z_d, z_s, \{z_d, z_s\}$, respectively;
2. $\text{Im}(D_{u_4\overline{\partial}}) + E_d = L^2_{m,d}(\Sigma_d \setminus \{z_d\}; u_d^*TX \otimes \Lambda^{0,1});$
3. $\text{Im}(D_{u_3\overline{\partial}}) + E_s = L^2_{m,d}(\Sigma_s \setminus \{z_s\}; u_s^*TX \otimes \Lambda^{0,1});$
4. $\text{Im}(D_{u_2\overline{\partial}}) + E_D = L^2_{m,D}(\Sigma_D; u_D^*TD \otimes \Lambda^{0,1}).$

We also require them to satisfy the mapping transversality condition of Definition 4.2.

**Definition 4.2.** Let

$$\mathcal{E}V_d : W^2_{m+1,\delta}(\Sigma_d \setminus \{z_d\}; (u_d^*TX, u_d^*TL)) \rightarrow T_{u_d(z_d)}D$$

be the continuous linear map that associates to a triple $(V, (r_\infty, s_\infty), v)$ the vector $v$. The map

$$\mathcal{E}V_s : W^2_{m+1,\delta}(\Sigma_s \setminus \{z_s\}; u_s^*TX) \rightarrow T_{u_s(z_s)}D,$$

is defined similarly. Finally, let:

$$\mathcal{E}V_D := (\mathcal{E}V_{D,d}, \mathcal{E}V_{D,s}) : L^2_{m+1}(\Sigma_D; u_D^*TD) \rightarrow T_{u_4(z_d)}D \oplus T_{u_3(z_s)}D.$$

be the map that associates to $V \in L^2_{m+1}(\Sigma_D; u_D^*TD)$ the pair of vectors $(V(z_d), V(z_s))$. We say that $E_s, E_d, E_D$ of the previous definition satisfy the **mapping transversality condition**, if the following map is surjective:

$$(\mathcal{E}V_d + \mathcal{E}V_{D,d}, \mathcal{E}V_s + \mathcal{E}V_{D,s}) :$$

$$(D_{u_4\overline{\partial}})^{-1}E_d \oplus (D_{u_3\overline{\partial}})^{-1}E_s \oplus (D_{u_2\overline{\partial}})^{-1}E_D \rightarrow T_{u_4(z_d)}D \oplus T_{u_3(z_s)}D.$$

We shall use these obstruction spaces to define Kuranishi neighborhoods of the elements represented by $u_d, u_s, u_D$, respectively. Note that at this stage we are studying three irreducible components separately. The process of gluing them will be discussed in the next stage.

The source curve $\Sigma_d$ of $u_d$ comes with one interior nodal point $z_d$ and one boundary marked point $z_0$. The group of isometries of $\Sigma_d$ preserving
z_d and z_0 is trivial and hence Σ_d together with these marked points is stable. Moreover, this source curve does not have any deformation parameter. However, the source curve Σ_s of u_s comes with only one interior nodal point z_s. Therefore, it is unstable and we add two extra marked points w_{s,1}, w_{s,2} such that it becomes stable with no deformation parameter. Similarly, the source curve Σ_D of u_D comes with two interior nodal points z_s, z_d and is unstable. We add one marked point w_D so that it becomes stable without any deformation parameter. We follow the method of [FOd, appendix] to use transversal submanifolds for the purpose of killing the extra freedom of moving the auxiliary marked points. (See also [FOOO4, Section 20], [FOOO8, Subsection 9.3].) Namely, we fix submanifolds N_{s,1}, N_{s,2} and N_D with the following properties:

**Condition 4.3.**

1. N_{s,1}, N_{s,2} are codimension 2 smooth submanifolds of X, and N_D is a codimension 2 smooth submanifold of D.

2. For i = 1, 2, there exists an open neighborhood V_i(w_{s,i}) of w_{s,i} such that V_i(w_{s,i}) \cap u_*^{-1}(N_{s,i}) = \{w_{s,i}\} and u_i|_{V_i(w_{s,i})} is transversal to N_{s,i} at w_{s,i}.

3. There exists an open neighborhood V_D(w_D) of w_D such that V_D(w_D) \cap u_D^{-1}(N_D) = \{w_D\} and u_D|_{V_D(w_D)} is transversal to N_D at w_D.

We now define (Kuranishi) neighborhoods of u_d, u_s, u_D as follows. Let u_d' : (Σ_d \{z_d\}, ∂Σ_d) → (X \D, E) (resp. u_s' : Σ_s \{z_s\} → X \D, u_D' : Σ_D → D) be a L^2_{m+1,loc} map such that:

\[
d(u_d(x), u_d(x)) \leq \epsilon \quad (\text{resp. } d(u_s(x), u_s'(x)) \leq \epsilon,)
\]

\[
d(u_D(x), u_D'(x)) \leq \epsilon.)
\]

for any x ∈ Int(Σ_d \{z_d\}) (resp. x ∈ Σ_s \{z_s\}, x ∈ Σ_D). Here d is defined with respect to the metric g on X \D or the metric g' on D, introduced at the beginning of Section 3. We wish to define:

\[
E_d(u_d') \subset L^2_m(Σ_d \{z_d\}; (u_d')^*TX \otimes Λ^{0,1})
\]

\[
E_s(u_s') \subset L^2_m(Σ_s \{z_s\}; (u_s')^*TX \otimes Λ^{0,1})
\]

\[
E_D(u_D') \subset L^2_m(Σ_D; (u_D')^*TD \otimes Λ^{0,1})
\]

which are finite dimensional subspaces consisting of elements with compact supports. Firstly we need to impose an additional constraint on E_d, E_s, E_D.

**Condition 4.4.** If x ∈ Σ_d (resp. x ∈ Σ_s, x ∈ Σ_D) is in the support of an element of E_d (resp. E_s, E_D), then u_d (resp. u_s, u_D) is an immersion at x.

This condition in particular implies that E_d (resp. E_s, E_D) is zero, if u is constant on Σ_d (resp. Σ_s, Σ_D). Using the fact that Σ has genus 0, we can always take E_d, E_s, E_D satisfying this additional condition.

We denote by Supp(E_d) (resp. Supp(E_s), Supp(E_D)) the union of the supports of elements of E_d (resp. E_s, E_D). We define a map I_d : Supp(E_d) → Σ_d (resp. I_s : Supp(E_s) → Σ_s, I_D : Supp(E_D) → Σ_D) as follows. (Here t
stands for target.) For \( x \in \text{Supp}(E_d) \) (resp. \( x \in \text{Supp}(E_s) \), \( x \in \text{Supp}(E_D) \)), the point \( I_d^\epsilon(x) \) (resp. \( I_s^\epsilon(x) \), \( I_D^\epsilon(x) \)) is given by the following conditions:

**Condition 4.5.**

1. We require that the distance between \( x \) and \( I_d^\epsilon(x) \) (resp. \( I_s^\epsilon(x) \), \( I_D^\epsilon(x) \)) is smaller than the constant \( \epsilon \). We choose \( \epsilon \) small enough such that (4.2) and this condition imply:

\[
\begin{align*}
  d(u_d(x), u_d'(I_d^\epsilon(x))) &\leq o, \quad \text{(resp. } d(u_s(x), u_s'(I_s^\epsilon(x))) \leq o, \\
  d(u_D(x), u_D'(I_D^\epsilon(x))) &\leq o)
\end{align*}
\]

where \( o \) is a constant smaller than the injectivity radii of \( X \setminus \mathcal{D} \) and \( \mathcal{D} \).

2. The condition in (1) implies that there exists a unique minimal geodesic \( \gamma_d : [0, 1] \to X \setminus \mathcal{D} \) (resp. \( \gamma_s : [0, 1] \to X \setminus \mathcal{D}, \gamma_D : [0, 1] \to \mathcal{D} \)) joining \( u_d(x) \) to \( u_d'(I_d^\epsilon(x)) \) (resp. \( u_s(x) \) to \( u_s'(I_s^\epsilon(x)) \), \( u_D(x) \) to \( u_D'(I_D^\epsilon(x)) \)).

We require that the vector \( (d\gamma_d/dt)(0) \) (resp. \( (d\gamma_s/dt)(0), (d\gamma_D/dt)(0) \)) is perpendicular to the image of \( u_d \) (resp. \( u_s, u_D \)) at \( t = 0 \).

We fix a unitary connection on \( T(X \setminus \mathcal{D}) \), whose restriction to \( \mathcal{U} \) is given by the direct sum of the trivial connection on \( \mathbb{C} \) and a unitary connection on \( T\mathcal{D} \). In particular, this connection is invariant with respect to the partial \( \mathbb{C}_s \)-action. The parallel transport along the geodesics \( \gamma_d, \gamma_s \) with respect to this unitary connection induces complex linear maps:

\[
T_{u_d(x)}X \to T_{u_d'(I_d^\epsilon(x))}X, \quad T_{u_s(x)}X \to T_{u_s'(I_s^\epsilon(x))}X.
\]

We thus obtain bundle maps:

\[
u_d^*TX \to (u_d' \circ I_d^\epsilon)^*TX, \quad u_s^*TX \to (u_s' \circ I_s^\epsilon)^*TX.
\]

By differentiating and projecting to the \((0, 1)\) part, we also obtain bundle maps:

\[
d^{0,1}I_d^\epsilon : \Lambda^{0,1} \to (I_d^\epsilon)^*\Lambda^{0,1}, \quad d^{0,1}I_s^\epsilon : \Lambda^{0,1} \to (I_s^\epsilon)^*\Lambda^{0,1}.
\]

We may assume that these are isomorphisms by choosing \( \epsilon \) to be small enough. Taking tensor product gives rise to the maps:

\[
u_d^*TX \otimes \Lambda^{0,1} \to (u_d' \circ I_d^\epsilon)^*TX \otimes (I_d^\epsilon)^*\Lambda^{0,1},
\]

\[
u_s^*TX \otimes \Lambda^{0,1} \to (u_s' \circ I_s^\epsilon)^*TX \otimes (I_s^\epsilon)^*\Lambda^{0,1}.
\]

which induce linear maps:

\[
\mathcal{P}\mathcal{A}\mathcal{L} : L_m^2(\text{Supp} E_d; u_d^*TX \otimes \Lambda^{0,1}) \to L_m^2(\text{Supp} E_d; (u_d')^*TX \otimes (I_d^\epsilon)^*\Lambda^{0,1}),
\]

\[
\mathcal{P}\mathcal{A}\mathcal{L} : L_m^2(\text{Supp} E_s; u_s^*TX \otimes \Lambda^{0,1}) \to L_m^2(\text{Supp} E_s; (u_s')^*TX \otimes (I_s^\epsilon)^*\Lambda^{0,1}).
\]

---

5Note that \( \Sigma_d, \Sigma_s, \Sigma_D \) together with marked points are stable without automorphisms. In a more general case, some of the irreducible components might have automorphisms. In that case the analogues of the map \( I_d^\epsilon \) are required to be equivariant with respect to the automorphisms of the source curve.

6Here the geodesics are defined with respect to the metric \( g \) on \( X \setminus \mathcal{D} \) and the metric \( g' \) on \( \mathcal{D} \).
We now define
\[(4.5)\quad E_d(u_d') = \mathcal{P}\mathcal{A}\mathcal{L}(E_d), \quad E_s(u_s') = \mathcal{P}\mathcal{A}\mathcal{L}(E_s).\]

**Definition 4.6.** We denote by \(U_d\) (resp. \(U_s\)) the set of \(L^2_{m+1,loc}\) maps \(u_d' : (\Sigma_d \setminus \{z_d\}, \partial \Sigma_d) \to (X \setminus \mathcal{D}, L)\) (resp. \(u_s' : \Sigma_s \setminus \{z_s\} \to X \setminus \mathcal{D}\)) with the following properties:

1. The \(C^2\)-distance between \(u_d\) and \(u_d'\) (resp. \(u_s\) and \(u_s'\)) is less than \(\epsilon\).
2. The equation
   \[\bar{\partial}u_d' \in E_d(u_d'),\quad \text{(resp. } \bar{\partial}u_s' \in E_s(u_s'))\]
   is satisfied.
3. There exists \(p \in \mathcal{D}\) such that
   \[\lim_{x \to z_d} u_d'(x) = p,\quad \text{(resp. } \lim_{x \to z_s} u_s'(x) = p)\]
4. In the latter case, \(u_s'(w_{s,1}) \in \mathcal{N}_{s,1}\) and \(u_s'(w_{s,2}) \in \mathcal{N}_{s,2}\).

We define \(U_d^+\) to be the set of maps \(u_d'\) satisfying (1), (2) and (3), but not necessarily (4).

Note that standard regularity results imply that elements of \(U_d\) and \(U_s\) are smooth.

In the same way as in the case of \(u_d', u_s'\), for \(u_D' : \Sigma_D \to \mathcal{D}\) with
\[d(u_D(x), u_D'(x)) \leq \epsilon,\]
we define:
\[(4.6)\quad \mathcal{P}\mathcal{A}\mathcal{L} : L^2_m(\text{Supp} E_D; u_D'^*T\mathcal{D} \otimes \Lambda^{0,1}) \to L^2_m(\Sigma_D; (u_D')^*T\mathcal{D} \otimes \Lambda^{0,1})\]
using the map \(I_D^1\) and parallel transport with respect to the chosen unitary connection on \(T\mathcal{D}\). We also define:
\[(4.7)\quad E_D(u_D') = \mathcal{P}\mathcal{A}\mathcal{L}(E_D).\]

**Remark 4.7.** Since \(\Sigma_d, \Sigma_s\) and \(\Sigma_D\) with the marked points are stable, we can use the identity map instead of \(I_d^1, I_s^1\) and \(I_D^1\). This is the approach used in [FOn, FOOO2] and many other places in the literature. We call our choice here the **target space parallel transportation**.\(^7\) This method works better for our construction of Subsection 10.3.

**Definition 4.8.** We denote by \(U_D\) the set of \(L^2_m\) maps \(u_D' : \Sigma_D \to \mathcal{D}\) with the following properties:

1. The \(C^2\)-distance between \(u_D\) and \(u_D'\) is less than \(\epsilon\).
2. The equation
   \[\bar{\partial}u_D' \in E_D(u_D')\]
   is satisfied.
3. \(u_D'(w_D) \in \mathcal{N}_D\).

We define \(U_D^+\) to be the set of maps \(u_D'\) satisfying (1) and (2), but not necessarily (3).

\(^7\)A similar method was used in [FOOO3, page 250, Condition 4.3.27].
We define maps:

\[ ev_d : U_d \rightarrow D, \quad ev_s : U_s \rightarrow D, \quad (ev_{D,d}, ev_{D,s}) : U_D \rightarrow D \times D, \]

by

\[
ev_d(u'_d) := u'_d(z_d), \quad ev_s(u'_s) := u'_s(z_s),
\]

\[
ev_d(u'_d) := u'_D(z_d), \quad ev_s(u'_D) := u'_D(z_s).
\]

We summarize their properties as follows.

**Lemma 4.9.** If \( \varepsilon \) is small enough, then we have:

1. \( U_d, U_D, U_s \) are smooth manifolds.
2. The maps \( ev_d, ev_{D,d}, ev_{D,s}, ev_s \) are smooth.
3. The fiber product

\[ U_d ev_D ev_{D,d} ev_{D,s} ev_s U_s \]

is transversal.

**Proof.** Part (1) is a consequence of the implicit function theorem using the assumptions in Definition 4.1. Part (2) follows from the way we set up Fredholm theory. Part (3) follows from the surjectivity of the map (4.1).

The fiber product (4.9) describes a Kuranishi neighborhood of any element \( [\Sigma, z_0, u] \) of the stratum of \( \mathcal{M}^{\text{RGW}}_1(L; \beta) \), consisting of objects with the combinatorial data given in Section 2. Next, we include the gluing construction and construct a Kuranishi neighborhood of \( [\Sigma, z_0, u] \) in the moduli space \( \mathcal{M}^{\text{RGW}}_1(L; \beta) \). Let \( D^2 \) be the unit disk in the complex plane and \( D^2(r) \) denote \( r \cdot D^2 \). We fix coordinate charts:

\[
\varphi_d : \text{Int}(D^2) \rightarrow \Sigma_d, \quad \varphi_{D,d} : \text{Int}(D^2) \rightarrow \Sigma_D,
\]

\[
\varphi_{D,s} : \text{Int}(D^2) \rightarrow \Sigma_D, \quad \varphi_s : \text{Int}(D^2) \rightarrow \Sigma_s,
\]

which are bi-holomorphic maps onto the image and \( \varphi_d(0) = z_d, \varphi_{D,d}(0) = z_{D,d}, \varphi_{D,s}(0) = z_{D,s}, \varphi_s(0) = z_s \). We assume that the marked points \( w_D, w_{s,i} \) do not belong to the image of the above coordinate charts. For \( \sigma_1, \sigma_2 \in D^2 \setminus \{0\} \), we form the disk \( \Sigma(\sigma_1, \sigma_2) \) as follows. Consider the disjoint union:

\[
\begin{align*}
(\Sigma_d \setminus \varphi_d(D^2(|\sigma_1|))) & \cup (\Sigma_D \setminus (\varphi_{D,d}(D^2(|\sigma_1|)) \cup \varphi_{D,s}(D^2(|\sigma_2|)))) \\
& \cup (\Sigma_s \setminus \varphi_s(D^2(|\sigma_2|))).
\end{align*}
\]

and define the equivalence relation \( \sim \) on (4.12) as follows:

- (gl-i) If \( z_1z_2 = \sigma_1, z_1, z_2 \in D^2 \), then \( \varphi_d(z_1) \sim \varphi_{D,d}(z_2) \).
- (gl-ii) If \( z_1z_2 = \sigma_2, z_1, z_2 \in D^2 \), then \( \varphi_s(z_1) \sim \varphi_{D,s}(z_2) \).

Then \( \Sigma(\sigma_1, \sigma_2) \) is the quotient space of (4.12) by this equivalence relation. See Figure 3 below. The above definition can be extended to the case that \( \sigma_1 \) or \( \sigma_2 \) vanishes. For example, if \( \sigma_2 = 0 \), then (4.12) is replaced with:

\[
(\Sigma_d \setminus \varphi_d(D^2(|\sigma_1|))) \cup (\Sigma_D \setminus \varphi_{D,d}(D^2(|\sigma_1|))) \cup \Sigma_s.
\]

where we use the identification in (gl-i), and the identification in (gl-ii) is replaced with \( \varphi_s(0) \sim \varphi_{D,s}(0) \).
By construction, there exist bi-holomorphic embeddings:

\[ u_0 : \Sigma_d \rightarrow \Sigma(\sigma_1, \sigma_2), \quad I_0 : \Sigma_s(\sigma_2) \rightarrow \Sigma(\sigma_1, \sigma_2). \]

(4.14)

Let \( u'_d : \Sigma_d(\sigma_1) \rightarrow X \setminus D, u'_s : \Sigma_s(\sigma_2) \rightarrow X \setminus D, U'_D : \Sigma_D(\sigma_1, \sigma_2) \rightarrow \mathcal{U} \subset X \setminus D \) be \( L^2_{m+1} \) maps such that \( u'_d, u'_s, u'_D := \pi \circ U'_D \) are close to the restrictions of \( u_d, u_s, u_D \) in the same sense as in (4.2). We define:

\[ E_d(u'_d) \subset L^2_m(\Sigma_d(\sigma_1); (u'_d)^*TX \otimes \Lambda^{0,1}) \]

(4.15)

\[ E_s(u'_s) \subset L^2_m(\Sigma_s(\sigma_2); (u'_s)^*TX \otimes \Lambda^{0,1}) \]

similar to (4.5), using target space parallel transportations. Next, we define:

\[ E_D(U'_D) \subset L^2_m(\Sigma_D(\sigma_1, \sigma_2); (U'_D)^*TX \otimes \Lambda^{0,1}) \]

(4.16)

as follows. Since \( u'_D \) is close to \( u_D \), we can use the same construction as in (4.7) to define:

\[ E_D(u'_D) \subset L^2_m(\Sigma_D; (u'_D)^*TD \otimes \Lambda^{0,1}) \]

Then the decomposition in (3.4) allows us to define:

\[ E_D(U'_D) \subset L^2_m(\Sigma_D; (U'_D)^*TX \otimes \Lambda^{0,1}). \]

(4.17)

By construction, we have isomorphisms

\[ \mathcal{P}_d : E_d(u'_d) \rightarrow E_d, \quad \mathcal{P}_s : E_s(u'_s) \rightarrow E_s, \quad \mathcal{P}_D : E_D(u'_D) \rightarrow E_D. \]

(4.18)

Recall that we fixed a codimension 2 submanifold \( \mathcal{N}_D \subset D \). We define \( \hat{\mathcal{N}}_D \subset X \) to be its inverse image in the tubular neighborhood \( \mathcal{U} \) of \( D \) in \( X \) by the projection map \( \pi \). In the following definition \( \epsilon \) is the same constant as in Lemma 4.9. We may make this constant smaller as we move through the paper whenever it is necessary.
Definition 4.10. We denote by $U_0$ the set of all triples $(u', \sigma_1, \sigma_2)$ where $\sigma_1, \sigma_2 \in D^2(\epsilon)$. In the case that $\sigma_1$ and $\sigma_2$ are non-zero, $(u', \sigma_1, \sigma_2)$ needs to satisfy the following properties:

1. $u' : \Sigma(\sigma_1, \sigma_2) \to X \setminus D$ is a smooth map.
2. Let:
   $$u'_d := u' \circ I_d, \quad u'_s := u' \circ I_s, \quad U'_D = u' \circ I_D.$$
   Then the $C^2$ distance of $u'_d$ (resp. $u'_s$) with the restriction of $u_d$ (resp. $u_s$) to $\Sigma_d(\sigma_1)$ (resp. $\Sigma_s(\sigma_2)$) is less than $\epsilon$. The maps $U'_D$ and $U_D$ are also $C^2$-close to each other in the sense that the image of $U'_D$ is contained in the open set $U$ and there is a constant $r$ such that the $C^2$ distance of $U'_D$ and $Dil_r \circ U_D$, restricted to $\Sigma_D(\sigma_1, \sigma_2)$, is less than $\epsilon$.

3. (Modified non-linear Cauchy-Riemann equation) $u'_d, u'_s, U'_D$ satisfy the equations:
   $$(4.19) \quad \overline{\partial} u'_d \in E_d(u'_d), \quad \overline{\partial} u'_s \in E_s(u'_s), \quad \overline{\partial} U'_D \in E_D(U'_D).$$
4. (Transversal constraints) We also require:
   $$(4.20) \quad u'(w_D) \in \widehat{N}_D, \quad u'(w_{s,1}) \in N_{s,1}, \quad u'(w_{s,2}) \in N_{s,2}.$$

Here we use $I_D, I_s$ to regard $w_D, w_{s,i}$ as elements of $\Sigma(\sigma_1, \sigma_2)$. In the case that one of the constants $\sigma_1$ and $\sigma_2$ vanishes, the other one is also zero, and $u'$ is an element of the fiber product (4.9).

One might hope that the space $U_0$ is cut down transversely by (4.19) and (4.20), and hence it could be used to define a Kuranishi neighborhood of $[\Sigma, z_0, u]$ in $\mathcal{M}_1^{\text{RGW}}(L; \beta)$. However, this naive expectation does not hold. Roughly speaking, if that would hold, then one should obtain a solution for any element of the fiber product (4.9) close to $[\Sigma, z_0, u]$ and any small values of $\sigma_1, \sigma_2$. On the other hand, as a consequence of [DF2, Proposition 3.63], the stratum in (4.9) has real codimension 2 in our case, which is a contradiction. Note that this is in contrast with the stable map compactification, where a fiber product of the form (4.9) has codimension 4. To resolve this issue, we introduce a space $U$ larger than $U_0$ such that $U$ is a smooth manifold and $U_0$ is cut out from $U$ by an equation of the following form:

$$(4.21) \quad \sigma_1^2 = c \sigma_2^3.$$

The space $U$ is realized as the moduli space of \textit{inconsistent solutions}, which will be defined in the next section. Note that the set of solutions of (4.21) has a singularity at the locus $\sigma_1 = \sigma_2 = 0$.

---

8 The $C^2$ distance in part (2) of the definition are defined with respect to the metric $g$ on $X \setminus D$ and the metric on $N_D(X) \setminus D$ which has the form in (3.2).
5. INCONSISTENT SOLUTIONS AND THE MAIN ANALYTICAL RESULT

In this section, we discuss the main step where the construction of the Kuranishi chart in our situation is different from the case of the stable map compactification.

**Definition 5.1.** For \( \sigma_1, \sigma_2 \in D^2(\epsilon) \), an **inconsistent solution** is a 7-tuple \((u'_d, u'_s, U'_D, \sigma_1, \sigma_2, \rho_1, \rho_2)\) satisfying the following properties:

1. \( u'_d : \Sigma_D(\sigma_1) \rightarrow X \setminus D, \ u'_s : \Sigma_D(\sigma_2) \rightarrow X \setminus D, \ U'_D : \Sigma_D(\sigma_1, \sigma_2) \rightarrow N_D(X) \setminus D \). The \( C^2 \) distances of \( u'_d, u'_s \) and \( U'_D \) with \( u_d, u_s \) and \( U_D \) are less than \( \epsilon \).

2. The following equations are satisfied:

\[
\begin{align*}
\partial u'_d & \in E_d(u'_d), \quad \partial u'_s \in E_s(u'_s), \quad \partial U'_D \in E_D(U'_D). \\
\end{align*}
\]

Here \( E_d(u'_d) \), \( E_d(u'_s) \) and \( E_d(U'_D) \) are defined as in (4.15) and (4.17) using target parallel transport.

3. We require the following transversal constraints:

\[
\begin{align*}
\pi \circ U'_D(w_D) & \in N_D, \quad u'_s(w_{s,1}) \in N_{s,1}, \quad u'_s(w_{s,2}) \in N_{s,2}. \\
\end{align*}
\]

4. Let \( z_1, z_2 \in D^2 \).

(a) If \( z_1 z_2 = \sigma_1 \), then:

\( u'_d(\varphi_d(z_1)) = (\text{Dil}_{\rho_1} \circ U'_D)(\varphi_{D,1}(z_2)). \)

In particular, we assume that the left hand side is contained in the open neighborhood \( \mathcal{U} \) of \( D \).

(b) If \( z_1 z_2 = \sigma_2 \), then:

\( u'_s(\varphi_s(z_1)) = (\text{Dil}_{\rho_2} \circ U'_D)(\varphi_{D,2}(z_2)). \)

We say two inconsistent solutions \((u_s^{(j)}, u_d^{(j)}, U_D^{(j)}, \sigma_1^{(j)}, \sigma_2^{(j)}, \rho_1^{(j)}, \rho_2^{(j)}), \ j = 1, 2, \) are **equivalent** if the following holds:

(i) \( u_d^{(1)} = u_d^{(2)}, \ u_s^{(1)} = u_s^{(2)}, \ \sigma_1^{(1)} = \sigma_1^{(2)}, \ \sigma_2^{(1)} = \sigma_2^{(2)}. \)

(ii) There exists a nonzero complex number \( c \) such that:

\( U_D^{(2)} = \text{Dil}_{1/c} \circ U_D^{(1)}, \ \rho_1^{(2)} = c \rho_1^{(1)}, \ \rho_2^{(2)} = c \rho_2^{(1)}. \)

We will write \( \mathcal{U} \) for the set of all equivalence classes of inconsistent solutions.

**Remark 5.2.** In the above definition, we include the case that \( \sigma_1 \) or \( \sigma_2 \) is 0 in the following way:

1. If \( \sigma_1 = 0 \) (resp. \( \sigma_2 = 0 \)), then the condition (4) (a) (resp. (b)) is replaced by the condition that \( u'_d(\varphi_d(0)) = \pi \circ U'_D(\varphi_{D,1}(0)) \) (resp. \( u'_s(\varphi_s(0)) = \pi \circ U'_D(\varphi_{D,1}(0)) \)).

2. If \( \sigma_1 = 0 \) (resp. \( \sigma_2 = 0 \)), then \( \rho_1 = 0 \) (resp. \( \rho_2 = 0 \)).

In the case that exactly one of \( \sigma_1 \) and \( \sigma_2 \) is zero, the source curve \( \Sigma(\sigma_1, \sigma_2) \) has only one node. Such source curves do not appear in \( \mathcal{U}_0 \). However, there are elements of this form in \( \mathcal{U} \).

\(^9\)Here we use the same convention as in Definition 4.10 to defined the \( C^2 \) -distances.
Below we state our main analytic results about $U$:

**Proposition 5.3.** If $\epsilon$ is small enough, then the moduli space $U$ is a smooth manifold diffeomorphic to\(^{10}\):

\[ (U_{\text{ev}d} \times_{\text{ev}_{D,s}} U_D \times_{\text{ev}_s} U_s) \times D^2(\epsilon) \times D^2(\epsilon). \]

The diffeomorphism has the following properties:

1. This diffeomorphism identifies the projection to the factor $D^2(\epsilon) \times D^2(\epsilon)$ with:

\[ [u'_h, u'_d, D, \sigma_1, \rho_1, \rho_2] \mapsto (\sigma_1, \sigma_2). \]

2. There exists $\hat{\rho}_1 : U \to \mathbb{C}$ such that any element $q$ of $U$ has a representative whose $\rho_i$ component is equal to $\hat{\rho}_i(q)$. The functions $\hat{\rho}_i$ are smooth. Moreover, there exists a homeomorphism:

\[ U_0 \cong \{ \eta \in U \mid \hat{\rho}_1(\eta) = \hat{\rho}_2(\eta) \}. \]

This homeomorphism is given as follows. Let:

\[ \eta = [u'_h, u'_d, u'_s, \sigma_1, \sigma_2, \hat{\rho}_1, \hat{\rho}_2] \]

be an element of the right hand side of (5.4) with $c = \hat{\rho}_1 = \hat{\rho}_2$. Then we can glue the three maps $u'_d, u'_s, \text{Dil}_c \circ U'_D$ as in (gl-i),(gl-ii) to obtain a map $\eta' : \Sigma(\sigma_1, \sigma_2) \to X$. This gives the desired element of the left hand side of (5.4).

**Remark 5.4.** We can take our diffeomorphism so that its restriction to $U_{\text{ev}d} \times_{\text{ev}_{D,s}} U_D \times_{\text{ev}_s} U_s \times \{(0,0)\}$ is the obvious one. We can also specify the choice of $\hat{\rho}_i$ in (2) above by requiring

\[ \hat{\rho}_1 = \sigma_1^2. \]

From now on, we will take this choice unless otherwise mentioned explicitly. The proof we will give implies that:

\[ \hat{\rho}_2(\eta) = f(\eta)\sigma_2^3, \]

where $f$ is a nonzero smooth function.

The next proposition is the exponential decay estimate similar to those in the case of the stable map compactification. (See [FOOO5] for the detail of the proof of this exponential decay estimate in the case of the stable map compactification.) To state our exponential decay estimate, we need to introduce some notations. We define $T_i \in [0, \infty)$, $\theta_i \in {\mathbb{R}}/{2\pi\sqrt{-1}\mathbb{Z}}$ by the formula:

\[ \sigma_i = \exp(-(T_i + \sqrt{-1}\theta_i)). \]

The exponential decay estimate is stated in terms of $T_i$ and $\theta_i$.

---

\(^{10}\)See Remark 5.6 for the definition of the smooth structure of $D^2(\epsilon)$. 
We define a homeomorphism from a neighborhood of the origin in $D^2$ as follows:

\[ z = \exp(-(T + \sqrt{-1}\theta)). \]

We define a smooth structure on $D^2$ different from the standard one as follows. For $z \in D^2$, let $T, \theta$ be defined by the following identity:

\[ z = \exp(-(T + \sqrt{-1}\theta)). \]

We define a smooth structure on $D$, temporarily denoted by $D_{\text{new}}$, such that $z \mapsto w$ becomes a diffeomorphism from $D_{\text{new}}$ to $D^2$ with the standard...
smooth structure. This new smooth structure \( D^2_{\text{new}} \) is used to define a smooth structure on the factors \( D^2 \) in (5.3). (We drop the term ‘new’ from \( D^2_{\text{new}} \) hereafter.) The Proposition 5.5 implies smoothness of various maps at the origin of \( D^2 \) with respect to the new smooth structure. See for example [FOOO4, Lemma 22.6], [FOOO5, Subsection 8.2], [FOOO8, Section 10] for further discussions related to this point.

6. Kuranishi Charts: a Special Case

In this section we use Propositions 5.3 and 5.5 to obtain a Kuranishi chart at the point \([\Sigma, z_0, u] \in \mathcal{M}_1^{\text{RGW}}(L; \beta)\). By definition, a Kuranishi chart of a point \( p \) in a space \( M \) consists of \((V_p, \Gamma_p, \mathcal{E}_p, s_p, \psi_p)\) where \( V_p \), the Kuranishi neighborhood, is a smooth manifold containing a distinguished point \( \tilde{p} \), \( \Gamma_p \), the isotropy group, is a finite group acting on \( V_p \), \( \mathcal{E}_p \), the obstruction bundle is a vector bundle over \( V_p \) and \( s_p \), the Kuranishi map, is a section of \( \mathcal{E}_p \) over \( V \). Moreover, the action of \( \Gamma_p \) at \( \tilde{p} \) is trivial and the action of this group on \( V_p \) is lifted to \( \mathcal{E}_p \). The section \( s_p \) is \( \Gamma_p \)-equivariant and vanishes at \( \tilde{p} \). Finally, \( \psi_p \) is a homeomorphism from \( s_p^{-1}(0)/\Gamma_p \) to a neighborhood of \([\Sigma, z_0, u] \) in \( \mathcal{M}_1^{\text{RGW}}(L; \beta) \), which maps \( \tilde{p} \) to \( p \).

In the present case, we define the Kuranishi neighborhood to be the manifold \( \mathcal{U} \) in Proposition 5.3, and define the isotropy group to be the trivial one. The obstruction bundle \( \mathcal{E} \) on \( \mathcal{U} \) is a trivial bundle whose fiber is \( E_d \oplus E_D \oplus E_s \oplus \mathbb{C} \).

The Kuranishi map
\[
s = (s_d, s_D, s_s, s_\rho) : \mathcal{U} \to E_d \oplus E_D \oplus E_s \oplus \mathbb{C}
\]
is defined by
\[
\begin{align*}
s_d(\xi, T_1, T_2, \theta_1, \theta_2) &= \mathcal{P}_d(\overline{\partial}u_D^\xi(T_1, T_2, \theta_1, \theta_2; \cdot)) \\
s_p(\xi, T_1, T_2, \theta_1, \theta_2) &= \mathcal{P}(\overline{\partial}u_D^\xi(T_1, T_2, \theta_1, \theta_2; \cdot)) \\
s_s(\xi, T_1, T_2, \theta_1, \theta_2) &= \mathcal{P}_s(\overline{\partial}u_s^\xi(T_1, T_2, \theta_1, \theta_2; \cdot)) \\
s_\rho(\xi, T_1, T_2, \theta_1, \theta_2) &= \sigma_1^2 - \rho_2(\xi, T_1, T_2, \theta_1, \theta_2).
\end{align*}
\]
(6.2)

Here \( \mathcal{P}_d, \mathcal{P}, \mathcal{P}_s \) are as in (4.18). The maps \( u_D^\xi, u_s^\xi, u_\rho^\xi \) are as in (5.7). Therefore, \( \overline{\partial}u_s^\xi(T_1, T_2, \theta_1, \theta_2; \cdot) \in E_s(u_s^\xi(T_1, T_2, \theta_1, \theta_2; \cdot)) \) is a consequence of (4.19). Since \( E_s(u_s^\xi(T_1, T_2, \theta_1, \theta_2; \cdot)) \) is in the domain of \( \mathcal{P}_s \), the first map is well-defined. Similarly, we can show that the second and the third maps are also well-defined. The last map is equivalent to \( \hat{\rho}_1 - \hat{\rho}_2 \), because of (5.5).

Lemma 6.1. The map \( s_\rho \) is smooth.

Proof. Proposition 5.3 implies that \( s_\rho \) is smooth. Smoothness of the maps \( s_s, s_d, s_p \) for non-zero values of \( \sigma_1 \) and \( \sigma_2 \) is a consequence of standard elliptic regularity. Smoothness for \( \sigma_1 = 0 \) follows from part (1) of Proposition 5.5. For similar results in the context of the stable map compactification, see [FOOO4, Lemma 22.6], [FOOO5, Theorem 8.25], [FOOO8, Proposition
We finally construct the parametrization map

\[ \psi : s^{-1}(0) \to M_{1}^{\text{RGW}}(L; \beta). \]

Let \( \mathfrak{x} = [u_d', u_D', u_s', \sigma_1, \sigma_2, \rho_1, \rho_2] \in \mathcal{U} \) be an element such that \( s(\mathfrak{x}) = 0 \). Firstly, let \( \sigma_1 \) and \( \sigma_2 \) be both non-zero. Equation \( s_{\mathfrak{x}}(\mathfrak{x}) = 0 \) implies that \( \rho_1 = \rho_2 \). Therefore, we can glue \( u_d', u_D', u_s' \), as in Proposition 5.3 (2), to obtain \( u' : \Sigma(\sigma_1, \sigma_2) \to X \setminus D \). We use \( s_{\mathfrak{x}}(\mathfrak{x}) = s_{\mathfrak{x}}(\mathfrak{x}) = s_{\mathfrak{x}}(\mathfrak{x}) = 0 \) to conclude that \( u' \) is \( J \)-holomorphic. We define \( \psi(\mathfrak{x}) \in M_{1}^{\text{RGW}}(L; \beta) \) to be the element determined by \( u' \) and \( z_0 \in \partial D \subset \partial \Sigma(\sigma_1, \sigma_2) \). In the case that \( \sigma_1 = 0 \), \( \rho_1 \) vanishes by definition. Equation \( s(\mathfrak{x}) = 0 \) implies that \( \rho_2 \) is also zero. We can also conclude from Definition 5.1 that \( \sigma_2 = 0 \). Finally the first three equations in (6.2) imply that \( \mathfrak{x} \) determines an element of \( M_{1}^{\text{RGW}}(L; \beta) \) in the stratum described in Section 2. The case that \( \sigma_2 = 0 \) can be treated similarly. It is easy to see that \( \psi \) is a homeomorphism to a neighborhood of \([\Sigma, z_0, u]\) in \( M_{1}^{\text{RGW}}(L; \beta) \). Given Propositions 5.3 and 5.5, we thus proved the following result:

**Proposition 6.2.** \( (\mathcal{U}, \mathcal{E}, s, \psi) \) provides a Kuranshi chart for the moduli space \( M_{1}^{\text{RGW}}(L; \beta) \) at \([\Sigma, z_0, u]\).

7. **Proof of the Main Analytical Result**

The purpose of this section is to prove Proposition 5.3. The proofs are similar to the arguments in [FOOO5]. However, there is one novel point, which is related to the fact that we need the notion of inconsistent solutions. In this section, we go through the construction of the required family of inconsistent solutions, emphasizing on this novel point. Then the estimates claimed in Proposition 5.5 can be proved in the same way as in [FOOO5, Section 6].

Throughout this section, we use a different convention for our figures to sketch pseudo holomorphic curves in \( X \). In our figures in this section (e.g. Figure 4), we regard the divisor \( D \) as a vertical line on the right. This is in contrast with our convention in Figure 3 and [DF2], where we regard the divisor as a horizontal line on the bottom. Our new convention is more consistent with the previous literature, especially [FOOO5].

**7.1. Cylindrical Coordinates.** In (4.10), we fix coordinate charts on \( \Sigma_d \), \( \Sigma_s \), \( \Sigma_D \) near the nodal points and parametrized by the disc \( \text{Int}(D^2) \). In this section, it is convenient to use a cylindrical coordinates on the domain of these coordinate charts. Thus we modify the definition of the maps in (4.10) as follows:

\[
\begin{align*}
\varphi_d : [0, \infty) \times S^1 & \to \Sigma_d, \\
\varphi_s : [0, \infty) \times S^1 & \to \Sigma_s, \\
\varphi_{D,d} : (-\infty, 0] \times S^1 & \to \Sigma_D, \\
\varphi_{D,s} : (-\infty, 0] \times S^1 & \to \Sigma_D,
\end{align*}
\]
where
\[ \varphi_d(r'_1, s'_1), \quad \varphi_{D,d}(r''_1, s''_1), \]
\[ \varphi_s(r'_2, s'_2), \quad \varphi_{D,s}(r''_2, s''_2), \]
for \((r'_i, s'_i) \in [0, \infty) \times S^1, (r''_i, s''_i) \in (-\infty, 0) \times S^1,\)
is defined to be what we denoted by
\[ \varphi_d(\exp(-(r'_1 + i s'_1))), \quad \varphi_{D,d}(\exp(r''_1 + i s''_1)), \]
\[ \varphi_s(\exp(-(r'_2 + i s'_2))), \quad \varphi_{D,s}(\exp(r''_2 + i s''_2)), \]
in Section 4.

The equations
\[ z_1z_2 = \sigma_1 \quad \text{or} \quad z_1z_2 = \sigma_2 \]
appearing in (gl-i) and (gl-ii)\(^{11}\) can be rewritten as:
\[ r''_1 = r'_1 - 10T_1, \quad s''_1 = s'_1 - \theta_1, \]
\[ r''_2 = r'_2 - 10T_2, \quad s''_2 = s'_2 - \theta_2, \]
where\(^{12}\)
\[ \sigma_i = \exp(-(10T_i + \sqrt{-1}\theta_i)). \]

We define
\[ r_i = r'_i - 5T_i = r''_i + 5T_i, \quad s_i = s'_i - \theta_i/2 = s''_i + \theta_i/2. \]

We also slightly change our convention for the polar coordinate of \(\rho_i\) of Definition 5.1 \((i = 1, 2)\) and define \(\mathfrak{R}_i, \eta_i\) as follows:
\[ \rho_i = \exp(-(10\mathfrak{R}_i + \sqrt{-1}\eta_i)). \]

See Figure 4 below and compare with [FOOO5, (6.2) and (6.3)]\(^{13}\).

---

\(11\) See the discussion about the construction of \(\Sigma(\sigma_1, \sigma_2)\) around (4.12).

\(12\) We use the coefficient 10 here to be consistent with [FOOO5]. Otherwise, they are not essential.

\(13\) In [FOOO5], the letter \(\tau\) is used for the variables that we denote by \(r_i\) here. In this paper, we use \(\tau\) to denote the \(\mathbb{R}\) factor appearing in the target space.
7.2. **Bump Functions.** For the purpose of constructing approximate solutions (pre-gluing) and for each step of the Newton’s iteration used to solve our variant of non-linear Cauchy-Riemann equation, we use bump functions. Here we review various bump functions that we need. We may use the maps \( \varphi \) of \( \Sigma(\sigma_1, \sigma_2) \):

\[
X_{i,T_i} = [1, 1]_{r_i} \times S_{s_i}^1 = [5T_i - 1, 5T_i + 1]_{r'_i} \times S_{s'_i}^1 = [-5T_i - 1, -5T_i + 1]_{r''_i} \times S_{s''_i}^1,
\]

\[
A_{i,T_i} = [-T_i - 1, -T_i + 1]_{r_i} \times S_{s_i}^1 = [4T_i - 1, 4T_i + 1]_{r'_i} \times S_{s'_i}^1 = [-6T_i - 1, -6T_i + 1]_{r''_i} \times S_{s''_i}^1,
\]

\[
B_{i,T_i} = [T_i - 1, T_i + 1]_{r_i} \times S_{s_i}^1 = [6T_i - 1, 6T_i + 1]_{r'_i} \times S_{s'_i}^1 = [-4T_i - 1, -4T_i + 1]_{r''_i} \times S_{s''_i}^1.
\]

Using \( \varphi_d \) (resp. \( \varphi_s \)), the spaces \( X_{1,T_1}, A_{1,T_1}, B_{1,T_1} \) (resp. \( X_{2,T_2}, A_{2,T_2}, B_{2,T_2} \)) can be identified with subspaces of \( \Sigma_d \setminus \{z_d\} \) (resp. \( \Sigma_s \setminus \{z_s\} \)). Similarly, the map \( \varphi_{D,d} \) (resp \( \varphi_{D,s} \)) allows us to regard \( X_{1,T_1}, A_{1,T_1}, B_{1,T_1}, X_{2,T_2}, A_{2,T_2}, B_{2,T_2} \) as subspaces of \( \Sigma_D \setminus \{z_d, z_s\} \). (See Figure 5 below.)

![Figure 5](image)

**Figure 5.** \( X_{i,T_i}, A_{i,T_i}, B_{i,T_i} \)

We fix a non-increasing smooth function \( \chi : \mathbb{R} \to [0, 1] \) such that

\[
\chi(r) = \begin{cases} 
1 & r < -1 \\
0 & 1 < r,
\end{cases}
\]

and \( \chi(-r) = 1 - \chi(r) \). We now define

\[
\begin{aligned}
\chi_{i,X}(r_i, s_i) &= \chi(r_i), & \chi_{i,X}(r_i, s_i) &= \chi(-r_i), \\
\chi_{i,A}(r_i, s_i) &= \chi(r_i + T_i), & \chi_{i,A}(r_i, s_i) &= \chi(-(r_i + T_i)), \\
\chi_{i,B}(r_i, s_i) &= \chi(r_i - T_i), & \chi_{i,B}(r_i, s_i) &= \chi(-(r_i - T_i)).
\end{aligned}
\]
The functions $\chi_{1,X}^{-}, \chi_{1,A}^{-}$ and $\chi_{1,B}^{-}$ can be extended to smooth functions on $\Sigma_d$ which are locally constant outside of the spaces $X_{1,T_1}$, $A_{1,T_1}$ and $B_{1,T_1}$, respectively. We use the same notations to denote these extensions. Similarly, we can define functions $\chi_{2,X}^{-}, \chi_{2,A}^{-}$ and $\chi_{2,B}^{-}$ on $\Sigma_s$. These functions can be also regarded as functions defined on $\Sigma(\sigma_1, \sigma_2)$ in the obvious way.

We use $\chi_i^{-}$ (resp. $\chi_i^{+}$ and $\chi_i$), for $i = 1, 2$, to define a smooth function $\chi_i^{-}$ (resp. $\chi_i^{+}$ and $\chi_i$) on $\Sigma(\sigma_1, \sigma_2)$ as follows. On the neck regions where the coordinate $(r_i, s_i)$, for $i = 1, 2$, is defined, we set $\chi_i^{-}$ (resp. $\chi_i^{+}$, $\chi_i$) to be the function $\chi_i^{-}(r_i, s_i)$ (resp. $\chi_i^{+}(r_i, s_i)$ and $\chi_i(r_i, s_i)$) given in (7.5). This function is defined to be locally constant on the complement of the above space. See Figures 6 and 7.

![Figure 6. $\chi_{2,X}^{-}, \chi_{2,A}^{-}, \chi_{2,B}^{-}$](image)

Note that the supports of the first derivatives of $\chi_i^{-}, \chi_i^{+}, \chi_i$ are subsets of $X_{i,T_i}, A_{i,T_i}, B_{i,T_i}$, respectively. The supports of the first derivatives of $\chi_i^{-}$, $\chi_i^{+}$, $\chi_i$ are subsets of $X_{1,T_1} \cup X_{2,T_2}$, $A_{1,T_1} \cup A_{2,T_2}$, $B_{1,T_1} \cup B_{2,T_2}$, respectively.

### 7.3 Weighted Sobolev Norms

In Section 3, we define weighted Sobolev norms on several function spaces on $\Sigma_d$, $\Sigma_s$, $\Sigma_D$. Here we use weighted Sobolev norms to define a function space on $\Sigma(\sigma_1, \sigma_2)$. Since $\Sigma(\sigma_1, \sigma_2)$ is compact and the weight functions that we will define are smooth, the resulting weighted Sobolev norm is equivalent to the usual Sobolev norm. In other words, the ratio between the two norms is bounded as long as we fix $\sigma_1, \sigma_2$. However, the ratio depends on $\sigma_1, \sigma_2$ and is unbounded as $\sigma_1, \sigma_2$ go to zero. Therefore, using weighted Sobolev norm is crucial to show that various estimates are independent of $\sigma_1, \sigma_2$.

We decompose $\Sigma(\sigma_1, \sigma_2)$ as follows:

$$
\Sigma(\sigma_1, \sigma_2) = \left( \Sigma_d \setminus \text{Im} \varphi_d \right) \cup \left( \Sigma_s \setminus \text{Im} \varphi_s \right) \cup \left( \Sigma_D \setminus (\text{Im} \varphi_{D,d} \cup \text{Im} \varphi_{D,s}) \right) \\
\cup \left( [-5T_1, 5T_1]_{r_1} \times S_{s_1}^1 \right) \cup \left( [-5T_2, 5T_2]_{r_2} \times S_{s_2}^1 \right).
$$
Here we identify $[-5T_1, 5T_1]_{r_1} \times S^1_{s_1}$ and $[-5T_2, 5T_2]_{r_2} \times S^1_{s_2}$ with their images in $\Sigma(\sigma_1, \sigma_2)$. We also introduce the following notations for various subspaces of $\Sigma(\sigma_1, \sigma_2)$: (See Figures 8, 9 and 10.)

\begin{align}
\Sigma_{\varphi_d}^+(\sigma_1, \sigma_2) &= \Sigma_d \setminus \text{Im}\varphi_d, \\
\Sigma_{\varphi_d}^- (\sigma_1, \sigma_2) &= \Sigma_d \setminus \varphi_d(D^2(\sigma_1)), \\
\Sigma_{\varphi_s}^+(\sigma_1, \sigma_2) &= \Sigma_s \setminus \text{Im}\varphi_s, \\
\Sigma_{\varphi_s}^- (\sigma_1, \sigma_2) &= \Sigma_s \setminus \varphi_s(D^2(\sigma_2)), \\
\Sigma_D^- (\sigma_1, \sigma_2) &= \Sigma_D \setminus (\text{Im}\varphi_{D,d} \cup \text{Im}\varphi_{D,s}), \\
\Sigma_D^+ (\sigma_1, \sigma_2) &= \Sigma_D \setminus (\varphi_{D,d}((\sigma_1) \cup \text{Im}\varphi_{D,s}(\sigma_2))).
\end{align}

(7.6)

Note that $\Sigma_{\varphi_d}^+(\sigma_1, \sigma_2)$, $\Sigma_{\varphi_s}^+(\sigma_1, \sigma_2)$ and $\Sigma_D^+(\sigma_1, \sigma_2)$ are respectively equal to the spaces $\Sigma_d(\sigma_1)$, $\Sigma_s(\sigma_2)$ and $\Sigma_D(\sigma_1, \sigma_2)$ defined in (4.13).

**Definition 7.1.** Let $e^\sigma_{\delta_1, \delta_2} : \Sigma(\sigma_1, \sigma_2) \to [0, \infty)$ be a smooth function satisfying the following properties (see Figure 11):

(i) If $x \in \Sigma_{\varphi_d}^- (\sigma_1, \sigma_2) \cup \Sigma_{\varphi_s}^- (\sigma_1, \sigma_2) \cup \Sigma_D^- (\sigma_1, \sigma_2)$, then $e^\sigma_{\delta_1, \delta_2} (x) = 1$;
(ii) If $r_i \in [1 - 5T_i, -1]$, then $e^\sigma_{\delta_1, \delta_2} (r_i, s_i) = e^{\delta(r_i + 5T_i)}$;
(iii) If $r_i \in [1, 5T_i - 1]$, then $e^\sigma_{\delta_1, \delta_2} (r_i, s_i) = e^{\delta(-r_i + 5T_i)}$. 

---

**Figure 7.** $\chi^\sigma_A$, $\chi^\sigma_A$, $\chi^\sigma_B$
(iv) If $|r_i| - 5T_i| \leq 1$, then $e_{\delta}^{\sigma_1,\sigma_2}(r_i, s_i) \in [1, 10]$;
(v) If $|r_i| \leq 1$, then $e_{\delta}^{\sigma_1,\sigma_2}(r_i, s_i) \in [e^{5T_1\delta}, 10e^{5T_2\delta}]$.

We fix a smooth map $u' : \Sigma(\sigma_1, \sigma_2) \to X \setminus D$ and assume that the diameters of:

\begin{equation}
(7.7) \quad u'([-5T_1, 5T_1]r_1 \times S^1_{S_1}) \quad \text{and} \quad u'([-5T_2, 5T_2]r_2 \times S^1_{S_2})
\end{equation}

with respect to the metric $g$ are less than a given positive real number $\kappa$. We require that the above sets are contained in $U$, introduced in the beginning of Section 2, where the partial $\mathbb{C}^*$-action is defined. Assuming $\kappa$ is small enough, to any:

$$V \in L^2_m(\Sigma(\sigma_1, \sigma_2); u'^*TX).$$
we associate sections \( \hat{v}_1 \) and \( \hat{v}_2 \) of \( u'^*TX \) over the subspaces \([-5T_1, 5T_1]_{r_1} \times S^1_{s_1} \) and \([-5T_2, 5T_2]_{r_2} \times S^1_{s_2} \) in the following way.

Let \((0, 0)_i \in [-5T_1, 5T_1]_{r_1} \times S^1_{s_1} \) be the point whose \( r_i, s_i \) coordinates are 0. By choosing \( m \) to be greater than 1, the following vector is well-defined:

\begin{equation}
(7.8) \quad v_i = V((0, 0)_i) \in T_{u'((0,0)_i)}X.
\end{equation}

Suppose \( v_i = v_{i,R} + v_{i,S1} + v_{i,D} \) is the decomposition of this vector with respect to (3.4). If \( \kappa \) is small enough, we can assume that the distance between any two points of the projection of (7.7) to \( D \) is less than the injectivity radius of \( D \). In particular, we can extend \( v_{i,D} \) to a vector field \( \hat{v}_{i,D} \) in a neighborhood of \((0, 0)_i\) using parallel transport along geodesics based at \( u'((0,0)_i) \) with respect to the unitary connection on \( TD \), which we fixed before. Then the vector \( \hat{v}_i \) is defined to be:

\begin{equation}
(7.9) \quad \hat{v}_i = v_{i,R} + v_{i,S1} + \hat{v}_{i,D}.
\end{equation}

Now we define

\begin{equation}
(7.10)
\begin{aligned}
&\|V\|_{W_{m,\delta}^2}^2 \\
= &\|V\|_{L^2_m((\Sigma_d \setminus \text{Im}\varphi_d) \cup (\Sigma_u \setminus \text{Im}\varphi_u) \cup (\Sigma_D \setminus (\text{Im}\varphi_D,d \cup \text{Im}\varphi_D,s)))}^2 \\
&+ \sum_{i=1}^{m} \sum_{j=0}^{m} \int_{[-5T_1,5T_1]_{r_i} \times S^1_{s_i}} e_{\sigma_1,\sigma_2}^\delta (r_i,s_i) |\nabla^j (V - \hat{v}_i)|^2 dr_i ds_i \\
&+ |v_1|^2 + |v_2|^2.
\end{aligned}
\end{equation}

We use the cylindrical metric on \( \Sigma(\sigma_1, \sigma_2) \) and the metric \( g \) on \( X \setminus D \) to define norms in the first and the second lines of the right hand side of (7.10).
This definition is analogous to (3.6). The space of all \( V \) as above with finite \( \| \cdot \|_{W_{m,\delta}^2} \) norm which satisfies the boundary condition:

\[
V(z) \in T_{u'(z)}L \quad \forall z \in \partial \Sigma(\sigma_1, \sigma_2)
\]

forms a Hilbert space, which we denote by:

\[(7.11) \quad W_{m,\delta}^2((\Sigma(\sigma_1, \sigma_2), \partial \Sigma(\sigma_1, \sigma_2)); (u'^*TX, u'|_\partial^*TL)).\]

Next, let:

\[
V \in L_{m}^2(\Sigma(\sigma_1, \sigma_2); u'^*TX \otimes \Lambda^{0,1})
\]

and define:

\[(7.12) \quad \|V\|_{L_{m,\delta}^2}^2 = \sum_{j=0}^m \int_{\Sigma(\sigma_1, \sigma_2)} e_{\sigma_1, \sigma_2}^j(z) |\nabla^j V(z)|^2 vol_{\Sigma(\sigma_1, \sigma_2)}.
\]

We use the cylindrical metric on \( \Sigma(\sigma_1, \sigma_2) \) and the metric \( g \) on \( X \setminus D \) to define the norm and the volume element \( vol_{\Sigma(\sigma_1, \sigma_2)} \). The set of all such \( V \) with \( \|V\|_{L_{m,\delta}^2} < \infty \) forms a Hilbert space, which we denote by

\[(7.13) \quad L_{m,\delta}^2(\Sigma(\sigma_1, \sigma_2); u'^*TX \otimes \Lambda^{0,1}).\]

As a topological vector space, this is the same space as the standard space of Sobolev \( L_{m}^2 \) sections. However, the ratio between the above \( L_{m,\delta}^2 \) norm and the standard Sobolev \( L_{m}^2 \) norm is unbounded while \( \sigma_1, \sigma_2 \) go to 0.

Finally, we can use the above Sobolev spaces, to define the linearization of the non-linear Cauchy-Riemann equation at \( u' \), which is a Fredholm operator:

\[(7.14) \quad D_{u'} : W_{m+1,\delta}^2((\Sigma(\sigma_1, \sigma_2), \partial \Sigma(\sigma_1, \sigma_2)); (u'^*TX, u'|_\partial^*TL)) \rightarrow L_{m,\delta}^2(\Sigma(\sigma_1, \sigma_2); u'^*TX \otimes \Lambda^{0,1}).\]

7.4. Pre-gluing. Suppose \( \xi = (u_D^\xi, u_D^\xi, u_D^\xi) \) is an element of the following space:\(^{14}\)

\[(7.15) \quad U_{d}^{ev_d} \times_{evD,d} U_{d}^{+} \times_{ev_s} U_{d}^{+}.
\]

In this subsection, for each choice of \( \sigma_1 \) and \( \sigma_2 \), we shall construct an approximate inconsistent solution and approximate the error for this approximate solution.

By assumption, the pull back bundle \( (u_D^\xi)^*\mathcal{N}_D(X) \) has a meromorphic section \( \mathfrak{s}^\xi \) which has poles of order 2 and 3 at \( z_4 \) and \( z_3 \), respectively. As in (2.1), \( \mathfrak{s}^\xi \) gives rise to a map

\[(7.16) \quad U_D^\xi : D \setminus \{z_d, z_s\} \rightarrow \mathcal{N}_D(X) \setminus D.
\]

A priori, the section \( \mathfrak{s}^\xi \) is well-defined up to the action of \( \mathbb{C}_s \) and for each \( \xi \) in (7.15), we fix one such section such that \( U_D^\xi \) depends smoothly on \( \xi \). Later

\(^{14}\)Recall that \( \Sigma_d \) together with the marked points \( z_0 \) and \( z_d \) is already source stable and we did not need to introduce auxiliary marked points on this space. This is the reason that the first factor is \( U_d \), rather than \( U_d^+ \).
we will pin down the choice of sections such that (5.5) is satisfied. Recall that a neighborhood of the zero section in $N_{D}(X)$ with the neighborhood $U$ of $D$ in $X$. For now, we assume that the section $s^\xi$ is chosen such that the image of $U^\xi_{D}$ on the domain $\Sigma^+_{D}(\sigma_1, \sigma_2)$ belongs to this neighborhood of the zero section of $N_{D}(X)$. Recall that $\Sigma^+_{D}(\sigma_1, \sigma_2)$ is defined in (7.6).

Next, we shall glue the three maps $u^\xi_\partial, u^\xi_\xi, U^\xi_{D}$ by a partition of unity. One should beware that the output of this construction is an approximate inconsistent solution. In particular, it will not be a globally well-defined map from $\Sigma(\sigma_1, \sigma_2)$ to $X$. In order to describe this process, we need to fix an exponential map.

There is a map

$$
(7.17) \quad \text{Exp} : T(X \setminus D) \to \mathfrak{g}(\Delta)
$$

with $\mathfrak{g}(\Delta)$ being a neighborhood of the diagonal $\Delta$ in $(X \setminus D) \times (X \setminus D)$ such that:

(i) For $p \in X \setminus D$ and $V \in T_p(X \setminus D)$, the first component of $\text{Exp}(p, V)$ is $p$.

(ii) $\text{Exp}$ maps $(p, 0) \in T_p(X \setminus D)$ to $(p, p) \in (X \setminus D) \times (X \setminus D)$. Moreover, at the point $(p, 0)$, the derivative of $\text{Exp}$ in the fiber direction given by $T_p(X \setminus D) \subset T(X \setminus D)$ is equal to $(0, \text{id})$ where $\text{id}$ is the identity map from $T_p(X \setminus D)$ to itself.

(iii) Recall that we defined partial $\mathbb{C}^*$ actions for a pair of an almost complex manifold $Y$ and a complex submanifold $D$ of (complex) codimension 1 in [DF2, Subsection 3.2]. This notion can be generalized to the case of complex submanifolds of arbitrary codimension in an obvious way. For example, the derivative of the partial $\mathbb{C}^*$ action for the pair $(X, D)$ determines a partial $\mathbb{C}^*$ action for the pair $(TX, TD)$. Moreover, the product of two copies of partial $\mathbb{C}^*$ actions for the pair $(X, D)$ induces a partial $\mathbb{C}^*$ action on $(X \times X, D \times D)$. We require that the map (7.17) is equivariant with respect to these two partial $\mathbb{C}^*$ actions.

(iv) For a positive real number $\kappa$, let $D_\kappa TL$ denote the tangent vectors to $L$ whose norms are smaller than $\kappa$. There is $\kappa$ such that:

$$
\text{Exp}(D_\kappa TL) \subset L \times L
$$

Let $\exp$ be the exponential map with respect to the metric $g$. The map $(\text{id}, \exp)$, defined on a neighborhood of the zero section of $T(X \setminus D)$, satisfies (i)-(iii). We can modify this map and extend it to a map on $T(X \setminus D)$ which satisfies (iv). We denote the inverse of (7.17) by

$$
E : \mathfrak{g}(\Delta) \to T(X \setminus D).
$$

We now define $\rho^{\xi}_{i(0)} \in \mathbb{C}^*$ $(i = 1, 2)$ as follows. We define the composition

$$
u^{\xi}_{i} \circ \varphi_{\partial} : D^2 \to X \setminus D,$$
We take a (holomorphic) trivialization $\Pi : N_D(X) \to \mathbb{C}$ of the normal bundle $N_D(X)$ in a neighborhood of $u_0^d(z_d)$. Note that $u_d^\xi(z_d) \in D$ is in a small neighborhood of $u_0^d(z_d)$. Therefore, $u_d^\xi \circ \varphi_d$ induces a holomorphic function

$$\Pi \circ u_d^\xi \circ \varphi_d : D^2(o) \to \mathbb{C}$$

for a small $o > 0$. By assumption $\Pi \circ u_d^\xi \circ \varphi_d$ has a zero of order 2 at $z_d$. We define $c_d^\xi \in \mathbb{C}^*$ by

$$(7.18) \quad (\Pi \circ u_d^\xi \circ \varphi_d)(z) = c_d^\xi z^2 + f(z)z^3$$

where $f(z)$ is holomorphic at 0.

Using the trivialization $\Pi$, we may regard the meromorphic section $s^\xi \circ \varphi_d$ as a meromorphic function which has a pole of order 2 at $z_d$. In particular, there is a constant $c_{D,d}^\xi \in \mathbb{C}^*$ such that $\Pi \circ s^\xi \circ \varphi_d : D^2(o) \setminus \{0\} \to \mathbb{C}$ has the following form:

$$(7.19) \quad (\Pi \circ s^\xi \circ \varphi_{D,d})(w) = c_{D,d}^\xi w^{-2} + \frac{g(w)}{w},$$

where $g$ is holomorphic at 0. We now define:

$$(7.20) \quad \rho_{1,0}^\xi(\sigma_1, \sigma_2) = \frac{c_d^\xi \sigma_1^2}{c_{D,d}^\xi}.$$ 

Note that $\rho_{1,0}^\xi$ is independent of the choice of the trivialization of $N_D(X)$, because an alternative choice affects the numerator and the denominator of the right hand side by multiplying with the same number. The constant $\rho_{1,0}^\xi$ has the property that if $zw = \sigma_1$, then:

$$(7.21) \quad (u_d^\xi \circ \varphi_d)(z) \sim (\text{Dil}_{\rho_{1,0}^\xi} \circ U_D^\xi \circ \varphi_{D,d})(w)$$

where $\sim$ means the coincidence of the lowest order term.

We define $\rho_{2,0}^\xi$ in a similar way using the behavior of $u_s$ and $u_{D,s}$ in a neighborhood of $z_s$. Namely, we replace (7.18) and (7.19) by:

$$\quad (\Pi \circ u_s^\xi \circ \varphi_s)(z) = c_s^\xi z^3 + f(z)z^4,$$

$$(7.23) \quad (\Pi \circ s^\xi \circ \varphi_{D,s})(w) = c_{D,s}^\xi w^{-3} + \frac{g(w)}{w^2},$$

respectively and define:

$$(7.24) \quad \rho_{2,0}^\xi(\sigma_1, \sigma_2) = \frac{c_s^\xi \sigma_2^2}{c_{D,s}^\xi}.$$ 

Now we define a map

$$u_{\sigma_1, \sigma_2, 0}^{t, \xi} : \Sigma(\sigma_1, \sigma_2) \to X$$
as follows. Roughly speaking, $u_{\sigma_1,\sigma_2,0}^{t,\xi,i}$ is obtained by gluing the three maps $u_d^\xi, u_s^\xi, \text{Dil}_D^\xi \circ U_D^\xi$, using bump functions $\chi_{\overline{\sigma_i}X}, \chi_{\overline{\sigma_i}X}^\prime$. From now on, we write $\rho_{i,0}^\xi$ instead of $\rho_{i,0}^\xi(\sigma_1,\sigma_2)$ when the dependence on $\sigma_i$ is clear.

**Definition 7.2.**

1. If $z \in \Sigma_d^- (\sigma_1,\sigma_2)$, then:
   
   $$u_{\sigma_1,\sigma_2,0}^{t,\xi,1}(z) = u_{\sigma_1,\sigma_2,0}^{t,\xi,2}(z) = u_d^\xi(z).$$

2. If $z \in \Sigma_s^- (\sigma_1,\sigma_2)$, then:
   
   $$u_{\sigma_1,\sigma_2,0}^{t,\xi,1}(z) = u_{\sigma_1,\sigma_2,0}^{t,\xi,2}(z) = u_s^\xi(z).$$

3. If $z \in \Sigma_D^- (\sigma_1,\sigma_2)$, then:
   
   $$u_{\sigma_1,\sigma_2,0}^{t,\xi,i}(z) = (\text{Dil}_{\rho_{i,0}^\xi} \circ U_D^\xi)(z)$$
   for $i = 1, 2$.

4. Suppose $z = (r_1, s_1) \in [-5T_1, 5T_1] \times S_{s_1}^1$. We define
   
   $$u_{\sigma_1,\sigma_2,0}^{t,\xi,i}(z) = \text{Exp}_2 \left( u_d^\xi(z), \chi_{\overline{\sigma_i}X}^\prime(z) E(u_d^\xi(z), (\text{Dil}_{\rho_{i,0}^\xi} \circ U_D^\xi)(z)) \right).$$

   Here $\text{Exp}_2$ denotes the composition of $\text{Exp}$ and projection map from $(X \setminus \mathcal{D}) \times (X \setminus \mathcal{D})$ to the second factor.

5. Suppose $z = (r_2, s_2) \in [-5T_2, 5T_2] \times S_{s_2}^1$. We define
   
   $$u_{\sigma_1,\sigma_2,0}^{t,\xi,i}(z) = \text{Exp}_2 \left( u_s^\xi(z), \chi_{\overline{\sigma_i}X}^\prime(z) E(u_s^\xi(z), (\text{Dil}_{\rho_{i,0}^\xi} \circ U_D^\xi)(z)) \right).$$

**Remark 7.3.** In part (4), if $r_1$ is close to $-5T_1$, then the right hand side is $u_d^\xi$, and if $r_1$ is close to $5T_1$, then the right hand side is $\text{Dil}_{\rho_{i,0}^\xi} \circ U_D^\xi$. A similar property holds for the definition in part (5). In particular, our definition is well-defined.

**Step 0-3 (Error estimate)**

The next lemma provides an estimate of $\tilde{\Sigma} u_{\sigma_1,\sigma_2,0}^{t,\xi,i}$ modulo the obstruction space

$$(E_d \oplus E_s \oplus E_D)(u_{\sigma_1,\sigma_2,0}^{t,\xi,i}).$$

In the case that $\rho_{1,0}^\xi \neq \rho_{2,0}^\xi$, we need to restrict the domain in the following way to obtain an appropriate estimate. We put

$$\Sigma(\sigma_1,\sigma_2)_i^- = \begin{cases} 
\Sigma(\sigma_1,\sigma_2) \setminus ([-5T_2, 5T_2] \times S_{s_2}^1) & \text{if } i = 1, \\
\Sigma(\sigma_1,\sigma_2) \setminus ([-5T_1, 5T_1] \times S_{s_1}^1) & \text{if } i = 2.
\end{cases}$$

We consider the $L^2_{m,\delta}$ norm of the restriction of maps to $\Sigma(\sigma_1,\sigma_2)_i^-$ and denote it by $L^2_{m,\delta,\tilde{\Sigma}}^{\pm,i}.$

\[\text{[FOOO2, Section A1.4] and [FOOO5].}\]
Lemma 7.4. There exist constants $\delta_1, C_m$ (for any integer $m$) and vectors $\xi_{d,0}(0) \in E_d(u_{\sigma_1,\sigma_2}(0)), \xi_{s,0}(0) \in E_s(u_{\sigma_1,\sigma_2}(0)), \xi_{D,0}(0) \in E_D(u_{\sigma_1,\sigma_2}(0))$ such that $\delta_1, C_m$ are independent of $\sigma_1, \sigma_2, \xi$, and we have the following inequalities:

\begin{align}
(1) \quad & \|\overline{\partial}u_{\sigma_1,\sigma_2}(0) - \xi_{d,0}(0) - \xi_{s,0}(0) - \xi_{D,0}(0)\|_{L^{2,1,-}} \leq C_m e^{-\delta_1 T_1}. \\
(2) \quad & \|\overline{\partial}u_{\sigma_1,\sigma_2}(0) - \xi_{d,0}(0) - \xi_{s,0}(0) - \xi_{D,0}(0)\|_{L^{2,2,-}} \leq C_m e^{-\delta_1 T_2}.
\end{align}

We can be more specific about the value of the constant $\delta_1$ as in (3.5) and (3.9). However, the actual choices do not matter for the details of our construction. So we do not give an exact value for this constant.

Proof. We define:

\[ \xi_{d,0}(0) := \overline{\partial} u_d \in E_d(u_{\sigma_1,\sigma_2}(0)), \]
\[ \xi_{s,0}(0) := \overline{\partial} u_s \in E_s(u_{\sigma_1,\sigma_2}(0)), \]
\[ \xi_{D,0}(0) := \overline{\partial} (\text{Dil}_{\rho_{\xi,0}} \circ U_{\Sigma}) \in E_D(u_{\sigma_1,\sigma_2}(0)). \]

Then by construction the support of $\overline{\partial} u_{\sigma_1,\sigma_2}(0) - \xi_{d,0}(0) - \xi_{s,0}(0) - \xi_{D,0}(0)$ is contained in $([-5T_1,5T_1](1) \times S^1_{1}) \cup [\{5T_2,5T_2\}(1) \times S^1_{2}) \cup [\{5T_1,5T_1\}(1) \times S^1_{1}]$. Therefore, it suffices to estimate $\overline{\partial} u_{\sigma_1,\sigma_2}(0)$ on $[-5T_1,5T_1](1) \times S^1_{1}$. Below we discuss the case $i = 1$.

The other case is similar.

Let $z = \varphi_{\Sigma}(r_1')$ be the coordinate on $\Sigma_d$ used to denote points in a neighborhood of $z_d$ and $w = \varphi_{\Sigma_d}(r_1', s_1')$ be the coordinate on $\Sigma_D$ used to denote points in a neighborhood of $z_d$. In order to obtain $\Sigma(\sigma_1, \sigma_2)$, the equation:

\[ zw = \sigma_1 \]

is used to glue $\Sigma_d$ and $\Sigma_D$. Note that the supports of the derivatives of the bump functions $\chi_{1,1}^\pm, \chi_{2,1}^\pm$ are in $X_1(T_1)$. (Here we look at the restriction of the function $\chi_{1,1}^\pm$ to $\Sigma(\sigma_1, \sigma_2)_1$. Otherwise, part of the support of the derivite of this function is contained in $X_2(T_2)$.) Therefore, the support of $\overline{\partial} u_{\sigma_1,\sigma_2}(0)$ is contained in the same subspace.

Firstly we wish to show that the maps $f_1 := u_d$ and $f_2 := \text{Dil}_{\rho_{\xi,0}} \circ U_{\Sigma}$, as maps from from $X_{1,T_1}$ to $\Omega \setminus D \subset X \setminus D$, are close to each other in the $C^m$ metric. In fact, analogues of the inequalities in (3.5) and (3.9) show that there are constants $C'_m$ and $\delta_0$ independent of $\sigma_1, \sigma_2$ and $\xi$ such that:

\[ d_{C^m}(f_1, f_2) \leq C'_m e^{-5\delta_0 T_1} \]

where $d_{C^m}$ is computed with respect to the cylindrical metric $g$. To be a bit more detailed, this inequality holds because the leading terms of $f_1$ and $f_2$ agree with each other, and $f_1$ and $f_2$ are both holomorphic.
Let \( h_1, h_2 : X_{1,T_1} \to \mathcal{U} \) are maps such that their \( C^0 \) distance is less than or equal to a constant \( \kappa \). If \( \kappa \) is small enough, then the following map is well defined:

\[
F(h_1, h_2) = \text{Exp}_2 \left( h_1, \chi_{X'} \cdot E(h_1, h_2) \right).
\]

Clearly there is a constant \( K \) such that:

\[
\left\| \bar{\partial}F(h_1, h_2) - \bar{\partial}F(h_1, h_2') \right\|_{L^2_m} \leq K \cdot d_{C^{m+1}}(h_2, h_2')
\]

Since \( F(f_1, f_1) = f_1 \), the above inequality together with (7.27) implies that there is a constant \( C_m \) such that:

\[
\left\| \bar{\partial}u_{\xi,1}^{\xi,1} \right\|_{L^2_m(X_{1,T_1})} \leq C_m e^{-\delta_0 T_1}
\]

Therefore, if we pick \( \delta \) and \( \delta_1 \) such that \( \delta + \delta_1 < 5 \delta_0 \), then the desired inequality holds.

7.5. Why Inconsistent Solutions? We already hinted at the necessity of inconsistent solutions at the end of Section 4. In this section we elaborate on this point with an eye toward modifying the approximate solution of the previous section to a solution. We firstly sketch our approach for this modification which is based on Newton’s iteration method. Next, we explain the main point where the proof in the case of the RGW compactification diverges from the case of the stable map compactification. The discussion of this subsection is informal, and the actual proof will be carried out in the next two subsections.

Suppose \( u_{\xi,1}^{\xi,1} \) and \( u_{\xi,2}^{\xi,2} \) are the approximate solutions of the previous subsection associated to the element \( \xi = (u^\xi_0, u^\xi_D, u^\xi_s) \) of (7.15). We assume that \( \sigma_1 \) and \( \sigma_2 \) are chosen such that \( \rho_{1,0}(\sigma_1, \sigma_2) = \rho_{2,0}(\sigma_1, \sigma_2) \). In particular, \( u_{\sigma_1,\sigma_2,0} = u_{\sigma_1,\sigma_2,0}^{\xi,1} \) and we denote these maps by \( u' \). Lemma 7.4 gives the following estimate:

\[
\left\| \bar{\partial}u' \right\|_{L^2_m(\Sigma(\sigma_1, \sigma_2))/E(u')} \leq C e^{-\delta_1 \min\{T_1, T_2\}}.
\]

Here \( E(u') = E_0(u') \oplus E_2(u') \oplus E_D(u') \), and the norm on the left hand side is the induced norm on the quotient space. The next step would be to find:

\[
V \in W^2_{m+1,\delta}(\Sigma(\sigma_1, \sigma_2), \partial \Sigma(\sigma_1, \sigma_2)); (u'^sTX, u'_sT_L).
\]

which satisfies the equation:

\[
(7.28) \quad (D_{u'} \bar{\partial})V + \bar{\partial}u' \equiv 0 \mod E(u').
\]

and

\[
\left\| V \right\|_{W^2_{m+1,\delta}} \leq C \left\| \bar{\partial}u' \right\|_{L^2_m(\Sigma(\sigma_1, \sigma_2))/E(u')}.
\]

Then we could define our first modified approximate solution as follows:

\[
u''(z) = \text{Exp}(u'(z), V(z))
\]

This modified solution would satisfy the following inequality:

\[
\left\| \bar{\partial}u'' \right\|_{L^2_m(\Sigma(\sigma_1, \sigma_2))/E(u'')} \leq \mu \left\| \bar{\partial}u' \right\|_{L^2_m(\Sigma(\sigma_1, \sigma_2))/E(u')}
\]

\[
for a fixed $0 < \mu < 1$ if $\sigma_1, \sigma_2$ are sufficiently small (or equivalently, $T_1, T_2$ are sufficiently large).

We could then continue to obtain $u^{(i)}$ such that
\[
\|\overline{\mathcal{T}} u^{(i)}\|_{L^2_m(\Sigma(\sigma_1, \sigma_2))/E(u^{(i)})} \leq \mu^i \|\overline{\mathcal{T}} u'\|_{L^2_m(\Sigma(\sigma_1, \sigma_2))/E(u')}
\]
and for fixed constants $C$ and $c$, the $W^{2,1}_{m+1,\delta}$-distance between $u^{(i)}$ and $u^{(i+1)}$ is bounded by $Ce^{-c\min(T_1, T_2)}$. Then $u^{(i)}$ would be convergent to a map $u$, and it would be the required solution of the equation:
\[
(7.29) \quad \overline{\mathcal{T}} u \equiv 0 \mod E(u).
\]
This is the standard Newton’s iteration method to solve a nonlinear equation using successive solutions to the linearized equation. However, the RGW compactification is singular at the starting point of our construction, the element $\zeta$ of (7.15). So we cannot expect the above Newton’s iteration method works without some adjustments. We fix our approach by thickening the solution set of (7.29) to the set of inconsistent solutions.

The main reason that we will work with this larger moduli space lies in the step that we find the solution $V$ of the equation (7.28). To solve this equation, we need to find a right inverse to the following operator modulo $E(u')$:
\[
D_{u'}\overline{\mathcal{T}}: W^2_{m+1,\delta}(\Sigma(\sigma_1, \sigma_2), \partial\Sigma(\sigma_1, \sigma_2); (u'^*TX, u'|^*TL)) \to L^2_{m,\delta}(\Sigma(\sigma_1, \sigma_2); u'^*TX \otimes \Lambda^{0,1})/E(u').
\]
The standard approach to construct this right inverse is to glue the right inverses of the linearized operators $D_{u_d}\overline{\mathcal{T}}$, $D_{u_s}\overline{\mathcal{T}}$ and $D_{U_{\partial}}\overline{\mathcal{T}}$. The linearized operator $D_{u'}\overline{\mathcal{T}}$ over the cylinder $[-5T_i, 5T_i]_{r_i} \times S^1_{s_i}$ is modeled by an operator of the form
\[
\frac{\partial}{\partial r_i} + P_{r_i}.
\]
The relevant operators $P_{r_i}$ in our setup have non-trivial kernel and our gluing construction is of “Bott-Morse” type. As it was clarified by Mrowka’s Mayer-Vietoris principle [Mr], to have a well-behaved gluing problem we need to assume certain ‘mapping transversality conditions’.

To be more specific, the zero eigenspace of the operator $P_{r_i}$ can be identified with:
\[
(7.30) \quad (\mathbb{R} \oplus \mathbb{R}) \oplus T_{u_d(z_i)}D \quad \text{if } i = 1, \\
R \oplus \mathbb{R} \oplus T_{u_s(z_i)}D \quad \text{if } i = 2.
\]
Here $\mathbb{R} \oplus \mathbb{R}$ is the tangent space to $\mathbb{C}_+$. The mapping transversality condition we introduced in Definition 4.2 concerns the summand $T_{u_d(z_i)}D$. Therefore, it is not sufficient for the Mayer-Vietoris principle in our setup. However, working with inconsistent solutions allows us to enlarge the tangent spaces and obtain the required transversality condition. A byproduct of using inconsistent solutions is that we might end up with inconsistent solutions
throughout Newton’s iterations, even if the starting approximate solution has \( \rho_1 = \rho_2 \).

7.6. Inconsistent Maps and Linearized Equations. In Section 5, the notion of holomorphic maps was extended to inconsistent solutions of the Cauchy-Riemann equation. It is also convenient to define generalizations of maps from \( \Sigma(\sigma_1, \sigma_2) \) to \( X \setminus \mathcal{D} \):

**Definition 7.5.** A 7-tuple \( u' = (u'_1, u'_s, U'_D, \sigma_1, \sigma_2, \rho_1, \rho_2) \) is an inconsistent map if it satisfies only parts (1) and (4) of Definition 5.1. In other words, we do not require that the 7-tuple satisfies the Cauchy-Riemann equation in (5.1) and the constraint in (5.2). We define equivalence of inconsistent maps in the same way as in Definition 5.1.

An example of inconsistent maps can be constructed using the maps:

\[
\begin{align*}
\xi_X \in (\xi, X) & \quad \text{of Subsection 7.4 which are associated to an element } \xi = (u_1^\xi, u_D^\xi, u_2^\xi) \text{ of (7.15)}.
\end{align*}
\]

We use these two maps to define:

\[
\begin{align*}
u_{d,1,\sigma_2,(0)}^\xi & := u_{d,1,\sigma_2,(0)}^{t,1} \big|_{\Sigma_1^+(\sigma_1, \sigma_2)} \\
u_{s,1,\sigma_2,(0)}^\xi & := u_{s,1,\sigma_2,(0)}^{t,2} \big|_{\Sigma_1^+(\sigma_1, \sigma_2)} \\
u_{D,1,\sigma_2,(0)}^\xi & := \begin{cases} 
\text{Dil}_{1/\rho_1} \circ u_{d,1,\sigma_2,(0)}^{t,1} & \text{on } \Sigma_1^+(\sigma_1, \sigma_2) \setminus \Sigma_1^+(\sigma_1, \sigma_2) \\
\text{Dil}_{1/\rho_2} \circ u_{s,1,\sigma_2,(0)}^{t,2} & \text{on } \Sigma_1^+(\sigma_1, \sigma_2) \setminus \Sigma_1^+(\sigma_1, \sigma_2) \\
\text{Dil}_{1/\rho_2} \circ u_{s,1,\sigma_2,(0)}^{t,2} & \text{on } \Sigma_1^+(\sigma_1, \sigma_2) \setminus \Sigma_1^+(\sigma_1, \sigma_2)
\end{cases}
\end{align*}
\]

Note that \( \text{Dil}_{1/\rho_1} \circ u_{d,1,\sigma_2,(0)}^{t,1} = \text{Dil}_{1/\rho_2} \circ u_{s,1,\sigma_2,(0)}^{t,2} \) on \( \Sigma_1^+(\sigma_1, \sigma_2) \). The following lemma is obvious from the construction:

**Lemma 7.6.** The 7-tuple:

\[
\begin{align*}
u_{d,1,\sigma_2,(0)} = (u_{d,1,\sigma_2,(0)}, u_{s,1,\sigma_2,(0)}, U_{D,1,\sigma_2,(0)}, \sigma_1, \sigma_2, \rho_1, \rho_2)
\end{align*}
\]

is an inconsistent map.

The inconsistent map \( u_{d,1,\sigma_2,(0)}^\xi \) of Lemma 7.6 is the approximate solution at the 0-th step. In order to obtain an actual inconsistent solution, we keep modifying this approximate solution into better approximate solutions. To be a bit more detailed, we firstly use \( u_{d,1,\sigma_2,(0)}^\xi \) and our bump functions to obtain a triple \( u_{d,1,\sigma_2,(0)}^{\xi,\prime} = (u_{d,1,\sigma_2,(0)}, u_{s,1,\sigma_2,(0)}, u_{s,1,\sigma_2,(0)}) \) as follows:

\[
\begin{align*}
(7.31) \quad u_{d,1,\sigma_2,(0)}^{\xi,\prime} : \Sigma_d \setminus \{z_d\} \rightarrow X \setminus \mathcal{D}, \\
u_{s,1,\sigma_2,(0)}^{\xi,\prime} : \Sigma_s \setminus \{z_s\} \rightarrow X \setminus \mathcal{D},
\end{align*}
\]

\[
(7.32) \quad U_{D,1,\sigma_2,(0)}^{\xi,\prime} : \Sigma_D \setminus \{z_d, z_s\} \rightarrow \mathcal{N}_D(X).
\]
which are close to \((\xi_1^\xi, \xi_2^\xi, U^\xi_D)\). In fact, the smaller the values of \(\sigma_1\) and \(\sigma_2\) are, the closer \(u_{\sigma_1,\sigma_2}^\xi(0)\) is to \(\xi\). Thus we can exploit this to conclude that an appropriate version of the Cauchy-Riemann operator associated to \(u_{\sigma_1,\sigma_2}^\xi(0)\) has a right inverse. (See Lemma 7.17.) This allows us to find a modified inconsistent map \(u_{\sigma_1,\sigma_2}^{\xi,i}\). We repeat the same process to construct a sequence of inconsistent maps \(\{u_{\sigma_1,\sigma_2}^{\xi,i}(i)\}\) which are approximate solutions and they converge to an inconsistent solution. This sequence of modified inconsistent solution is constructed using Newton’s iteration, and it also has some components of the “alternating method”. In this method we solve the equation in various pieces and glue them together.

In order to carry out the above plan, we need to introduce norms to quantify the distance between two inconsistent maps and to measure how good an approximate solution is. Such norms are given in the following definitions:

**Definition 7.7.** Let \(u' = (u'_d, u'_s, U'_D, \sigma_1, \sigma_2, \rho_1, \rho_2)\) be an inconsistent map. We consider a triple \(V = (V_d, V_s, V_D)\) with

\[
V_d \in L^2_{m+1}(\Sigma^+(\sigma_1, \sigma_2), (u'_d)^*TX),
\]

\[
V_s \in L^2_{m+1}(\Sigma^+(\sigma_1, \sigma_2), (u'_s)^*TX),
\]

\[
V_D \in L^2_{m+1}(\Sigma^+(\sigma_1, \sigma_2), (U'_D)^*(TN_D(X))).
\]

We assume \(V_d(\xi) \in T\Sigma^+(\sigma_1, \sigma_2)\). Moreover, we assume that there exist \((a_d, b_d)\), \((a_s, b_s)\) \in \(\mathbb{R} \oplus \mathbb{R}\) such that

\[
V_d - V_D = (a_d, b_d) \quad \text{on} \quad [-5T_1, 5T_1]_1 \times S^1_{s_1}
\]

\[
V_s - V_D = (a_s, b_s) \quad \text{on} \quad [-5T_2, 5T_2]_2 \times S^2_{s_2}
\]

Here we regard \(\mathbb{R} \oplus \mathbb{R}\) as the vector field on the neighborhood \(\mathcal{U}\) of \(D\) given by the \(\mathbb{C}_*\) action.

Define \(v_i = V_D((0, 0)_i)\) where \((0, 0)_i\) is the same as in (7.8). We then define \(\hat{v}_i\) in the same way as in (7.9). We now define \(||V||^2_{L^2_{m,\delta}}\) as follows:

\[
||V_d||^2_{L^2_{m}(\Sigma^+(\sigma_1, \sigma_2))} + ||V_s||^2_{L^2_{m}(\Sigma^+(\sigma_1, \sigma_2))} + ||V_D||^2_{L^2_{m}(\Sigma^+(\sigma_1, \sigma_2))} + \sum_{j=0}^{m} \int_{[-5T_j, 5T_j]_1 \times S^1_{s_1}} e^{\xi_{1,\sigma,2}} |\nabla^j (V_D - \hat{v}_1)|^2 \, dr_1 ds_1
\]

\[
+ \sum_{j=0}^{m} \int_{[-5T_j, 5T_j]_2 \times S^2_{s_2}} e^{\xi_{1,\sigma,2}} |\nabla^j (V_D - \hat{v}_2)|^2 \, dr_2 ds_2
\]

\[+ ||v_1||^2 + ||v_2||^2.\]

---

*See, for example, [Fu, Sublemma 8.6]. Application of alternating method for gluing analysis of this kinds is initiated by Donaldson [Do]. He applied alternating method directly to a nonlinear equation.*
Let \( V = (V_d, V_s, V_D) \), \( V' = (V'_d, V'_s, V'_D) \) be as above. We say they are equivalent if \( V_d = V'_d \), \( V_s = V'_s \) and \( V_D - V'_D \in \mathbb{R} \oplus \mathbb{R} \), where \( \mathbb{R} \oplus \mathbb{R} \) is the set of vector fields generated by the \( C_\gamma \) action. We put

\[
\|V\|_2^{2}_{m, \delta} = \inf \{\|V'\|_2^{2}_{m, \delta} \mid \text{\( V' \) is equivalent to \( V \)}\}.
\]

**Definition 7.8.** For \( j = 1, 2 \), let \( u'_j \) be an inconsistent map. We assume that there is a representative \((u'_{d,(j)}, u'_{s,(j)}, U'_{D,(j)}, \sigma_1, \sigma_2, \rho_1, \rho_2)\) for \( u'_j \) such that \((u'_{d,(1)}, u'_{s,(1)}, U'_{D,(1)})\) is \( C^0\)-close to \((u'_{d,(2)}, u'_{s,(2)}, U'_{D,(2)})\). Define \( V_d, V_s, V_D \) by the following properties:

\[
\begin{align*}
\text{Exp}(u'_{d,(1)}, V_d) &= u'_{d,(2)}, \\
\text{Exp}(u'_{s,(1)}, V_s) &= u'_{s,(2)}, \\
\text{Exp}(U'_{D,(1)}, V_D) &= U'_{D,(2)}.
\end{align*}
\]

Let \( V = (V_d, V_s, V_D) \), and define:

\[d_{W^2_{m, \delta}}(u'_1(1), u'_2(2)) = \inf \{\|V\|_2^{2}_{m, \delta} \},\]

where the infimum is taken over all representatives for \( u'_1 \) and \( u'_2 \) which are close enough to each other in the \( C^0 \) metric such that the vectors in (7.33) exist. Therefore, \( d_{W^2_{m, \delta}} \) is a well defined distance between two equivalence classes of inconsistent maps.

For any inconsistent map \( u' = (u'_d, u'_s, U'_D, \sigma_1, \sigma_2, \rho_1, \rho_2) \), we may use a similar parallel transport construction as in Definition 5.1 to define obstruction spaces for \( u' \). That is to say, we define maps \( \mathcal{PAC} \) as in (4.4). Then the images of \( E_d \) and \( E_s \) with respect to these maps give rise to the obstruction spaces \( E_d(u'_d) \) and \( E_s(u'_s) \). Similarly, we define \( E_D(U'_D) \) by replacing \( u'_D \) with \( \pi \circ U'_D \) in (4.6) and using the decomposition (3.4). We will write \( E(u') \) for the direct sum of the vector spaces \( E_d(u'_d) \) and \( E_s(u'_s) \) and \( E_D(U'_D) \). Note that \( E_d(u'_d) \), \( E_s(u'_s) \) and \( E_D(U'_D) \) are identified with \( E_d \) and \( E_s \) and \( E_D \). Therefore, we drop \( u'_d, u'_s \) and \( U'_D \) from our notation for these obstruction spaces if it does not make any confusion.

**Definition 7.9.** Let \( u' = (u'_d, u'_s, U'_D, \sigma_1, \sigma_2, \rho_1, \rho_2) \) be an inconsistent map and \( \epsilon = (\epsilon_d, \epsilon_s, \epsilon_D) \in E_d \oplus E_s \oplus E_D \). Then we define \( \|\overline{\partial}u' - \epsilon\|_2^{2}_{L_m, \delta} \) to be the following sum:

\[
\begin{align*}
\|\overline{\partial}u'_d - \epsilon_d\|_{L_m^2, (\Sigma_d - (\sigma_1, \sigma_2))}^2 + \|\overline{\partial}u'_s - \epsilon_s\|_{L_m^2, (\Sigma_s - (\sigma_1, \sigma_2))}^2 \\
+ \|\overline{\partial}U'_D - \epsilon_D\|_{L_m^2, (\Sigma_D - (\sigma_1, \sigma_2))}^2 \\
+ \sum_{j=0}^{m} \int_{[5T_1, 5T_1]} e^2_{\delta, \sigma_1, \sigma_2} \left| \nabla^j \overline{\partial}U'_D \right|^2 dr_1 ds_1, \\
+ \sum_{j=0}^{m} \int_{[5T_2, 5T_2]} e^2_{\delta, \sigma_1, \sigma_2} \left| \nabla^j \overline{\partial}U'_D \right|^2 dr_2 ds_2.
\end{align*}
\]
We remark that the first 3 terms in the above definition are the Sobolev norms of $\bar{\partial}u - \epsilon$ in the \textit{thick part}. The fourth and the fifth terms are its weighted Sobolev norms in the neck region. Because of our choice of cylindrical metrics on $U$, the partial $\mathbb{C}_*$-action induces isometries and preserves the almost complex structure. Therefore, the above sum is well-defined and only depends on the equivalence class of $u'$.

The process of the modifications of our approximate solutions are performed by finding solutions to the linearization of the modified Cauchy-Riemann equations in (5.1). Since our equation has terms induced by the obstruction bundle, the linearized operator has an extra term in addition to $D_{u'}\bar{\partial}$. The equations in (5.1) can be regarded as an equation for an inconsistent map $u' = (u'_d, u'_s, U'_D, \sigma_1, \sigma_2, \rho_1, \rho_2)$ and $(\epsilon_d, \epsilon_s, \epsilon_D) \in E_d \oplus E_s \oplus E_D$:

$$\bar{\partial}u'_d - \epsilon_d = 0, \quad \bar{\partial}u'_s - \epsilon_s = 0, \quad \bar{\partial}U'_D - \epsilon_D = 0.$$  

Suppose $V = (V_d, V_s, V_D)$ is an element of the Hilbert space introduced in Definition 7.7. For each real number $\tau$ with $|\tau| < 1$, let $u^\tau$ be given by the triple $(u'_d, u'_s, U'_D)$ defined as:

$$u'_d := \text{Exp}(u'_d, \tau V_d), \quad u'_s := \text{Exp}(u'_s, \tau V_s), \quad U'_D := \text{Exp}(U'_D, \tau V_D).$$

We use parallel transport along minimal geodesics to obtain:

$$\mathcal{P} \mathcal{A} \mathcal{L} u'_d : L^2_{m,\delta}(\Sigma^+_d(\sigma_1, \sigma_2); u'_d TX \otimes \Lambda^{0,1}) \xrightarrow{\sim} L^2_{m,\delta}(\Sigma^+_d(\sigma_1, \sigma_2); u'_d TX \otimes \Lambda^{0,1}).$$

and maps $\mathcal{P} \mathcal{A} \mathcal{L} u'_s$ and $\mathcal{P} \mathcal{A} \mathcal{L} u'_D$. Then for $\epsilon = (\epsilon_d, \epsilon_s, \epsilon_D) \in E_d \oplus E_s \oplus E_D$, we define:

$$\frac{d}{d\tau} \bigg|_{\tau=0} ((\mathcal{P} \mathcal{A} \mathcal{L} u'_d)^{-1}(\epsilon_d)),$$

$$\frac{d}{d\tau} \bigg|_{\tau=0} ((\mathcal{P} \mathcal{A} \mathcal{L} u'_s)^{-1}(\epsilon_s)),$$

$$\frac{d}{d\tau} \bigg|_{\tau=0} ((\mathcal{P} \mathcal{A} \mathcal{L} u'_D)^{-1}(\epsilon_D)).$$

We also reserve the following notation for the triple given by the above vectors:

$$D_{u'}(\epsilon, V) = (D_{u'_d}(\epsilon, V_d), (D_{u'_s}(\epsilon, V_s), (D_{U'_D}(\epsilon, V_D)).$$

The linearizations of the Cauchy-Riemann equations in (7.34) at $(u', \epsilon)$ evaluated at $V$ as above and $\bar{\partial} \mathcal{J}(V) - (D_{u'}(\epsilon, V) - \bar{\partial} \mathcal{J}(V)).$

where:

$$D_{u'}(\epsilon, V) = (D_{u'_d}(\epsilon, V_d), D_{u'_s}(\epsilon, V_s), D_{U'_D}(\epsilon, V_D)).$$
7.7. **Newton’s Iteration.** Now we are ready to carry out the strategy which is discussed in the previous subsection. In the following, we use the maps constructed in Subsection 7.4.

**(Step 0-4) (Separating error terms into three parts)**

We firstly fix notations for the error terms of our first approximation $u_{\sigma_1,\sigma_2,0}^{\xi}$:

\[
\text{Err}_{d,\sigma_1,\sigma_2,0}(0) = \chi_d x (D u_{\sigma_1,\sigma_2,0}(0) - e_{d,0}^{\xi}),
\]

\[
\text{Err}_{s,\sigma_1,\sigma_2,0}(0) = \chi_s x (D u_{\sigma_1,\sigma_2,0}(0) - e_{s,0}^{\xi}),
\]

\[
\text{Err}_{D,\sigma_1,\sigma_2,0}(0) = \chi_D x (D u_{\sigma_1,\sigma_2,0}(0) - e_{D,0}^{\xi}).
\]

(7.39)

where $e_{d,0}^{\xi}$, $e_{s,0}^{\xi}$, $e_{D,0}^{\eta}$ are defined in (7.26).

**(Step 1-1) (Approximate solution for linearization)**

Next we define:

\[
u_{\sigma_1,\sigma_2,0}^{\xi} = (u_{d,\sigma_1,\sigma_2,0}(0), u_{s,\sigma_1,\sigma_2,0}(0), u_{D,\sigma_1,\sigma_2,0}(0))
\]

whose entries have the form given in (7.31) and (7.32). Let:

\[
u_d^{\xi} = p_{d,\sigma_1,\sigma_2,0}(0) = p_{D,d,\sigma_1,\sigma_2,0}(0),
\]

\[
u_s^{\xi} = p_{s,\sigma_1,\sigma_2,0}(0) = p_{D,s,\sigma_1,\sigma_2,0}(0).
\]

(7.41)

We take $c_{d,0}^{\xi}$, $c_{s,0}^{\xi}$, $c_{D,0}^{\xi}$, as in (7.18), (7.22), (7.19), (7.23), respectively. We regard $c_{d,0}^{\xi}z^{-2}$ as an element of the fiber of $N_{\sigma}(X)$ at $p_{d,\sigma_1,\sigma_2,0}(0)$ and hence as an element of $X \setminus D$. We define:

\[
u_{d,\sigma_1,\sigma_2,0}^{\eta}(r_1, s_1) := \text{Exp}(c_{d,0}^{\xi}z^{-2}, \chi_{1,x}(r_1 - T_1, s_1)E(c_{d,0}^{\xi}z^{-2}, u_{\sigma_1,\sigma_2,0}(0)(r_1, s_1)))
\]

if $(r_1, s_1) \in [-5T_1, \infty) \times S_{\sigma_1} \subset \Sigma_d \setminus \{z_d\}$. If $\overline{\mathfrak{s}} \in \Sigma_d \setminus \{z_d\}$ is an element in the complement of $[-5T_1, \infty) \times S_{\sigma_1}^2$, then we define:

\[
u_{d,\sigma_1,\sigma_2,0}^{\eta}(\overline{\mathfrak{s}}) := \nu_{d,\sigma_1,\sigma_2,0}^{\eta}(\mathfrak{s}).
\]

This completes the definition of $u_{d,\sigma_1,\sigma_2,0}^{\eta}$ as a map from $\Sigma_d \setminus \{z_d\}$ to $X \setminus D$.

Similarly, we define:

\[
u_{s,\sigma_1,\sigma_2,0}^{\eta}(r_2, s_2) := \text{Exp}(c_{s,0}^{\xi}z^{-3}, \chi_{2,x}(r_2 - T_2, s_2)E(c_{s,0}^{\xi}z^{-3}, u_{\sigma_1,\sigma_2,0}(0)(r_2, s_2)))
\]

if $(r_2, s_2) \in [-5T_2, \infty) \times S_{\sigma_2} \subset \Sigma_s \setminus \{z_s\}$. Here we regard $c_{s,0}^{\xi}z^{-3}$ as an element of the fiber of $N_{\eta}(X)$ at $p_{s,\sigma_1,\sigma_2,0}(0)$ and hence as an element of $X \setminus D$. If $\overline{\mathfrak{s}} \in \Sigma_s \setminus \{z_s\}$ is an element in the complement of $[-5T_2, \infty) \times S_{\sigma_2}^2$, then we define:

\[
u_{s,\sigma_1,\sigma_2,0}^{\eta}(\overline{\mathfrak{s}}) := \nu_{s,\sigma_1,\sigma_2,0}^{\eta}(\mathfrak{s}).
\]
It is easy to see from the definitions that \( u_{d,\sigma_1,\sigma_2}(0) \) and \( u_{s,\sigma_1,\sigma_2}(0) \) satisfy Condition 7.10:

**Condition 7.10.** We require that the map \( u_0'' : (\Sigma_\delta \setminus \{z_d\}, \partial \Sigma_\delta) \to (X \setminus D, L) \) (resp. \( u'' : \Sigma_\delta \setminus \{z_s\} \to X \setminus D \)) satisfies the following conditions:

1. \( u_0'' \) (resp. \( u'' \)) maps \([3T_1, \infty)_1 \times S_{s_1}^1 \) (resp. \([3T_2, \infty)_2 \times S_{s_2}^1 \)) to \( \mathcal{U} \).

There exist \( p_d, p_s \in D \) such that the restriction of \( \pi \circ u_0'' \) (resp. \( \pi \circ u'' \)) to \([3T_1, \infty)_1 \times S_{s_1}^1 \) (resp. \([3T_2, \infty)_2 \times S_{s_2}^1 \)) is a constant map to \( p_d \) (resp. \( p_s \)).

2. After an appropriate trivialization of the normal bundle \( \mathcal{N}_D(X) \) at the points \( p_d, p_s \), there exist \( c_d, c_s \in \mathbb{C} \) such that the restriction of \( u_0'' \circ \varphi_d \) to \([3T_1, \infty)_1 \times S_{s_1}^1 \) (resp. \( u'' \circ \varphi_s \) to \([3T_2, \infty)_2 \times S_{s_2}^1 \)) is

\[
(7.44) \quad (u_0'' \circ \varphi_d)(z) = c_d z^2, \quad \text{(resp.} \quad (u'' \circ \varphi_s)(z) = c_s z^3)\]

Next, we define the map \( U_{D,\sigma_1,\sigma_2}(0) : \Sigma_\delta \setminus \{z_d, z_s\} \to \mathcal{N}_D(X) \). The trivializations of the fibers of \( \mathcal{N}_D(X) \) at the points \( p_d, p_s \) and \( p_d, p_s \) allow us to identify \( c_{D,\sigma} w^{-2} \) and \( c_{D,\sigma} w^{-3} \) with elements of \( \mathcal{N}_D(X) \setminus D = \mathbb{R} \times SN_D(X) \). We define:

\[
U_{D,\sigma_1,\sigma_2}(0)(r_1, s_1) \quad \text{and} \quad U_{D,\sigma_1,\sigma_2}(0)(r_2, s_2) \quad \text{if} \quad (r_1, s_1) \in \mathcal{S}_{s_1} \text{ and } \mathcal{S}_{s_2} \quad \text{and}\]

\[
U_{D,\sigma_1,\sigma_2}(0)(r_1, s_1) \quad \text{and} \quad U_{D,\sigma_1,\sigma_2}(0)(r_2, s_2) \quad \text{if} \quad (r_2, s_2) \in \mathcal{S}_{s_1} \text{ and } \mathcal{S}_{s_2} \quad \text{and}\]

Note that we can equivalently use the term \( (\text{Dil}_{1/\rho_1(0)} \circ \rho_1(0)) \) on the right hand side of the above definition. We remark that the ‘highest order’ terms of the maps \( (\text{Dil}_{1/\rho_1(0)} \circ \rho_1(0)) \) agree with each other on \([-5T_1, 5T_1] \times S_{s_1}^2 \). Similarly, \( U_{D}(\varphi_D, d(w)) \) (resp. \( U_{D}(\varphi_D, s(w)) \)) and \( c_{D,\sigma} w^{-2} \) (resp. \( c_{D,\sigma} w^{-3} \)) have the same highest order terms on \([-5T_1, 5T_1] \times S_{s_1}^2 \).

It is easy to see from definition that \( U_{D,\sigma_1,\sigma_2}(0) \) satisfies Condition 7.11:

**Condition 7.11.** We require \( U_0'' : \Sigma_\delta \setminus \{z_d, z_s\} \to X \setminus D \) satisfies the following conditions:
(1) There exist $p_{D,d}, p_{D,s} \in D$ such that the restriction of $\pi \circ U''_{D}$ to $(-\infty, -3T_1]_{r_1} \times S_{1}^{1}_{s_1}$ (resp. $(-\infty, -3T_2]_{r_2} \times S_{1}^{1}_{s_2}$) is a constant map to $p_{D,d}$ (resp. $p_{D,d}$).

(2) There exist $c_{D,d}, c_{D,s} \in C_s$ such that the restriction of $U''_{D} \circ \varphi_{D,d}$ to $(-\infty, -3T_1]_{r_1} \times S_{1}^{1}_{s_1}$ (resp. $U''_{D} \circ \varphi_{D,s}$ to $(-\infty, -3T_2]_{r_2} \times S_{1}^{1}_{s_2}$) is

\[(7.47) \quad (U''_{D} \circ \varphi_{D,d})(w) = c_{D,d}w^{-2}, \quad \text{resp. } (U''_{D} \circ \varphi_{D,s})(w) = c_{D,s}w^{-3}).\]

Let $u'' = (u''_d, u''_s, U''_{D})$ be a triple of maps satisfying Conditions 7.10, 7.11. We also assume:

\[(7.48) \quad p_d = p_{D,d}, \quad p_s = p_{D,s}.\]

**Definition 7.12.** Let $W_{m,d}^{2}(u'', U''_{D}; TX)$ be the set of all $V = (V_d, V_s, V_D)$ satisfying the following properties:

1. $V_d = (V_d, (\tau_{\infty,d}, g_{\infty,d}, \nu), v_d) \in W_{m,d}^{2}(\Sigma_d \setminus \{z_d\}; ((u''_{d})^*T_X, (u''_{d})^*TL)).$ (This function space is introduced in Definition 3.3.)

2. $V_s = (V_s, (\tau_{\infty,s}, g_{\infty,s}), v_s) \in W_{m,s}^{2}(\Sigma_s \setminus \{z_s\}; (u''_{s})^*T_X).$ (This function space is introduced in Definition 3.6.)

3. $V_D = (V_D, (\tau_{\infty,D,d}, g_{\infty,D,d}), (\tau_{\infty,D,s}, g_{\infty,D,s}), v_{D,d}, v_{D,s}) \in W_{m,d}^{2}(\Sigma_D \setminus \{z_d, z_s\}; (U''_{D})^*T(\mathbb{R}_s \times SN_D(X))).$ (This function space is introduced in Definition 3.8.)

4. We assume $v_d = v_{D,d}, \quad v_s = v_{D,s}.$

The space $W_{m,d}^{2}(u'', TX)$ is a linear subspace of finite codimension of the direct sum of three Hilbert spaces defined in Definitions 3.3, 3.6, 3.8. Therefore, it is also a Hilbert space.

We regard $\mathbb{R} \oplus \mathbb{R}$ as the subspace of $W_{m,d}^{2}(\Sigma_D \setminus \{z_d, z_s\}; (U''_{D})^*T(\mathbb{R} \times SN_D(X)))$ given by constant sections with values in $\mathbb{R} \oplus \mathbb{R} \subset T(\mathbb{R}_s \times SN_D(X)).$ Thus $\mathbb{R} \oplus \mathbb{R}$ can be also regarded as a subspace of $W_{k,d}^{2}(u'', TX).$

We define $W_{m,d}^{2}(u'', TX)$ to be the quotient space of $W_{m,d}^{2}(u'', TX)$ by this copy of $\mathbb{R} \oplus \mathbb{R}.$

**Remark 7.13.** We do not assume $\tau_{\infty,d} = \tau_{\infty,D,d}$ or $\tau_{\infty,s} = \tau_{\infty,D,s}.$ The fact that we might have $\tau_{\infty,d} \neq \tau_{\infty,D,d}$ or $\tau_{\infty,s} \neq \tau_{\infty,D,s}$ is related to the shift of $\rho_1, \rho_2,$ which we mentioned in Subsection 7.5.

**Definition 7.14.** Let $L_{m,d}^{2}(u'', TX \otimes \Lambda^{0,1})$ be the direct sum of the three Hilbert spaces:

\[L_{m,d}^{2}(\Sigma_d \setminus \{z_d\}; (u''_{d})^*T_X \otimes \Lambda^{0,1}) \quad \oplus \quad L_{m,s}^{2}(\Sigma_s \setminus \{z_s\}; (u''_{s})^*T_X \otimes \Lambda^{0,1}) \quad \oplus \quad L_{m,d}^{2}(\Sigma_D \setminus \{z_d, z_s\}; (U''_{D})^*T(\mathbb{R}_s \times SN_D(X)) \otimes \Lambda^{0,1}),\]

introduced in Definitions 3.4, 3.6, 3.8. The three operators (3.8), (3.11), (3.15) together induce a Fredholm operator:

\[(7.49) \quad D_w^{\mathcal{J}} : W_{m+1,d}^{2}(u'', TX) \to L_{m,d}^{2}(u'', TX \otimes \Lambda^{0,1}).\]
Remark 7.15. If $u''_s, u''_a, u''_D$ are $C^1$-close to $u'_s, u'_a, u'_D$, then the surjectivity of (3.8), (3.11), (3.15) (for $u'_s, u'_a, u'_D$) modulo $E_d(u'_a) \oplus E_s(u'_s) \oplus E_D(u'_D)$ and the mapping transversality condition of Definition 4.2 imply that (7.49) is surjective modulo the obstruction space $E_d(u''_a) \oplus E_s(u''_a) \oplus E_D(u''_D)$. (See also Lemma 7.17.)

Lemma 7.16. The triple:

$$\Err^{\xi}_{\sigma_1,\sigma_2,0} := (\Err^{\xi}_{d,\sigma_1,\sigma_2,0}, \Err^{\xi}_{s,\sigma_1,\sigma_2,0}, \Err^{\xi}_{D,\sigma_1,\sigma_2,0})$$

determines an element of $L^2_{m,\delta}(u''_{\sigma_1,\sigma_2,0}; TX \otimes \Lambda^{0,1})$. The terms above are defined in (7.39).

Proof. It follows from the fact that the map $u''_{d,\sigma_1,\sigma_2,0}$ (resp. $u''_{s,\sigma_1,\sigma_2,0}$, $u''_{D,\sigma_1,\sigma_2,0}$) coincides with $u'_{s,\sigma_1,\sigma_2,0}$ (resp. $u'_{s,\sigma_1,\sigma_2,0}$, $(\text{Dil}_{\xi,s})^{-1} \circ u'_{s,\sigma_1,\sigma_2,0}$) on the support of $\Err^{\xi}_{d,\sigma_1,\sigma_2,0}$ (resp. $\Err^{\xi}_{s,\sigma_1,\sigma_2,0}$, $\Err^{\xi}_{D,\sigma_1,\sigma_2,0}$). \hfill \Box

Lemma 7.17. Let the linear operator

$$L : W^2_{m+1,\delta}(u''_{\sigma_1,\sigma_2,0}; TX) \oplus E_d \oplus E_s \oplus E_D \rightarrow L^2_{m,\delta}(u''_{\sigma_1,\sigma_2,0}; TX \otimes \Lambda^{0,1})$$

be given as follows:

$$L(\mathbf{V}, \mathbf{f}) = D_{u''_{\sigma_1,\sigma_2,0}} \mathfrak{J}(\mathbf{V}) - (D_{u''_{\sigma_1,\sigma_2,0}} E)(\mathbf{c}^{\xi}_{(0)}, \mathbf{V}) - \mathbf{f}$$

where the terms of $\mathbf{c}^{\xi}_{(0)} := (\mathfrak{c}^{\xi}_{d,(0)}, \mathfrak{c}^{\xi}_{s,(0)}, \mathfrak{c}^{\xi}_{D,(0)})$ are defined in (7.26). The term $(D_{u''_{\sigma_1,\sigma_2,0}} E)(\mathbf{c}^{\xi}_{(0)}, \mathbf{V})$ is defined similar to the corresponding term in (7.38). If $\sigma_1$ and $\sigma_2$ are small enough, then there is a continuous operator

$$Q : L^2_{m,\delta}(u''_{\sigma_1,\sigma_2,0}; TX \otimes \Lambda^{0,1}) \rightarrow W^2_{m+1,\delta}(u''_{\sigma_1,\sigma_2,0}; TX) \oplus E_d \oplus E_s \oplus E_D$$

which is a right inverse to $L$. Let $(\mathcal{Q}, Q_d, Q_s, Q_D)$ be the components of $Q$ with respect to the decomposition of the target of $Q$. There is also constant $C$, independent of $\sigma_1, \sigma_2$ and $\xi$, such that for any $z \in L^2_{m,\delta}(u''_{\sigma_1,\sigma_2,0}; TX \otimes \Lambda^{0,1})$:

$$||\mathcal{Q}(z)||_{W^2_{m+1,\delta}} + |Q_d(z)| + |Q_s(z)| + |Q_D(z)| \leq C ||z||_{L^2_{m,\delta}}. \tag{7.50}$$

Moreover, we can make this choose of $Q$ unique by demanding that its image is $L^2$-orthogonal\textsuperscript{17} to the subspace $\ker(L)$.\textsuperscript{18}

\textsuperscript{17}We use the $L^2$ norm on the target of $Q$ given by $W^2_{m,\delta}$ with $m = 0$ and $\delta = 0$.

\textsuperscript{18}The last condition is similar to [FOOO5, Definition 5.9].
Proof. Using Definition 4.1 (2), (3), (4) and Definition 4.2, we can construct a continuous operator:

\[ Q_0 = (Q_{0,d}, Q_{0,s}, Q_{0,D}, Q_{0,E}) : L^2_m(\Sigma_d \setminus \{z_d\}; u_d^* TX \otimes \Lambda^{0,1}) \]
\[ \oplus L^2_m(\Sigma_s \setminus \{z_s\}; u_s^* TX \otimes \Lambda^{0,1}) \]
\[ \oplus L^2_m(\Sigma_D \setminus \{z_d, z_s\}; U_D^* TX \otimes \Lambda^{0,1}) \]
\[ \rightarrow W^2_{m+1,\delta}(\Sigma_d \setminus \{z_d\}; (u_d^* TX, u_d^* TL)) \]
\[ \oplus W^2_{m+1,\delta}(\Sigma_s \setminus \{z_s\}; u_s^* TX) \]
\[ \oplus W^2_{m+1,\delta}(\Sigma_D \setminus \{z_d, z_s\}; U_D^* T(\mathbb{R} \times SN_D(X))) \]
\[ \oplus E_d \oplus E_s \oplus E_D \]

such that:

\[ EV_{0,d} \circ Q_{0,d} = EV_{0,d} \circ Q_{0,D} \]
\[ EV_{0,s} \circ Q_{0,s} = EV_{0,d} \circ Q_{0,D} \]

and:

\[ (D_{u_d} \partial Q_{0,d}, D_{u_s} \partial Q_{0,s}, D_{U_D} \partial Q_{0,D}) = Q_{0,E}. \]

We use appropriate bump functions to obtain the operator:

\[ Q_1 : L^2_{m,\delta}(u_{\Sigma_1,\Sigma_2}^\xi(0); TX \otimes \Lambda^{0,1}) \rightarrow W^2_{m+1,\delta}(u_{\Sigma_1,\Sigma_2}^\xi(0); TX) \oplus E_d \oplus E_s \oplus E_D \]

(See [FOOO2, (7.1.23)] for more details. Note that [FOOO2, (7.1.23)] concerns with the case of gluing two irreducible components. Here we glue three components. However, the construction is essentially the same.) In the same way as in [FOOO2, Lemma 7.1.29], we can show:

\[ \|L \circ Q_1(z) - z\| \leq Ce^{-c_1 \min\{T_1, T_2\}}\|z\|, \quad \|Q_1(z)\| \leq C'\|z\|. \]

Here \(c, C, C' > 0\) are constants independent of \(\sigma_1\) and \(\sigma_2\). Thus for \(\sigma_1\) and \(\sigma_2\) small enough, we may define:

\[ Q_2 = \sum_{k=0}^{\infty} (-1)^k Q_1 \circ (id - L \circ Q_1)^k. \]

Then we have:

\[ (L \circ Q_2)(z) = id, \quad \|Q_2(z)\| \leq C''\|z\|. \]

(See [FOOO5, (6.68)]. This formula is used there to estimate derivatives of the right inverse with respect to the gluing parameter.) The operator \(Q_2\) has the required properties except the last one. To obtain the right inverse which also satisfies the last condition, we compose \(Q_2\) with projection to the orthogonal complement of the finite dimensional space \(\ker(L)\). \(\square\)

Let \(z := Err_{\Sigma_1,\Sigma_2}^\xi(0)\). By Lemma 7.16, we know that \(z\) belongs to the target of \(L\). Therefore, \(\overline{Q}(z)\) determines a triple as follows:

\[ V_{\Sigma_1,\Sigma_2}^\xi(1) = (V_{d,\Sigma_1,\Sigma_2}^\xi(1), V_{s,\Sigma_1,\Sigma_2}^\xi(1), V_{D,\Sigma_1,\Sigma_2}^\xi(1)) \in W^2_{m+1,\delta}(u_{\Sigma_1,\Sigma_2}^\xi(0); TX) \]
Moreover, we have:
\[
\Delta c_{d,s,\sigma_1,\sigma_2}(1) := Q_d(z), \quad \Delta c_{s,s,\sigma_1,\sigma_2}(1) := Q_s(z), \quad \Delta c_{D,s,\sigma_1,\sigma_2}(1) := Q_D(z).
\]

Lemmas 7.4 and 7.17 imply that:
\[
\|V^\xi_{\sigma_1,\sigma_2}(1)\|_{W^{m+1,\delta}} \leq Ce^{-c\delta_1\min\{T_1,T_2\}}
\]
and
\[
|\Delta c_{d,s,\sigma_1,\sigma_2}(1)|, |\Delta c_{s,s,\sigma_1,\sigma_2}(1)|, |\Delta c_{D,s,\sigma_1,\sigma_2}(1)| \leq Ce^{-c\delta_1\min\{T_1,T_2\}}.
\]

In summary, we obtained a solution of the linearized equation with appropriate decay properties.

(Step 1-2) (Gluing solutions)

In this step we will use \(V^\xi_{\sigma_1,\sigma_2}(1)\) to obtain an improved approximate inconsistent solution. Suppose the entries of \(V^\xi_{\sigma_1,\sigma_2}(1)\) are given as follows:
\[
\begin{align*}
V^\xi_{d,s,\sigma_1,\sigma_2}(1) &= (r^\xi_{d,s,\sigma_1,\sigma_2}(1), g^\xi_{d,d,s,\sigma_1,\sigma_2}(1), v^\xi_{d,d,s,\sigma_1,\sigma_2}(1), 1), \\
V^\xi_{s,s,\sigma_1,\sigma_2}(1) &= (r^\xi_{s,s,\sigma_1,\sigma_2}(1), g^\xi_{s,s,s,\sigma_1,\sigma_2}(1), v^\xi_{s,s,s,\sigma_1,\sigma_2}(1), 1), \\
V^\xi_{D,s,\sigma_1,\sigma_2}(1) &= (r^\xi_{D,s,\sigma_1,\sigma_2}(1), g^\xi_{D,D,s,\sigma_1,\sigma_2}(1), v^\xi_{D,D,s,\sigma_1,\sigma_2}(1), 1).
\end{align*}
\]

We also have the following identities:
\[
\begin{align*}
u^\xi_{\infty,d,s,\sigma_1,\sigma_2}(1) &= v^\xi_{\infty,D,d,s,\sigma_1,\sigma_2}(1), \quad \nu^\xi_{\infty,s,s,\sigma_1,\sigma_2}(1) = v^\xi_{\infty,D,s,s,\sigma_1,\sigma_2}(1).
\end{align*}
\]

by Definition 7.12 (4). We also define:
\[
\begin{align*}
\Delta r^\xi_{\infty,d,s,\sigma_1,\sigma_2}(1) &= r^\xi_{\infty,D,d,s,\sigma_1,\sigma_2}(1) - r^\xi_{\infty,d,d,s,\sigma_1,\sigma_2}(1), \\
\Delta g^\xi_{\infty,d,s,\sigma_1,\sigma_2}(1) &= g^\xi_{\infty,D,d,s,\sigma_1,\sigma_2}(1) - g^\xi_{\infty,d,d,s,\sigma_1,\sigma_2}(1), \\
\Delta v^\xi_{\infty,s,s,\sigma_1,\sigma_2}(1) &= v^\xi_{\infty,D,s,s,\sigma_1,\sigma_2}(1) - v^\xi_{\infty,d,s,s,\sigma_1,\sigma_2}(1).
\end{align*}
\]

Definition 7.18. We define \(u^\xi_{d,s,\sigma_1,\sigma_2}(\hat{z}) : \Sigma^+_d(\sigma_1,\sigma_2) \rightarrow X \setminus D\) as follows.

1. If \(\hat{z} \in \Sigma_d(\sigma_1,\sigma_2)\) then
   \[
   u^\xi_{d,s,\sigma_1,\sigma_2}(\hat{z}) = \Exp(u^\xi_{d,s,\sigma_1,\sigma_2,0}(\hat{z}), V^\xi_{d,s,\sigma_1,\sigma_2}(1)).
   \]

2. If \(\hat{z} = (r_1, s_1) \in [-5T_1, 5T_1] \subset S_{s_1}^1\) then
   \[
   u^\xi_{d,s,\sigma_1,\sigma_2}(\hat{z}) = \Exp(u^\xi_{d,s,\sigma_1,\sigma_2,0}(r_1, s_1), \cdot).
   \]
where

\[
\diamond = \chi_{1,2}(r_1, s_1)(V_{\iota_1, \iota_2, 0}^{\xi} - (r_\infty, d, \sigma_1, \sigma_2, 1(1)^{\xi}_{d, \sigma_1, \sigma_2, 1}) - \hat{\nu}_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi})
+ \chi_{A}(r_1, s_1)(V_{D, d, \sigma_1, \sigma_2, 1}^{\xi} - (r_\infty, d, \sigma_1, \sigma_2, 1(1)^{\xi}_{D, d, \sigma_1, \sigma_2, 1}) - \hat{\nu}_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi})
+ (r_\infty, d, \sigma_1, \sigma_2, 1(1)^{\xi}_{\infty, d, \sigma_1, \sigma_2, 1} + \hat{\nu}_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi}).
\]

Here and in Items (4), (6), (7), we extend \(v_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi}\) to \(\hat{\nu}_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi}\)
in the same way as in Definition 3.3.

We define \(u_{s, \sigma_1, \sigma_2, 1}^{\xi, \prime}: \Sigma_{s}^+(\sigma_1, \sigma_2) \rightarrow X \setminus D\) as follows.

(3) If \(\diamond \in \Sigma_{s}^-(\sigma_1, \sigma_2)\) then

\[
\diamond = \chi_{1,2}(r_1, s_1)(V_{s, \sigma_1, \sigma_2, 1}^{\xi} - (r_\infty, s, \sigma_1, \sigma_2, 1(1)^{\xi}_{s, \sigma_1, \sigma_2, 1}) - \hat{\nu}_{\infty, s, \sigma_1, \sigma_2, 1}^{\xi})
+ \chi_{A}(r_1, s_1)(V_{D, s, \sigma_1, \sigma_2, 1}^{\xi} - (r_\infty, s, \sigma_1, \sigma_2, 1(1)^{\xi}_{D, s, \sigma_1, \sigma_2, 1}) - \hat{\nu}_{\infty, s, \sigma_1, \sigma_2, 1}^{\xi})
+ (r_\infty, s, \sigma_1, \sigma_2, 1(1)^{\xi}_{\infty, s, \sigma_1, \sigma_2, 1} + \hat{\nu}_{\infty, s, \sigma_1, \sigma_2, 1}^{\xi}).
\]

We next define \(U_{D, \sigma_1, \sigma_2, 1}^{\xi, \prime}: \Sigma_{D}^+(\sigma_1, \sigma_2) \rightarrow X \setminus \mathcal{D}\) as follows.

(5) If \(\diamond \in \Sigma_{D}^-(\sigma_1, \sigma_2)\) then:

\[
U_{D, \sigma_1, \sigma_2, 1}^{\xi, \prime}(\diamond) = \exp(U_{D, \sigma_1, \sigma_2, 0}^{\xi, \prime}(\diamond), V_{D, \sigma_1, \sigma_2, 1}^{\xi})
\]

(6) If \(\diamond = (r_1, s_1) \in [-5T_1, 5T_1] \times S_{s_1}^1\) then:

\[
U_{D, \sigma_1, \sigma_2, 1}^{\xi, \prime}(\diamond) = \exp(U_{D, \sigma_1, \sigma_2, 0}^{\xi, \prime}(r_1, s_1), \diamond)
\]

where:

\[
\heartsuit = \chi_{1,2}(r_1, s_1)(V_{d, \sigma_1, \sigma_2, 1}^{\xi} - (r_\infty, d, \sigma_1, \sigma_2, 1(1)^{\xi}_{d, \sigma_1, \sigma_2, 1}) - \hat{\nu}_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi})
+ \chi_{A}(r_1, s_1)(V_{D, d, \sigma_1, \sigma_2, 1}^{\xi} - (r_\infty, d, \sigma_1, \sigma_2, 1(1)^{\xi}_{D, d, \sigma_1, \sigma_2, 1}) - \hat{\nu}_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi})
+ (r_\infty, d, \sigma_1, \sigma_2, 1(1)^{\xi}_{\infty, d, \sigma_1, \sigma_2, 1} + \hat{\nu}_{\infty, d, \sigma_1, \sigma_2, 1}^{\xi}).
\]

We remark that:

(7.55) \(\heartsuit - \diamond = (\Delta r_{\infty, d, \sigma_1, \sigma_2, 1(1)}^{\xi}, \Delta \sigma_{\infty, d, \sigma_1, \sigma_2, 1(1)}^{\xi})\).

(7) If \(\diamond = (r_2, s_2) \in [-5T_2, 5T_2] \times S_{s_2}^1\) then:

\[
U_{D, \sigma_1, \sigma_2, 1}^{\xi, \prime}(\diamond) = \exp(U_{D, \sigma_1, \sigma_2, 0}^{\xi, \prime}(r_2, s_2), \heartsuit)
\]
where:
\[ \blacklozenge = \chi_{2,g}(r_2, s_2) (V_{s,\sigma_1, \sigma_2, (1)}^\xi - (r_{\infty, s, \sigma_1, \sigma_2, (1)}^\xi, s_{\infty, s, \sigma_1, \sigma_2, (1)}) - \tilde{\nu}_{\infty, s, \sigma_1, \sigma_2, (1)}) + \chi_A(r_2, s_2) (V_{D, s, \sigma_1, \sigma_2, (1)}^\xi - (r_{\infty, D, s, \sigma_2, \sigma_2, (1)}^\xi, s_{\infty, D, s, \sigma_1, \sigma_2, (1)}) - \tilde{\nu}_{\infty, D, s, \sigma_2, \sigma_2, (1)}) + (r_{\infty, D, s, \sigma_1, \sigma_2, (1)}^\xi, s_{\infty, D, s, \sigma_1, \sigma_2, (1)}) + \tilde{\nu}_{\infty, D, s, \sigma_1, \sigma_2, (1)}) \]

We remark that:
\[ (7.56) \blacklozenge - \blacklozenge = (\Delta r_{\infty, s, \sigma_1, \sigma_2, (1)}, \Delta s_{\infty, s, \sigma_1, \sigma_2, (1)}) \]

Let:
\[ (7.57) \rho_{1,(1)}^\xi = \exp(- (\Delta r_{\infty, d, \sigma_1, \sigma_2, (1)}^\xi + \sqrt{-1}\Delta s_{\infty, d, \sigma_1, \sigma_2, (1)})) \in \mathbb{C}_* \]
\[ (7.58) \rho_{1,(1)}^\xi = \rho_{1,(0)}^\xi \rho_{1,(1)}^\Delta \in \mathbb{C}_* \]

for \( i = 1, 2 \). Finally, we define:
\[ (7.59) u_{\sigma_1, \sigma_2, (1)}^{\xi, t} = (u_d, u_s, u_{d, \sigma_1, \sigma_2, (1)}, u_{\sigma_1, \sigma_2, (1)}, u_{D, \sigma_1, \sigma_2, (1)}, \sigma_1, \sigma_2, \rho_{1,(1)}^\xi, \rho_{2,(1)}^\xi) \]

**Lemma 7.19.** The 7-tuple \( u_{\sigma_1, \sigma_2, (1)}^{\xi, t} \) is an inconsistent map in the sense of Definition 7.5.

**Proof.** This is a consequence of Lemma 7.6 and (7.55), (7.56). \( \square \)

**Remark 7.20.** We remark that if we change \( V_{D, s, \sigma_1, \sigma_2, (1)}^\xi \) by an element of \( \mathbb{R} \oplus \mathbb{R} \) (the tangent vector generated by the \( \mathbb{C}_* \) action), then \( \rho_{1, (1)}^\xi \) and \( \rho_{2, (1)}^\xi \) change by the same amount. Therefore, \( \blacklozenge \) and \( \blacklozenge \) do not change. On the other hand, \( \blacklozenge \) and \( \blacklozenge \) change by the same element in \( \mathbb{R} \oplus \mathbb{R} \). This implies that the equivalence class of \( u_{\sigma_1, \sigma_2, (1)}^{\xi, t} \) is fixed among all representatives for \( V_{D, s, \sigma_1, \sigma_2, (1)}^\xi \).

**Step 1-3 (Error estimate)**

\( u_{\sigma_1, \sigma_2, (1)}^{\xi, t} \) is our next approximate solution. Lemma 7.21 quantifies to what extent this inconsistent map improves the previous approximate solution \( u_{\sigma_1, \sigma_2, (0)}^{\xi, t} \).

**Lemma 7.21.** There is a constant \( C \) and for any positive number \( \mu \), there is a constant \( \eta \) such that the following holds. If \( \sigma_1, \sigma_2 \) are smaller than \( \eta \), then there exists \( \epsilon_{\sigma_1, \sigma_2, (1)}^\xi = (\epsilon_{d, \sigma_1, \sigma_2, (1)}^\xi, \epsilon_{s, \sigma_1, \sigma_2, (1)}^\xi, \epsilon_{D, \sigma_1, \sigma_2, (1)}^\xi, \epsilon_{\sigma_1, \sigma_2, (1)}^\xi) \) with \( \epsilon_{d, \sigma_1, \sigma_2, (1)}^\xi \in E_d, \epsilon_{s, \sigma_1, \sigma_2, (1)}^\xi \in E_s, \epsilon_{D, \sigma_1, \sigma_2, (1)}^\xi \in E_D \) such that the following holds:
\[ \text{(1)} \]
\[ (7.60) \| \nabla u_{\sigma_1, \sigma_2, (1)}^{\xi, t} - \epsilon_{\sigma_1, \sigma_2, (1)}^\xi \|_{L^2_{\mu, \delta}} \leq \mu \| \nabla u_{\sigma_1, \sigma_2, (1)}^{\xi, t} - \epsilon_{(0)}^\xi \|_{L^2_{\mu, \delta}} \]

where \( \epsilon_{(0)}^\xi = (\epsilon_{d,(0)}^\xi, \epsilon_{s,(0)}^\xi, \epsilon_{D,(0)}^\xi) \) is as in (7.26).
(2) \[ \| e_{\sigma_1,\sigma_2,(1)}^\xi - e_{(0)}^\xi \| \leq \mu C. \]

The square of the left hand side, by definition, is the sum of the squares of the factors associated to \( d, s, D \).

We define:

\[
\begin{align*}
\epsilon_{d,\sigma_1,\sigma_2,(1)} &= \epsilon_{d,\sigma_1,\sigma_2,(0)} + \Delta \epsilon_{d,\sigma_1,\sigma_2,(0)} \\
\epsilon_{s,\sigma_1,\sigma_2,(1)} &= \epsilon_{s,\sigma_1,\sigma_2,(0)} + \Delta \epsilon_{s,\sigma_1,\sigma_2,(0)} \\
\epsilon_{D,\sigma_1,\sigma_2,(1)} &= \epsilon_{D,\sigma_1,\sigma_2,(0)} + \Delta \epsilon_{D,\sigma_1,\sigma_2,(0)}. 
\end{align*}
\]

The proof of the estimates in the lemma is based on Lemma 7.17. That is to say, we use the estimate \((7.50)\) of Lemma 7.17 and the fact that \( V_{\sigma_1,\sigma_2,(1)} \) is given by solving the linearized equation. The details of this estimate is similar to the proof of [FOOO5, Proposition 5.17] and is omitted. In particular, to estimate the effect of the bump function appearing in Definition 7.18 (2), (4), (6) and (7), we use the ‘drop of the weight’ argument, which is explained in detail in [FOOO5, right above Remark 5.21].

Lemma 7.22 below concerns the estimate of the difference between \( u_{\sigma_1,\sigma_2,(1)}^{\xi'} \) and \( u_{\sigma_1,\sigma_2,(0)}^{\xi'} \).

**Lemma 7.22.** Let \( \sigma_1, \sigma_2 \) be small enough such that Lemma 7.17 holds. There is a fixed constant\(^{20}\) \( C \), independent of \( \sigma_1 \) and \( \sigma_2 \), such that:

\[ d_{W_{m+1,\delta}}(u_{\sigma_1,\sigma_2,(1)}^{\xi'}, u_{\sigma_1,\sigma_2,(0)}^{\xi'}) \leq C \| \partial u_{\sigma_1,\sigma_2,(0)}^{\xi'} - \epsilon_{\sigma_1,\sigma_2,(0)}^\xi \|_{L_{m,\delta}}. \]

**Proof.** This is a consequence of Lemma 7.17 and definitions. \( \square \)

**Step 1-4** (Separating error terms into three parts)

We put

\[
\begin{align*}
\text{Err}^{\xi}_{d,\sigma_1,\sigma_2,(1)} &= \chi_{d}(\partial u_{d,\sigma_1,\sigma_2,(1)}^{\xi'} - \epsilon_{d,\sigma_1,\sigma_2,(1)}) \\
\text{Err}^{\xi}_{s,\sigma_1,\sigma_2,(1)} &= \chi_{s}(\partial u_{s,\sigma_1,\sigma_2,(1)}^{\xi'} - \epsilon_{s,\sigma_1,\sigma_2,(1)}) \\
\text{Err}^{\xi}_{D,\sigma_1,\sigma_2,(1)} &= \chi_{D}(\partial u_{D,\sigma_1,\sigma_2,(1)}^{\xi'} - \epsilon_{D,\sigma_1,\sigma_2,(1)}) 
\end{align*}
\]

**Step 2-1** (Approximate solution for linearization)

We will next define

\[ u_{\sigma_1,\sigma_2,(1)}^{\xi''} = (u_{d,\sigma_1,\sigma_2,(1)}^{\xi''}, u_{s,\sigma_1,\sigma_2,(1)}^{\xi''}, u_{D,\sigma_1,\sigma_2,(1)}^{\xi''}) \]

---

\(^{19}\) This estimate is provided for the first step of Newton’s iteration. In the i-th step, a similar estimate appears where \( \mu C \) is replaced by \( \mu^i C \). It is important that \( C \) is independent of \( i \).

\(^{20}\) It is important that we can take the same constant for all the steps of inductive construction of Newton’s iteration. The dependence of those constants to various choices are studied in detail in [FOOO5]. So we do not repeat it here.
We next define formulas similar to (7.18), (7.19), (7.22) and (7.23) hold.

\[ p_{d,\sigma_1,\sigma_2}(1) = \exp(p_{d,\sigma_1,\sigma_2}(0) + \xi \cdot v_{\infty, d, \sigma_1, \sigma_2}) \]

(7.64)

\[ p_{s,\sigma_1,\sigma_2}(1) = \exp(p_{s,\sigma_1,\sigma_2}(0) + \xi \cdot v_{\infty, s, \sigma_1, \sigma_2}) \]

We next define

\[ c_{d(1)}^\xi = c_{d,\xi} \exp(-r_{\infty, d, \sigma_1, \sigma_2}(1) + \sqrt{-1}s_{\infty, d, \sigma_1, \sigma_2}(1)) \]

\[ c_{D,d(1)}^\xi = c_{D,d,\xi} \exp(-r_{\infty, D,d, \sigma_1, \sigma_2}(1) + \sqrt{-1}s_{\infty, D,d, \sigma_1, \sigma_2}(1)) \]

\[ c_{s(1)}^\xi = c_{s,\xi} \exp(-r_{\infty, s, \sigma_1, \sigma_2}(1) + \sqrt{-1}s_{\infty, s, \sigma_1, \sigma_2}(1)) \]

\[ c_{D,s(1)}^\xi = c_{D,s,\xi} \exp(-r_{\infty, D,s, \sigma_1, \sigma_2}(1) + \sqrt{-1}s_{\infty, D,s, \sigma_1, \sigma_2}(1)) \]

Then formulas similar to (7.18), (7.19), (7.22) and (7.23) hold.

In (7.42),(7.43),(7.45),(7.46), we replace \( c_{d}^\xi \) with \( c_{d(1)}^\xi \) and so on. In these formulas, we also replace (0) with (1). We thus define \( u_{d,\sigma_1,\sigma_2}^{\xi''} \), \( u_{s,\sigma_1,\sigma_2}^{\xi''} \), \( U_{D,\sigma_1,\sigma_2}^{\xi''} \) and \( u_{s,\sigma_1,\sigma_2}^{\xi''} \). Then \( (\bar{u}_{d,\sigma_1,\sigma_2}^{\xi''}(1), \bar{u}_{s,\sigma_1,\sigma_2}^{\xi''}(1), \bar{U}_{D,\sigma_1,\sigma_2}^{\xi''}(1)) \) determines an element of the space \( L_{1,2,\delta}^2(u_{\sigma_1,\sigma_2}^{\xi''}(1); TX \otimes \Lambda^{0,1}) \) (Lemma 7.16.)

We can then formulate an analogue of Lemma 7.17 where \( u_{\sigma_1,\sigma_2}^{\xi''}(0) \) is replaced by \( u_{\sigma_1,\sigma_2}^{\xi''}(1) \). Using this lemma, we can obtain \( V_{\sigma_1,\sigma_2}^{\xi''} \), \( \Delta_{D,\sigma_1,\sigma_2}^{\xi''} \), \( \Delta_{D,\sigma_1,\sigma_2}^{\xi''} \), \( \Delta_{D,\sigma_1,\sigma_2}^{\xi''} \). The counterpart of the estimate in (7.50) can be used to give appropriate bounds for these four terms. This completes (Step 2-1). (Step 2-2) and (Step 2-3) can be carried out in the same way as in (Step 1-2) and (Step 1-3).

More generally, we can perform (Step 1-1), (Step 1-2) and (Step 1-3) in the case that \( |\sigma_1|, |\sigma_2| \) are smaller than a positive number \( \epsilon_0 \) and obtain a sequence of inconsistent maps:

\[ u_{\sigma_1,\sigma_2}^{\xi}(i) := (u_{d,\sigma_1,\sigma_2}^{\xi}(i), u_{s,\sigma_1,\sigma_2}^{\xi}(i), U_{D,\sigma_1,\sigma_2}^{\xi}(i), \sigma_1, \sigma_2, \rho_1^{\xi}, \rho_2^{\xi}) \]

and a triple:

\[ \epsilon_{\sigma_1,\sigma_2}(i) = (\epsilon_{d,\sigma_1,\sigma_2}(i), \epsilon_{s,\sigma_1,\sigma_2}(i), \epsilon_{D,\sigma_1,\sigma_2}(i)) \in E_d \oplus E_s \oplus E_D \]

such that

\[ \| \bar{u}_{\sigma_1,\sigma_2}(i^{\xi}(i+1)) - \epsilon_{\sigma_1,\sigma_2}(i+1) \|^2_{L^2_{m,d}} \leq \mu \| \bar{u}_{\sigma_1,\sigma_2}(i^{\xi}(i+1)) - \epsilon_{\sigma_1,\sigma_2}(i) \|^2_{L^2_{m,d}} \]

(7.65)

and

\[ \| \bar{u}_{\sigma_1,\sigma_2}(i^{\xi}(i+1)) - \epsilon_{\sigma_1,\sigma_2}(i) \| \leq \mu^i C. \]

(7.66)
Moreover, we have:

\begin{equation}
(7.67) \quad d_{W^{2}_{m+1,\delta}}(u_{\sigma_{1},\sigma_{2},(i+1)}, u_{\sigma_{1},\sigma_{2},(i)}) \leq C \| \tilde{\partial u}_{\sigma_{1},\sigma_{2},(i)}^{\xi'} - e_{\sigma_{1},\sigma_{2}} \|_{L_{m,\delta}^{2}}.
\end{equation}

We make a remark that the constants \(c_0\) and \(C\) may be taken independent of \(i\). But these constants might depend on \(m\), the exponent in the weighted Sobolev space \(L_{m,\delta}^{2}\).

The estimates in (7.65) and (7.67) imply that the sequence \(\{u_{\sigma_{1},\sigma_{2},(i)}^{\xi'}\}\) converges in \(W^{2}_{m+1,\delta}\). We denote the limit by:

\begin{equation}
(7.68) \quad u_{\sigma_{1},\sigma_{2},(\infty)}^{\xi'} := (u_{d,\sigma_{1},\sigma_{2},(\infty)}, u_{s,\sigma_{1},\sigma_{2},(\infty)}, U_{D,\sigma_{1},\sigma_{2},(\infty)}, \sigma_{1}, \sigma_{2}, \rho_{D,\sigma_{1},\sigma_{2},(\infty)}^{\xi'}).
\end{equation}

The estimate in (7.66) implies that \(e_{d,\sigma_{1},\sigma_{2},(i)}^{\xi} \in E_{d,\sigma_{1},\sigma_{2},(i)}\), \(e_{s,\sigma_{1},\sigma_{2},(i)}^{\xi} \in E_{s}\), \(\xi_{D,\sigma_{1},\sigma_{2},(i)}^{\xi} \in E_{D}\) converges as \(i\) goes to infinity. We denote the limit by:

\[ e_{\sigma_{1},\sigma_{2},(\infty)}^{\xi} = (e_{d,\sigma_{1},\sigma_{2},(\infty)}^{\xi}, e_{s,\sigma_{1},\sigma_{2},(\infty)}^{\xi}, e_{D,\sigma_{1},\sigma_{2},(\infty)}^{\xi}). \]

As a consequence of (7.65), we have:

\[ \| \tilde{\partial u}_{\sigma_{1},\sigma_{2},(\infty)}^{\xi'} - e_{\sigma_{1},\sigma_{2}} \|_{L_{m,\delta}^{2}} = 0. \]

In other words, \(u_{\sigma_{1},\sigma_{2},(\infty)}^{\xi'}\) satisfies (4.19). Thus \(u_{\sigma_{1},\sigma_{2},(\infty)}^{\xi'}\) satisfies the requirements of an inconsistent solution (Definition 5.1) except possibly (4.20) (the transversal constraint).

7.8. Completion of the Proof. We are now in the position to complete the proof of Proposition 5.3. For any \(\xi \in U_{d,\sigma_{1},\sigma_{2},(\infty)}^{\xi} \times U_{D,\sigma_{1},\sigma_{2},(\infty)}^{\xi} \times U_{s,\sigma_{1},\sigma_{2},(\infty)}^{\xi}\) and sufficiently small \(\sigma_{1}, \sigma_{2} \in \mathbb{C}\), we defined an inconsistent map \(u_{\sigma_{1},\sigma_{2}}^{\xi'}\) in (7.68). For \((\sigma_{1}, \sigma_{2})\) we define

\[ \text{EV}_{\sigma_{1},\sigma_{2}} : U_{d,\sigma_{1},\sigma_{2},(\infty)}^{+} \times U_{D,\sigma_{1},\sigma_{2},(\infty)}^{+} \times U_{s,\sigma_{1},\sigma_{2},(\infty)}^{+} \to D \times X^{2} \]

by:

\[ \text{EV}_{\tau_{1},\tau_{2}}(\xi) = ((\pi \circ U_{D,\sigma_{1},\sigma_{2},(\infty)}^{\xi'})(w_{D,1}), (\pi \circ U_{D,\sigma_{1},\sigma_{2},(\infty)}^{\xi'})(w_{D,2}), U_{s,\sigma_{1},\sigma_{2},(\infty)}^{\xi'}(w_{s})). \]

Here \(w_{D,1}, w_{D,2}, w_{s}\) are as in Condition 4.3.

**Lemma 7.23.** \(\text{EV}_{\sigma_{1},\sigma_{2}}\) is transversal to \(\dot{N}_{D,1} \times N_{D,2} \times N_{s}\) for sufficiently small \(\sigma_{1}, \sigma_{2}\).

**Proof.** The map \(\text{EV}_{\sigma_{1},\sigma_{2}}\) converges to \(\text{EV}_{0,0}\) in the \(C^{1}\) sense as \(\sigma_{1}, \sigma_{2} \to 0\). Moreover, \(\text{EV}_{0,0}\) is transversal to \(\dot{N}_{D,1} \times N_{D,2} \times N_{s}\) by assumption (Definition 4.2). The lemma follows from these observations. \(\square\)
By definition
\[
\bigcup_{\sigma_1, \sigma_2} (EV_{w_{\sigma_1, \sigma_2}})^{-1}(N_{D,1} \times N_{D,2} \times N_s) \times \{(\sigma_1, \sigma_2)\}
\]
can be identified with \(U\), the set of inconsistent solutions. We also have:
\[(EV_{w_{0,0}})^{-1}(N_{D,1} \times N_{D,2} \times N_s) \cong U_{d_{ev_{D,1}}} \times U_{d_{ev_{D,2}}} \times U_s\]
by definition. Proposition 5.3 is a consequence of these facts.

Once Proposition 5.3 is proved the proof of Proposition 5.5 is similar to the proof of [FOOO5, Theorem 6.4]. We have written the proof of Proposition 5.3 so that the construction of the inconsistent solutions are parallel to the gluing construction in [FOOO5, Section 5]. Therefore, the proof of [FOOO5, Section 6] can be applied with almost no change to prove Proposition 5.5. This completes the construction of the Kuranishi chart at the point \([\Sigma, z_0, u]\).

8. Kuranishi Charts: the General Case

Up to point, we constructed a Kuranishi chart of the space \(M^{k+1}_{R,\mathbb{R},G,W}(L; \beta)\) at the particular point \([\Sigma, z_0, u]\) described in Section 2. In this section, we explain how this construction generalizes to an arbitrary point \(u\) of \(M^{k+1}_{R,\mathbb{R},G,W}(L; \beta)\). There is a DD-ribbon tree \(R = (R, c, \alpha, m, \lambda)\) such that \(u\) belongs to \(M^0(R) \subset M^{k+1}_{R,\mathbb{R},G,W}(L; \beta)\). (See [DF2, Subsection 3.6]). Let \((([\Sigma_v, z_v, u_v]; v \in C^0_{\text{int}}(R))\) be a representative for \(u\). In the case that \(c(v) = D\), the image of \(u_v\) is contained in \(D\), and we are given a meromorphic section \(U_v\) of \(u_v^*N_D(X)\). Recall that for each \(i\), the set of all sections \(\{U_v\}_{\lambda(v)=i}\) is well-defined up to an action of \(C_\ast\) [DF2, Formula (3.38)]. We firstly, associate a combinatorial object to \(u\) which is called a very detailed DD-ribbon tree and is the refinement of the notion of detailed DD-ribbon trees defined in [DF2, Subsection 3.6].

Let \(\hat{R}\) be the detailed tree associated to \(R\). Recall that each interior vertex \(v\) of \(\hat{R}\) corresponds to a possibly nodal Riemann surface \(\Sigma_v\). (See, for example, [DF2, Figure 9].) We refine the detailed DD-ribbon tree \(\hat{R}\) further to the very detailed DD-ribbon tree \(\hat{R}\) so that each vertex of \(\hat{R}\) corresponds to an irreducible component of \(\Sigma\). To be more detailed, for each \(v \in C^0_{\text{int}}(\hat{R})\), we form a tree \(Q_v\) such that:

(1) Each vertex corresponds to either an irreducible component of \(\Sigma_v\) or a marked point on it. The latter corresponds to an edge of \(\hat{R}\), which contains \(v\). We call any such vertex an exterior vertex.

(2) There are two types of edges in \(Q_v\). An edge of the first type joins two edges such that the corresponding irreducible components intersect. An edge of the second type is called an exterior edge and connects a vertex corresponding to a marked point to the vertex corresponding to the irreducible component containing the marked point.

We replace each interior vertex \(v\) of the detailed tree \(\hat{R}\) with \(Q_v\) and identify exterior edges of \(Q_v\) with the corresponding edges of \(\hat{R}\) containing \(v\). We
thus obtain a tree $\hat{R}$, called the very detailed DD-ribbon tree associated to $u$, or the very detailed tree associated to $u$ for short. Figure 12 sketches an element $u$ of our moduli space. The associated detailed DD-ribbon tree $\hat{R}$ and the very detailed DD-ribbon tree $\check{R}$ are given in Figures 13 and 14.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{An element of the moduli space $M_{k+1}^{RGW}(L; \beta)$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13.png}
\caption{The detailed DD-ribbon tree $\check{R}$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{The very detailed DD-ribbon tree $\check{R}$}
\end{figure}

We say an edge of $\hat{R}$ is a \textit{fine edge} if it does not correspond to an edge of the detailed DD-ribbon tree $\check{R}$. In Figure 14, the fine edges are illustrated...
by narrow lines and level 0 edges are illustrated by dotted lines. An edge of \( \tilde{R} \), which is not fine, is called a **thick edge**. We denote by \( C_{\text{th}}^{\text{int}}(\tilde{R}) \) and \( C_{\text{th}}^{\text{int}}(R) \) the set of all fine and thick edges of \( \tilde{R} \), respectively. The level of a vertex of \( \tilde{R} \) induced by a vertex of \( Q_v \) is defined to be \( \lambda(v) \). We do not associate a multiplicity number to a fine edge. Homology class of a vertex is the homology class of the map \( u \) on this component. The color of an interior vertex \( v \) of \( \tilde{R} \), denoted by \( c(v) \), is D if its level is positive. If this vertex has level 0, then its color is either s or d depending on whether \( \Sigma_v \) is a sphere or a disk.

The notion of level shrinking and level 0 edge shrinking for very detailed DD-ribbon trees can be defined as in the case of detailed DD-ribbon trees. We define a **fine edge shrinking** as follows. We remove a fine edge \( e \) and identify the two vertices connected to each other by \( e \). For two very detailed DD-ribbon trees \( \tilde{R}, \tilde{R}' \), we say \( \tilde{R}' \leq \tilde{R} \) if \( \tilde{R} \) is obtained from \( \tilde{R}' \) by a sequence of level shrinkings, level 0 edge shrinkings and fine edge shrinkings. Note that there might be a fine edge joining two vertices of level 0. We do **not** call any such edge a level 0 edge. The level 0 edges are limited to those joining vertices of color d.

We can stratify the moduli space \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) using very detailed DD-ribbon trees \( \tilde{R} \). Namely, we define \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)(\tilde{R}) \) to be the subset of \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) consists of elements \( u \) whose associated very detailed DD-ribbon tree is \( \tilde{R} \). If \( \tilde{R}' \leq \tilde{R} \), then the closure of \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)(\tilde{R}) \) contains \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)(\tilde{R}') \).

Let \( u = ((\Sigma_v, z_v, u_v); v \in C_{\text{th}}^{\text{int}}(\tilde{R})) \) be an element of \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)(\tilde{R}) \) as above. For an interior vertex \( v \) of \( \tilde{R} \), the triple \( u_v = (\Sigma_v, z_v, u_v) \) is stable by definition. If \( (\Sigma_v, z_v) \) is not stable, then we may add auxiliary interior marked points \( \tilde{w}_v \) so that \( (\Sigma_v, z_v \cup \tilde{w}_v) \) is stable. We assume that \( \tilde{w}_v \) is chosen such that the following symmetry assumption holds. Suppose \( \Gamma_u \) denotes the group of automorphisms of \( u \). Given \( \gamma \in \Gamma_u \), for each interior vertex \( v \), there exists a vertex \( \gamma(v) \) and a bi-holomorphic map \( \gamma_v : (\Sigma_v, z_v) \to (\Sigma_{\gamma(v)}, z_{\gamma(v)}) \) such that \( u_{\gamma(v)} \circ \gamma_v = u_v \). We assume \( \tilde{w}_v \) is mapped to \( \tilde{w}_{\gamma(v)} \) via \( \gamma_v \). Note that the case \( \gamma(v) = v \) is also included. For each member \( w_{v,i} \) of \( \tilde{w}_v \), we take a codimension 2 submanifold \( N_{v,i} \) of \( X \) (resp. \( D \)) if \( c(v) = d \) or s (resp. if \( c(v) = D \)). We assume that the same condition as Condition 4.3 holds for these choices of transversals. If \( \gamma \in \Gamma_u \) and \( \gamma_v(w_{v,i}) = w_{v',i'} \), then we require that \( N_{v,i} = N_{v',i'} \).

In order to define Cauchy-Riemann operators, we introduce function spaces similar to those of Section 3. For an interior vertex \( v \) of \( \tilde{R} \), if the color of \( v \) is d, s or D, the Hilbert space \( W^2_{m,\delta}(u_v; T) \) is respectively defined as in Definition 3.3, Definition 3.6 or Definition 3.8. Here \( T \) is a placeholder for the pull-back of the tangent bundle of \( X \) (if \( c(v) = d, s \)) or \( N_D X \setminus D \) (if \( c(v) = D \)). Similarly, for each \( v \), we define the Weighted Sobolev spaces \( L^2_{m,\delta}(u_v; T \otimes \Lambda^{0,1}) \) as in Section 3.
Remark 8.1. In the case that $c(v) = d$, the space $\Sigma_v$ may have boundary nodes. In that case we take cylindrical coordinates on a neighborhood of each boundary node and use a cylindrical metric on this neighborhood. The approach here is very similar to the case of interior nodes which is discussed in Section 3. See [FOOO5, Section 3] for the case of boundary nodes in the context of the stable map compactification.

We also need to fix cylindrical coordinates for nodes corresponding to fine edges. In this case the target of the corresponding cylindrical end is contained in a compact subset of $X \setminus D$ (for fine edges connecting level 0 vertices) or $N_D X \setminus D$ (for fine edges connecting positive level vertices). In the first case we use the metric $g$ given in Section 3. In the latter case, we use the metric on $N_D X \setminus D$ with the form given in (3.2).

Let $W_{m,d}^2,\sim(u; T)$ be the subspace of the direct sum:

$$\bigoplus_{v \in C_0^{\text{int}}(\tilde{R})} W_{m,d}^2(u_v; T)$$

consisting of elements $(V_v; v \in C_0^{\text{int}}(\tilde{R}))$ with the following properties. Let $e$ be an interior edge of $\tilde{R}$ joining $v_1$ and $v_2$. The source curve of the element $u_v$ contains a nodal point $z_{v_1,e}$ corresponding to the edge $e$. Suppose $e$ is not a level 0 edge or a fine edge. By definition $V_v$ has an asymptotic value $(r_{v_1,e}, s_{v_1,e}, v_{v_1,e}) \in \mathbb{R} \oplus \mathbb{R} \oplus T_{p_{v_1,e}} \mathcal{D}$ where $p_{v_1,e}$ is the point of $\mathcal{D}$ such that $u_v(z_{v_1,e}) = p_{v_1,e}$ and $\mathbb{R} \oplus \mathbb{R}$ corresponds to the tangent space of the partial $\mathbb{C}^*_+\text{-action}$. (See Definition 3.3.) We require:

$$v_{v_1,e} = v_{v_2,e}.$$

This condition is the counterpart of part (4) of Definition 7.12. In the case of a level 0 edge (resp. a fine edge), the corresponding asymptotic values are tangent vectors of $L$ (resp. tangent vectors of $X$ or $N_X \setminus D$) and we require that these two tangent vectors agree with each other. (See [FOOO5, Definition 3.4].)

Analogous to Definition 7.12, there is an action of $(\mathbb{R} \oplus \mathbb{R})^{\mid \tilde{R} \mid}$ on $W_{m,d}^2,\sim(u; T)$ with $\mid \tilde{R} \mid$ being the number of levels of $\tilde{R}$. We define $W_{m,d}^2(u; T)$ to be the quotient space with respect to this action. We also write $L_{m,d}^2(u; T \otimes \Lambda^{0,1})$ for the direct sum of $L_{m,d}^2(u_v; T \otimes \Lambda^{0,1})$ for $v \in C_0^{\text{int}}(\tilde{R})$.

The linearization of the Cauchy-Riemann equation associated to each vertex $v$ of the very detailed tree $\tilde{R}$, defines the linear operator:

$$D_u \mathcal{J} : W_{m+1,d}^2(u_v; T) \to L_{m,d}^2(u_v; T \otimes \Lambda^{0,1}).$$

The direct sum of these operators together determines a Fredholm operator:

$$D_u \mathcal{J} : W_{m+1,d}^2(u; T) \to L_{m,d}^2(u; T \otimes \Lambda^{0,1}).$$

In the case that this operator is not surjective, we need to introduce obstruction spaces as in Section 4. For each interior vertex $v$, we fix a vector space $E_v$ such that the following conditions are satisfied:
Condition 8.2.  

1. \( E_v \) is a finite dimensional subspace of \( L^2_{m,\delta}(\Sigma_v \setminus \vec{z}_v; u^*_v T(X \setminus D) \otimes \Lambda^{0,1}) \) if \( c(v) = d \) or \( s \), and is a finite dimensional subspace of \( L^2_m(\Sigma_v; u^*_v T D \otimes \Lambda^{0,1}) \) if \( c(v) = D \). Moreover, \( E_v \) consists of smooth sections. Using the decomposition in (3.4), we can also regard \( E_v \) as a subspace of \( L^2_m(\Sigma_v; u^*_v T \otimes \Lambda^{0,1}) \)

2. Elements of \( E_v \) have compact supports away from nodal points and boundary.

3. If \( u_v \) is a constant map, then \( E_v \) is 0.

4. If \( \gamma \in \Gamma_u \), then \( (\gamma_v)_* E_v = E_{\gamma(v)} \).

Here \( (\gamma_v)_* : L^2_m(\Sigma_v; T \otimes \Lambda^{0,1}) \to L^2_m(\Sigma_{\gamma(v)}; T \otimes \Lambda^{0,1}) \) is the map induced by \( \gamma_v : \Sigma_v \to \Sigma_{\gamma(v)} \). (Recall that \( u_{\gamma(v)} \circ \gamma_v = u_v \).)

5. The operator \( D_{uv} \partial \) in (8.2) is transversal to:

\[
E_0 = \bigoplus_{v \in C^\text{int}(\vec{R})} E_v \subset L^2_{m,\delta}(\Sigma_v; u^*_v T \otimes \Lambda^{0,1}).
\]

It is straightforward to see that there are obstruction spaces satisfying Condition 8.2. Since each operator \( D_{uv} \partial \) is Fredholm, we can fix \( E_v \), which satisfies part (1). This choice also would imply the required transversality in part (5). In the case that \( u_v \) is constant, we can pick \( E_v \) to be the trivial vector space because \( \Sigma_v \) has genus 0. (It is either a disk or a sphere.) Unique continuation implies that we can assume that the supports of the elements of \( E_v \) is contained in a compact subset of \( \Sigma_v \) away from the nodal points and boundary. By taking direct sums over the action of \( \Gamma_u \) if necessary, we may also assume that (4) holds. Using \( E_0 \), we can define a thickened moduli space which gives a Kuranishi neighborhood of \( u \) in a stratum of \( \mathcal{M}_k^{RGW}(L; \beta) \) which contains \( u \). (See Definition 8.3.) In the upcoming sections, we give a systematic construction of the obstruction spaces \( E_0 \) which satisfy further compatibility assumptions.

We next discuss the process of gluing the irreducible components of \( u \). We firstly need to explain how the deformation of source curves is parametrized. The mathematical content here is classical and we follow the approaches in [FOOO05, Section 8] and [FOOO08, Section 3].

For an interior vertex \( v \), we consider \( (\Sigma_v, \vec{z}_v \cup \vec{w}_v) \). This is a disk or a sphere with marked points, which is stable. We may regard it as an element of the moduli space \( \mathcal{M}_v^\text{source} \). The space \( \mathcal{M}_v^\text{source} \) is metrizable and we fix one metric on it for our purposes later. The moduli space \( \mathcal{M}_v^\text{source} \) comes with a universal family:

\[
\pi : c_v^\text{source} \to \mathcal{M}_v^\text{source}
\]

and \( \# \vec{z}_v + \# \vec{w}_v \) sections which are in correspondence with the marked points. (See, for example, [FOOO08, Section 2].) For \( \mathbf{r} \in \mathcal{M}_v^\text{source} \), the fiber \( \pi^{-1}(\mathbf{r}) \) together with the values of the sections at \( \mathbf{r} \) determines a representative for \( \mathbf{r} \), which we denote it by \( (\Sigma_{\mathbf{r},v}, \vec{z}_{\mathbf{r},v} \cup \vec{w}_{\mathbf{r},v}) \).
Since $\Sigma_v$ has no singularity, (8.4) is a $C^\infty$-fiber bundle near the point $(\Sigma_v, \vec{z}_v \cup \vec{w}_v)$. We fix a neighborhood $\mathcal{V}^{\text{source}}_v$ of $[\Sigma_v, \vec{z}_v \cup \vec{w}_v]$ and a trivialization 

$$\phi_v : \mathcal{V}^{\text{source}}_v \times \Sigma_v \to \mathcal{C}^{\text{source}}_v. \tag{8.5}$$

of (8.4) over this neighborhood. We assume that these trivializations are compatible with the automorphisms of $u$. For $\xi_v \in \mathcal{V}^{\text{source}}_v$, we define a complex structure $j_{\xi_v}$ on $\Sigma_v$ such that the restriction of the trivialization (8.5) to $\{\xi_v\} \times \Sigma_v$ defines a bi-holomorphic map $(\Sigma_v, j_{\xi_v}) \to \pi^{-1}(\xi_v)$.

Let $e$ be an interior edge of $\hat{R}$ containing $v$. There is a nodal point of $(\Sigma_v, \vec{z}_v)$ associated to $e$. Let $s_{v,e}$ be the section of (8.4) corresponding to this marked point. In the case of an interior node, an analytic family of coordinates at this nodal point is a holomorphic map

$$\varphi_{v,e} : \mathcal{V}^{\text{source}}_v \times \text{Int}(D^2) \to \mathcal{C}^{\text{source}}_v \tag{8.6}$$

such that for each $\xi \in \mathcal{V}^{\text{source}}_v$, we have $\varphi_{v,e}(\xi, 0) = s_{v,e}(\xi)$ and the restriction of $\varphi_{v,e}$ to $\{\xi\} \times \text{Int}D^2$ determines a holomorphic coordinate for $\Sigma_e = \pi^{-1}(\xi)$ around $\varphi_{v,e}(\xi, 0) = s_{v,e}(\xi)$. Thus $\varphi_{v,e}$ commutes with the projection map to $\mathcal{M}^{\text{source}}_v$ and $\varphi_{v,e}$ is a bi-holomorphic map onto an open subset of $\mathcal{C}^{\text{source}}_v$. When $z_{v,e}$ is a boundary node, we replace $D^2$ by $D_+^2 = \{z \in D^2 | \text{Im} z \geq 0\}$ and define the notion of analytic family of coordinates in a similar way. (See [FOOO8, Section 3] for more details.) We require that the images of the maps $\varphi_{v,e}$ are disjoint and away from the image of the sections of $\mathcal{C}^{\text{source}}_v$ corresponding to the auxiliary marked points $\vec{w}_v$. We also assume that the chosen analytic families are compatible with automorphisms of $u$.

We use analytic family of coordinates $\varphi_{v,e}$ to desingularize the nodal points as follows. Fix an element:

$$\bar{\sigma} = (\sigma_e; e \in C^1_{\text{int}}(\hat{R})) \in \prod_{e \in C^1_{\text{int}}(\hat{R})} \mathcal{V}_v^{\text{deform}}. \tag{8.7}$$

Here $\sigma_e \in D^2 =: \mathcal{V}_v^{\text{deform}}$ if $z_{v,e}$ is an interior node, and $\sigma_e \in [0, 1] =: \mathcal{V}_v^{\text{deform}}$ if $z_{v,e}$ is a boundary node. Let

$$\bar{\xi} = (\xi_v; v \in C^0_{\text{int}}(\hat{R})) \in \prod_{v \in C^0_{\text{int}}(\hat{R})} \mathcal{V}_v^{\text{source}}. \tag{8.8}$$

---

21Note that $z_{v,e}$ is a boundary node if and only if $e$ is a level 0 edge.
We put
\[
\Sigma^+_v(\vec{x}, \vec{\sigma}) = \Sigma_{\vec{x},\vec{v}} \ \backslash \ \bigcup_{(v,e):e \text{ is not a level 0 edge}} \varphi_{v,e}(x_v, D^2(|\sigma_e|)) \\
\bigcup_{(v,e):e \text{ is a level 0 edge}} \varphi_{v,e}(x_v, D^2(\sigma_e))
\]
(8.9)
\[
\Sigma^-_v(\vec{x}, \vec{\sigma}) = \Sigma_{\vec{x},\vec{v}} \ \backslash \ \bigcup_{(v,e):e \text{ is not a level 0 edge}} \varphi_{v,e}(x_v, D^2(1)) \\
\bigcup_{(v,e):e \text{ is a level 0 edge}} \varphi_{v,e}(x_v, D^2(1)).
\]

(8.10)
\[
\bigcup_{v \in C^\text{int}_1(\vec{R})} \Sigma^+_v(\vec{x}, \vec{\sigma})
\]
Recall that a fine edge $e$ connecting two level 0 vertices is not a level 0 edge by definition. The auxiliary marked points $\vec{w}_v$ determine a set of marked points on $\Sigma^-_v(\vec{x}, \vec{\sigma})$, which is also denoted by $\vec{w}_v$.

We define an equivalence relation $\sim$ on:

(8.10)
\[
\bigcup_{v \in C^\text{int}_1(\vec{R})} \Sigma^+_v(\vec{x}, \vec{\sigma})
\]
as follows. Let $e$ be an edge which is not a level 0 edge and connects the vertices $v_1, v_2$. Suppose $z_1, z_2 \in \text{Int}(D^2)$ with $z_1 z_2 = \sigma_e$. Then:
\[
\varphi_{v_1,e}(x_{v_1}, z_1) \sim \varphi_{v_2,e}(x_{v_2}, z_2).
\]
Let $e$ be a level 0 edge connecting the vertices $v_1, v_2$. Suppose $z_1, z_2 \in \text{Int}(D^2_\pm)$ with $z_1 z_2 = -\sigma_e$. Then:
\[
\varphi_{v_1,e}(x_{v_1}, z_1) \sim \varphi_{v_2,e}(x_{v_2}, z_2).
\]

We divide the space (8.10) by the equivalence relation $\sim$ and denote the quotient space by
\[
\Sigma(\vec{x}, \vec{\sigma}).
\]

(8.11)
\[
\Sigma(\vec{x}, \vec{\sigma}).
\]
Let:
\[
\sigma_e = \exp(-(10T_e + \theta_e \sqrt{-1})) \quad e \text{ is not a level 0 edge},
\]
\[
\sigma_e = \exp(-10T_e) \quad e \text{ is a level 0 edge}.
\]

For each $e \in C^\text{int}_1(\vec{R})$, there is a corresponding neck region in $\Sigma(\vec{x}, \vec{\sigma})$. We define coordinates $r_e, s_e$ on this region as follows. Suppose $e$ is not a level 0 edge. We choose $v_1, v_2$ so that $v_1$ and the root of $\vec{R}$ (corresponding to the zero-th exterior marked point $z_0$ of $u$) are in the same connected component of $\vec{R} \setminus e$. Let:
\[
\Sigma^+_v(\vec{x}, \vec{\sigma}) \setminus \Sigma^-_v(\vec{x}, \vec{\sigma}) = [-5T_e, 5T_e]_{r_e} \times S^1_{s_e}
\]
where
\[
(-5T_e, s_e) \in \text{Closure} \left( \Sigma^-_{v_1}(\vec{x}, \vec{\sigma}) \right) \quad \forall s_e \in S^1.
\]
The coordinate $r_e, s_e$ is defined in the same way as in (7.1), (7.2).

If $e$ is a level 0 edge, then we take $v_1, v_2$ so that $v_1$ and the root of $\vec{R}$ are in the same connected component of $\vec{R} \setminus e$. Then $\Sigma^+_v(\vec{x}, \vec{\sigma}) \setminus \Sigma^-_v(\vec{x}, \vec{\sigma})$ has
a connected component corresponding to each edge which is incident to \( v_1 \). We identify the connected component corresponding to the edge \( e \) with:

\[
\Sigma(\bar{\sigma}, \bar{\tau}) = \bigcup_{v \in C_0^\text{int}(\bar{R})} \Sigma_v(\bar{\sigma}, \bar{\tau}) \\
\bigcup_{e \in C_1^\text{int}(\bar{R}), e \text{ is not a level 0 edge}} [-5T_e, 5T_e] r_e \times [0, \pi] s_e \\
\bigcup_{e \in C_1^\text{int}(\bar{R}), e \text{ is a level 0 edge}} [-5T_e, 5T_e] r_e \times [0, \pi] s_e,
\]

(8.13) where the point \( \varphi_{v_1, e}(r_v, \exp(-(r_v + 5T_e) - \sqrt{-1}s_e)) \) is identified with \((r_e, s_e)\) in (8.12). Similarly, \( \Sigma^+_v(\bar{\sigma}, \bar{\tau}) \setminus \Sigma^-_v(\bar{\sigma}, \bar{\tau}) \) has a connected component corresponding to each edge which is incident to \( v_2 \). We identify the connected component corresponding to the edge \( e \) with (8.12) where the point \( \varphi_{v_2, e}(r_v, \exp((r_e - 5T_e) + \sqrt{-1}s_e)) \) is identified with \((r_e, s_e)\). These identifications are compatible with the equivalence relation \( \sim \).

We thus have the decomposition:

\[
\Sigma(\bar{\sigma}, \bar{\tau}) = \bigcup_{v \in C_0^\text{int}(\bar{R})} \Sigma_v(\bar{\sigma}, \bar{\tau}) \\
\bigcup_{e \in C_1^\text{int}(\bar{R}), e \text{ is not a level 0 edge}} [-5T_e, 5T_e] r_e \times [0, \pi] s_e \\
\bigcup_{e \in C_1^\text{int}(\bar{R}), e \text{ is a level 0 edge}} [-5T_e, 5T_e] r_e \times [0, \pi] s_e.
\]

This is the thick and thin decomposition which is used frequently in various kinds of Gromov-Witten theory. The inclusion of \( \bar{w}_v \) in \( \Sigma_v(\bar{\sigma}, \bar{\tau}) \) induces a set of marked points in \( \Sigma(\bar{\sigma}, \bar{\tau}) \), which is also denoted by \( \bar{w}_v \).

Now we introduce several thickened moduli spaces which are used in the definition of our Kuranishi structures. We firstly define the stratum corresponding to \( \bar{R} \):

**Definition 8.3.** Given a positive real number \( \kappa \), the space \( \tilde{\mathcal{U}}(u, \bar{R}, \kappa) \) consists of triples \((\bar{x}, u', U')\) with the following properties:

1. \( \bar{x} = (\bar{x}_v; v \in C_0^\text{int}(\bar{R})) \in \prod_v C_0^\text{int}(\bar{R}) \mathcal{V}_v^{\text{source}} \). For each interior vertex \( v \), \( \bar{x}_v \) belongs to the \( \kappa \)-neighborhood of the point of \( \mathcal{V}_v^{\text{source}} \) induced by \( u \). A representative \((\Sigma_v, \bar{z}_v, \bar{w}_v \cup \bar{w}'_v)\) of \( \bar{x}_v \) is also given where the irreducible component \( \Sigma_v \) is equipped with an almost complex structure \( \tilde{j}_v \).

2. \( u' : \Sigma \to X \) is a continuous map, whose restriction \( u'_v \) to \( \Sigma_v \) is smooth. If \( c(v) = D \), then we require that \( u'(\Sigma_v) \subset D \). Moreover, a meromorphic section \( U'_v \) of \( (u'_v)^* \mathcal{N}_D(X) \) is also fixed such that the data of the zeros and poles of \( U'_v \) is determined by the multiplicity of the thick edges connected to \( v \). If \( c(v) = d \), then the restriction of \( u'_v \) to the boundary of the disc \( \Sigma_v \) is mapped to \( L \).

3. The \( C^2 \)-distance\(^{22} \) between \( u'_v \) and \( u_v \) is less than \( \kappa \). If \( c(v) = D \), then the \( C^2 \)-distance\(^{23} \) between \( U'_v \) and \( U_v \) is less than \( \kappa \).

---

\(^{22}\)If \( c(v) = d \) or \( s \), then the \( C^2 \)-distance is defined using the metric \( g \) on \( X \), and if \( c(v) = D \), then the \( C^2 \)-distance is defined using the metric \( g' \) on \( D \).

\(^{23}\)The \( C^2 \)-distance is defined with respect to a metric which has the form given in (3.2). Note that the set of sections \( \{ U_v \}_v \) is defined up to action of \( C_1^N \), and here we mean that
(4) We require

\[ \bar{\partial}_{j_{v'}} u'_v \in E_v(u'_v). \]

Here \( j_{v'} \) is the complex structure of \( \Sigma_v \) corresponding to \( r_v \). (See the discussion proceeding (8.5).) Using the complex structure \( j_{v'} \) on \( \Sigma_v \), we may define the target space parallel transportation in the same way as in Section 4, and obtain \( E_v(u'_v) \) from \( E_v \). We also require:

\[ \bar{\partial}_{j_{v'}} U'_v \in E_v(u'_v). \]

if \( c(v) = D \). Here we use (3.4) to regard \( E_v(u'_v) \) as a subspace of \( L^2_{\kappa,\delta}(\Sigma_v; (U'_v)^*TN_D(X) \otimes \Lambda^{0,1}) \).

(5) We require

\[ u'_v(w_{v,i}) \in \mathcal{N}_{v,i}. \]

(6) If \( e \) is a fine edge connecting vertices \( v_1 \) and \( v_2 \) with color \( d \) or \( s \), then the values of \( u'_{v_1} \) and \( u'_{v_2} \) at the node \( \Sigma_{v_1} \cap \Sigma_{v_2} \) are equal to each other. If \( e \) is a fine edge connecting vertices \( v_1 \) and \( v_2 \) with color \( D \), then the values of \( U'_{v_1} \) and \( U'_{v_2} \) at the node \( \Sigma_{v_1} \cap \Sigma_{v_2} \) are equal to each other.

We define an equivalence relation \( \sim \) on \( \hat{U}(u, \bar{R}, \kappa) \) as follows. Let \( |\lambda| \) be the number of levels of the DD-ribbon tree associated to \( u \). For \( i = 1, \ldots, |\lambda| \), we take \( a_i \in \mathbb{C}^*_+ \). We define \( (\bar{r}, u', U'_{(0)}) \sim (\bar{r}, u', U'_{(1)}) \)

\[ U'_{(1),v} = \text{Dil}_{a_{\lambda(v)}} \circ U'_{(0),v}. \]

We denote the quotient space with respect to this equivalence relation by \( \hat{U}(u, \bar{R}, \kappa) \). The group of automorphisms \( \Gamma_u \) acts on \( \hat{U}(u, \bar{R}, \kappa) \) in an obvious way. We write \( \mathcal{U}(u, \bar{R}, \kappa) \) for the quotient space \( \hat{U}(u, \bar{R}, \kappa)/\Gamma_u \).

The space \( \mathcal{U}(u, \bar{R}, \kappa) \) is a generalization of \( \mathcal{U}_d \mathcal{U}_v \times_{\mathcal{D} \mathcal{D},d} \mathcal{U}_D \mathcal{U}_v \times_{\mathcal{S}} \mathcal{U}_s \) appearing in (5.3) and is a thickened version of a neighborhood of \( u \) in the stratum \( \mathcal{M}_1^0 \mathcal{R} \) of \( \mathcal{M}_{k+1}^\mathcal{RGW} \mathcal{L} ; \beta \) defined in [DF2, (3.40)]. The following lemma is a consequence of Condition 8.2 and the implicit function theorem.

**Lemma 8.4.** If \( \kappa \) is small enough, then \( \hat{U}(u, \bar{R}, \kappa) \) is a smooth manifold and \( \mathcal{U}(u, \bar{R}, \kappa) \) is a smooth orbifold.

**Remark 8.5.** Although it is not clear from the notation, the definition of \( \mathcal{U}(u, \bar{R}, \kappa) \) uses the choice of the additional marked points \( w_{v,i} \), the set of transversals \( \mathcal{N}_{v,i} \), the trivializations of the universal family \( \phi_v \) and analytic family of coordinates \( \varphi_{v,e} \). We call \( \Xi = (\bar{w}_v, (\mathcal{N}_{v,i}), (\phi_v), (\varphi_{v,e}), \kappa) \) a choice of trivialization and stabilization data (TSD) for \( u \). We define the size of \( \Xi \) to be the sum of \( \kappa \), the diameters of \( \mathcal{V}_v^\text{source} \) and the images of the maps \( \varphi_{v,e} \). When we say \( \Xi \) is small enough, we mean that the size of \( \Xi \) is small enough. Given this definition, a more accurate notation for \( \mathcal{U}(u, \bar{R}, \kappa) \) would be \( \mathcal{U}(u, \bar{R}, \Xi) \).

there is a representative for \( \{U_v\}_v \) such that the distance between \( U'_v \) and \( U_v \) is less than \( \kappa \).
We next introduce the generalization of the space $\mathcal{U}_0$ in Definition 4.10.

**Definition 8.6.** Let $\Xi = (\bar{w}_v, (\mathcal{N}_{v,i}), (\phi_v), (\varphi_{v,e}), \kappa)$ be a TSD at the element $u$ of $\mathcal{M}_k^{\text{RGW}}(L, \beta)$. The space $\tilde{\mathcal{U}}_0(u, \Xi)$ consists of $(\bar{\gamma}, \bar{\sigma}, u')$ with the following properties:

1. $\bar{f}(\bar{x}; v) \in C^\text{int}_0(\bar{R}) \cap \prod_{e \in C^\text{int}_0(\bar{R})} V_v^\text{source}$ and $\bar{\sigma} = (\sigma_e; e \in C^\text{int}_1(\bar{R})) \in \prod_{e \in C^\text{int}_1(\bar{R})} V_v^\text{deform}$. Furthermore, for each interior vertex $v$, $r_v$ belongs to the $\kappa$-neighborhood of the point of $V_v^\text{source}$ induced by $u$. Similarly, for each $e$, we have $|\sigma_e| < \kappa$.

2. $u' : (\Sigma(\bar{\gamma}, \bar{\sigma}), \partial(\Sigma(\bar{\gamma}, \bar{\sigma}))) \to (X \setminus \mathcal{D}, L)$ is a continuous map and is smooth on each irreducible component.

3. If $e$ is (resp. is not) a level 0 edge, then the image of the restriction to $u'$ to $[-5T_e, 5T_e]_{v_e} \times [0, \pi]_{s_v}$ (resp. $[-5T_e, 5T_e]_{v_e} \times S_{v_e}^1$) has a diameter less than $\kappa$. If $c(v) = D$, then the restriction of $u'$ to $\Sigma_v^+(\bar{\gamma}, \bar{\sigma})$ is included in the open neighborhood $\Upsilon$ of $\mathcal{D}$.

4. If $c(v) = d$ or $s$, then the $C^2$-distance between the restrictions of $u'$ and $u_v$ to $\Sigma_v^-(\bar{\gamma}, \bar{\sigma})$ is less than $\kappa$. If $c(v) = D$, then the previous part implies that the restriction of $u'$ to $\Sigma_v^-(\bar{\gamma}, \bar{\sigma})$ may be regarded as a map to $\mathcal{N}_D(X) \setminus \mathcal{D}$. We also demand that the $C^2$-distance between this map and $U_v$ is less than $\kappa$.

5. We require

$$\bar{J}_{\bar{f}, \bar{\sigma}, u'} \in E_0(u').$$

Here $\bar{J}_{\bar{f}, \bar{\sigma}}$ is the complex structure of $\Sigma(\bar{\gamma}, \bar{\sigma})$, and $E_0(u')$ is defined from $E_v$ by target space parallel transportation in the same way as in Section 4.

6. We require

$$u'(w_{v,i}) \in \mathcal{N}_{v,i}.$$

The group of automorphisms $\Gamma_u$ acts on $\tilde{\mathcal{U}}_0(u, \Xi)$ in the obvious way. We write $\tilde{\mathcal{U}}_0(u, \Xi)$ for the quotient space $\tilde{\mathcal{U}}_0(u, \Xi)/\Gamma_u$.

**Remark 8.7.** The above definition needs to be slightly modified if some of the components of $\bar{\sigma}$ are zero. Let $e$ be an edge connecting a vertex of level $i$ to a vertex of level $i + 1$ such that $\sigma_e = 0$. If $e'$ is another edge that connects a vertex of level $i$ to a vertex of level $i + 1$, then $\sigma_{e'} = 0$. Next, we decompose $\bar{R}$ into several blocks such that $\sigma_e = 0$ for the edges $e$ joining two different blocks and $\sigma_e \neq 0$ for an edge $e$, which is inside a block and is not a fine edge. In each block, we use Definition 8.6 and join spaces associated to various blocks in the same way as in Definition 8.3. We omit the details of this process because the actual space we use for the

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²⁴The diameter is defined with respect to the metric $g_{NC}$.
²⁵Here again we use the convention that the distance between an object and $\{U_v\}_{\lambda(v) > 0}$ is defined to be the minimum of the relevant distance between that object and all representatives of $\{U_v\}_{\lambda(v) > 0}$. 

definition of our Kuranishi structure is not \( U_0(u, \Xi) \) but \( U(u, \Xi) \), introduced in Definition 8.8. We can also define \( U_0(U, \Xi) \) as a subspace of \( U(u, \Xi) \). We brought firstly Definition 8.6 because its geometric meaning is more clear.

The space \( U_0(u, \Xi) \) in general is singular (not an orbifold). We introduce the notion of inconsistent solutions to thicken \( U_0(u, \Xi) \) into an orbifold.

**Definition 8.8.** Let \( \Xi = (\tilde{w}_v, (\mathcal{N}_{v,i}), (\phi_v), (\varphi_{v,e}), \kappa) \) be a TSD at the element \( u \) of \( \mathcal{M}_{k+1}^{RGW}(L, \beta) \). We say that \((\tilde{f}_v, \tilde{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \) is an inconsistent solution near \( u \) with respect to \( \Xi \) if it satisfies the following properties:

1. \( \tilde{f}_v(v) \in C^0_\text{int}(\tilde{R}) \) \( \cap \prod_{e \in C^0_\text{int}(\tilde{R})} \mathcal{V}_{v_0} \), \( \tilde{\sigma} = (\sigma_v; e \in C^0_\text{deform}(\tilde{R})) \) is not a level 0 edge, and \( \rho_i \in D^2 \) for each level \( i = 1, \ldots, |\lambda| \). Furthermore, for each interior vertex \( v \), \( \tilde{f}_v \) belongs to the \( \kappa \)-neighborhood of the point of \( \mathcal{V}_{v_0} \) induced by \( u \). Similarly, for each edge \( e \), we have \( |\sigma_e| < \kappa \).
2. If \( c(v) = d \) (resp. \( s \)), then \( u'_v : (\Sigma^+_v(\tilde{f}, \tilde{\sigma}), \partial \Sigma^+_v(\tilde{f}, \tilde{\sigma})) \to (X \setminus D, L) \) (resp. \( u'_v : \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \to X \setminus D \)) is a smooth map.
3. If \( c(v) = D \), then \( U'_v : \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \to N_D X \setminus D \) is a smooth map, and \( u'_v = \pi \circ U'_v \).
4. \( \rho_e = 0 \) if and only if \( \sigma_e = 0 \).
5. Suppose \( e \) is an edge connecting vertices \( v_0 \) and \( v_1 \) such that \( \lambda(v_0) = 0 \) and \( \lambda(v_1) \geq 1 \). Then we require:

\[
(8.20) \quad u'_{v_0} = \text{Dil}_{\rho_e} \circ U'_{v_1}
\]

on \([-5T_e, 5T_e] \times S_{s_e} = \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \cap \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \) if \( \sigma_e \neq 0 \). In particular, we assume that the restriction of \( u'_{v_0} \) to \([-5T_e, 5T_e] \times S_{s_e} \) is contained in the open neighborhood \( \mathcal{U} \) of \( D \). If \( \sigma_e = 0 \), then the values of \( u'_{v_0} \) and \( \pi \circ U'_{v_1} \) at the nodal points corresponding to \( e \) are equal to each other.
6. Suppose \( e \) is an edge connecting vertices \( v_1 \) and \( v_2 \) such that \( \lambda(v_1) = i > 0 \) and \( \lambda(v_2) \geq i + 1 \). We require

\[
(8.21) \quad U'_{v_1} = \text{Dil}_{\rho_e} \circ U'_{v_2}
\]

on \([-5T_e, 5T_e] \times S_{s_e} = \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \cap \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \) if \( \sigma_e \neq 0 \). If \( \sigma_e = 0 \), then the values of \( U'_{v_1} \) and \( \pi \circ U'_{v_2} \) at the nodal points corresponding to \( e \) are equal.
7. Suppose \( e \) is a level 0 edge connecting the vertices \( v_1 \) and \( v_2 \). If \( \sigma_e \neq 0 \), then we require:

\[
(8.22) \quad u'_{v_1} = u'_{v_2}
\]

on \([-5T_e, 5T_e] \times [0, 1] \times S_{s_e} = \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \cap \Sigma^+_v(\tilde{f}, \tilde{\sigma}) \). If \( \sigma_e = 0 \), then (8.22) holds at the nodal point corresponding to \( e \).
8. Suppose \( e \) is a fine edge connecting the vertices \( v_1 \) and \( v_2 \) with level zero (resp. with the same positive level). If \( \sigma_e \neq 0 \), then we require:

\[
(8.23) \quad u'_{v_1} = u'_{v_2} \quad (\text{resp.} \quad U'_{v_1} = U'_{v_2})
\]
Let the above properties. We define an equivalence relation \( \sim \) for \( j \) representatives of \( \{ \lambda \} \). Let:

\[ \lambda = 1 \]

(ii) \( \rho_\ell((\lambda)) = d \) or \( s \), then we require:

\[ \lambda = \frac{2}{5} \]

(9) If \( e \) is (resp. is not) a level 0 edge, then the image of the restriction to \( u'_v \) to \([−5T_e, 5T_e]_{r_e} × \sigma_e \) has a diameter 26 less than \( \kappa \).

(10) If \( c(v) = d \) or \( s \), then the \( C^2 \)-distance between the restrictions of \( u'_v \) and \( u_v \) to \( \Sigma^{-}(\bar{r}, \bar{\sigma}) \) is less than \( \kappa \). If \( c(v) = D \), then we demand that the \( C^2 \)-distance between the restrictions of \( U'_v \) and \( U_v \) to \( \Sigma^{-}(\bar{r}, \bar{\sigma}) \) is less than \( \kappa \). 27

(11) If \( c(v) = d \) or \( s \), then we require:

\[ \bar{\partial}_{j,\bar{\sigma}} u'_v \in E_v(u'_v). \]

Here \( j_{\bar{r},\bar{\sigma}} \) is the complex structure of \( \Sigma(\bar{r}, \bar{\sigma}) \), and \( E_v(u'_v) \) is defined from \( E_v \) by target space parallel transportation in the same way as in Section 5.

(12) If \( c(v) = D \), then we require:

\[ \bar{\partial}_{j_v} U'_v \in E_v(U'_v). \]

Here \( j_{\bar{r},\bar{\sigma}} \) is the complex structure of \( \Sigma(\bar{r}, \bar{\sigma}) \), and we use (3.4) to obtain \( E_v(U'_v) \) from \( E_v(u'_v) \) as a subspace of \( L^2_{\bar{m},\bar{\sigma}}(\Sigma_v; (U'_v)^* T\mathcal{N}_D(X) \otimes \Lambda^{0,1}) \).

(13) We have:

\[ u'_v (w_{v,i}) \in \mathcal{N}_{v,i}. \]

We denote by \( \widetilde{U}(u, \Xi) \) the set of all \( (\bar{r}, \bar{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \) satisfying the above properties. We define an equivalence relation \( \sim \) on it as follows. Let \( x_j = (\bar{r}_j(\lambda), \bar{\sigma}(\lambda), (u'_v(\lambda)), (U'_v(\lambda)), (\rho_e(\lambda)), (\rho_i(\lambda))) \) be elements of \( \widetilde{U}(u, \Xi) \) for \( j = 1, 2 \). We say that \( x_1 \sim x_2 \) if there exists \( a_i \in \mathbb{C}_\ast \) \( (i = 1, \ldots, |\lambda|) \) with the following properties. Let:

\[ b_i = a_1 \cdots a_i \in \mathbb{C}_\ast. \]

(i) \( \bar{r}_1(\lambda) = \bar{r}_2(\lambda), \bar{\sigma}_1(\lambda) = \bar{\sigma}_2(\lambda), u'_v(\lambda_1) = u'_v(\lambda_2). \)

(ii) \( \rho_i(\lambda_1) = a_i \rho_i(\lambda_2). \)

(iii) \( U'_v(\lambda_1) = \text{Dil}_a \rho_i(\lambda_2) \circ U'_v(\lambda_2) \)

(iv) Suppose \( e \) is an edge connecting a vertex \( v_0 \) with \( \lambda(v_0) = 0 \) to a vertex \( v_1 \) with \( \lambda(v_1) \geq 1 \). Then we require:

\[ \rho_e(\lambda_2) = b_\lambda(v_2) \rho_e(\lambda_1). \]

(v) Suppose \( e \) is an edge connecting a vertex \( v_1 \) with \( \lambda(v_1) \geq 1 \) to a vertex \( v_2 \) with \( \lambda(v_2) \geq 2 \). Then we require:

\[ \rho_e(\lambda_2) = a_\lambda(v_1) \cdots a_\lambda(v_2) \rho_e(\lambda_1). \]

26 The diameter is defined with respect to the metric \( g_{NC} \).

27 Here again we use the convention that the distance between an object and \( \{ U_v \}_{\lambda(v)>0} \), is defined to be the minimum of the relevant distance between that object and all representatives of \( \{ U_v \}_{\lambda(v)>0} \).
We denote by \( \mathcal{U}(u, \Xi) \) the quotient space \( \mathcal{U}(u, \Xi) / \sim \). The group \( \Gamma_u \) acts on \( \mathcal{U}(u, \Xi) \) in an obvious way. We denote by \( U(u, \Xi) \) the quotient space \( \mathcal{U}(u, \Xi) / \Gamma_u \). We say an element of \( U(u, \Xi) \) is an \textit{inconsistent solution} near \( u \) with respect to \( \Xi \). When it does not make any confusion, the elements of \( \mathcal{U}(u, \Xi) \) or \( U(u, \Xi) \) are also called inconsistent solutions near \( u \) with respect to \( \Xi \).

**Remark 8.9.** Initially it might seem that the complex numbers \( \rho_i \) do not play any role in the definition of the elements of \( U(u, \Xi) \). However, later they make it slightly easier for us to define the obstruction maps.

Our generalization of Proposition 5.3 claims that \( U(u, \Xi) \) is a smooth orbifold. Before stating this result, we elaborate on the relationship between \( U(u, \Xi) \) and \( U_0(u, \Xi) \).

**Definition 8.10.** Let \( (\vec{x}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \) be an inconsistent solution near \( u \) with respect to \( \Xi \). We say that it satisfies \textit{consistency equation} if for each edge \( e \) connecting vertices \( v_1 \) and \( v_2 \) with \( 0 \leq \lambda(v_1) < \lambda(v_2) \), we have:

\[
\rho_e = \rho_{\lambda(v_1)+1} \cdots \rho_{\lambda(v_2)}.
\]

It is easy to see that the consistency equation (8.27) is independent of the choice of the representative with respect to the relations given by \( \sim \) and the action of \( \Gamma_u \).

**Lemma 8.11.** The set of inconsistent solutions near \( u \) satisfying consistency equation can be identified with \( U_0(u, \Xi) \).

**Proof.** Let \( (\vec{x}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \) be an inconsistent solution near \( u \) satisfying consistency equations. For the simplicity of exposition, we consider the case that all the components of \( \vec{\sigma} \) are nonzero.\(^{28}\) Define \( \tau_i \) to be the product \( \rho_1 \cdot \rho_2 \ldots \rho_i \). For each vertex \( v \) with \( c(v) = D \), we also define:

\[
U^m_v = \text{Dil}_{\tau_{\lambda(v)}} \circ U'_v.
\]

Then the maps \( U^m_v \) for \( c(v) = D \) and \( U'_v \) for \( c(v) = d \) or \( s \) are compatible on the overlaps and by gluing them together, we obtain an element of \( U_0(u, \Xi) \). The reverse direction is clear. \( \square \)

**Example 8.12.** We consider the case of detailed DD-ribbon tree in Figure 1. This tree has two edges (whose multiplicities are 2 and 3, respectively). We denote them by \( e_d \) and \( e_s \), respectively. Two parameters \( \rho_d \) and \( \rho_s \) are associated to these edges. (In Section 7, \( \rho_d \) and \( \rho_s \) are denoted by \( \rho_1 \) and \( \rho_2 \), respectively). The total number of levels is 1. So there is a parameter \( \rho \) associated to this level. The consistency equation (8.27) implies that:

\[
\rho_d = \rho = \rho_s,
\]

which is the same as the equation in (5.4).

\(^{28}\) This is the case that we gave a detailed definition of \( U_0(u, \Xi) \), after all. For other cases, this lemma can be used as the definition.
For any \( \ell \leq m - 2 \), we fix a \( C^\ell \) structure on \( \hat{U}(u, \Xi) \) in the following way. For an interior vertex \( v \) of \( \Sigma_v \), let \( \Sigma_v^- \) be the space \( \Sigma_v^- (\vec{\tau}, \vec{\sigma}) \) in the case that \( \vec{\sigma} = 0 \) and \( \vec{\tau} \) is induced by \( u \). The trivialization of the universal family allows us also to identify \( \Sigma_v^- \) with \( \Sigma_v^- (\vec{\tau}, \vec{\sigma}) \) for different choices of \( \vec{\tau}, \vec{\sigma} \). Define maps:

\[
(8.28) \quad \text{Res}_v : \hat{U}(u, \Xi) \to I_{m+1}^2 (\Sigma_v^-, X \setminus D) \quad \text{if } \lambda(v) = 0,
\]

\[
(8.29) \quad \text{Res}_v : \hat{U}(u, \Xi) \to I_{m+1}^2 (\Sigma_v^-, \mathcal{N}_D X \setminus D) \quad \text{if } \lambda(v) > 0,
\]

such that Res\(_v\)(\(\vec{\tau}, \vec{\sigma}, (u'_v), (\rho_e), (\rho_i)\)) is the restriction of \( u'_v \) or \( U'_v \) to \( \Sigma_v^- (\vec{\tau}, \vec{\sigma}) \approx \Sigma_v^- \). By unique continuation, Res\(_v\) and the obvious projection maps induce an embedding:

\[
\hat{U}(u, \Xi) \to \prod_{v \in C^0_{\text{int}}(R)} \mathcal{Y}_v^{\text{source}} \times \prod_{e \in C^{\text{ext}}_1(R)} \mathcal{Y}_e^{\text{deform}} \times (D^2)^{|\lambda|}
\]

\[
\times \prod_{v \in C^0_{\text{int}}(R), \lambda(v)=0} I_{m+1}^2 (\Sigma_v^-, X \setminus D)
\]

\[
\times \prod_{v \in C^0_{\text{int}}(R), \lambda(v)>0} I_{m+1}^2 (\Sigma_v^-, \mathcal{N}_D X \setminus D)
\]

\[
(8.30)
\]

We use this embedding to fix a \( C^\ell \)-structure on \( \hat{U}(u, \Xi) \). The group \( C^{|\lambda|}_\epsilon \) acts freely on the target and the domain of (8.30), and the above embedding is equivariant with respect to this action. We use the induced map at the level of the quotients to define a \( C^\ell \)-structure on \( \hat{U}(u, \Xi) \). Note that we can define a slice for \( \hat{U}(u, \Xi) \) using the following idea. For each \( 1 \leq i \leq |\lambda| \), we fix an interior vertex \( v_i \) with \( \lambda(v_i) = i \) and a base point \( x_i \in \Sigma_v \). We also trivialize the bundle \( \mathcal{N}_D(X) \) in a neighborhood of \( U_{v_i}(x_i) \). Each element of \( \hat{U}(u, \Xi) \) has a unique representative \( (\vec{\tau}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \) such that \( U'_v(x_i) = 1 \in \mathbb{C} \). Here we assume that \( \kappa \) is small enough such that \( U'_v(x_i) \) belongs to the neighborhood of \( U_{v_i}(x_i) \) that \( \mathcal{N}_D(X) \) is trivialized.

**Proposition 8.13.** The space \( \hat{U}(u, \Xi) \) is a \( C^\ell \)-manifold and \( U(u, \Xi) \) is a \( C^\ell \)-orbifold. There exists a \( \Gamma_u \)-invariant open \( C^\ell \)-embedding for \( \ell \leq m - 2 \):

\[
\Phi : \prod_{e \in C^{\text{ext}}_1(R)} \mathcal{Y}_e^{\text{deform}} \times \hat{U}(u, \tilde{R}, \Xi) \times D^2(\epsilon)^{|\lambda|} \to \hat{U}(u, \Xi)
\]

with the following properties:

1. \( \Phi(\vec{\tau}, \vec{\sigma}, (\rho)) = [\vec{\tau}, \vec{\sigma}, (u'_{\vec{\tau}, \vec{\sigma}, v}), (U'_{\vec{\tau}, \vec{\sigma}, v}), (\rho_e(\vec{\tau}, \vec{\sigma}, \xi)), (\rho_i)] \)

   Namely, the gluing parameters \( \vec{\tau} \) are preserved by the map \( \Phi \). Moreover, the deformation parameter \( \vec{\tau} \) is the same as the one for the source curve of \( \vec{\tau} \).

2. For each edge \( e \in C^{\text{ext}}_0(R) \) that is not a level 0 edge, there exists a nonzero smooth function \( f_e \) such that:

\[
\rho_e(\vec{\tau}, \vec{\sigma}) = f_e(\vec{\tau}, \vec{\sigma}) \sigma_e^{m(e)}
\]
where \( m(e) \) is the multiplicity of the edge \( e \).

(3) Let \( \xi = (\bar{f}, u', U') \in \tilde{U}(u, \tilde{R}, \Xi) \) and \( \bar{\sigma}_0 \) be the vector that \( \sigma_e \) are all zero. Then we have:

\[
\Phi(\bar{\sigma}_0, \xi, (\rho_i)) = [\bar{f}, \bar{\sigma}_0, (u'_{\bar{\sigma}_0, \xi, v}), (U'_{\bar{\sigma}_0, \xi, v}), (\rho_e(\bar{\sigma}_0, \xi)), (\rho_i)],
\]

where \( u'_{\bar{\sigma}_0, \xi, v} \) is the restriction \( u' \) of \( u' \), \( U'_{\bar{\sigma}_0, \xi, v} \) is the restriction of \( U' \) and \( \rho_e(\bar{\sigma}_0, \xi) = 0 \).

The proof of Proposition 8.13 is essentially the same as the proof of Proposition 5.3, and it is only notationally more involved.

We next state a generalization of Proposition 5.5. For a thick edge \( e \) which is not of level 0, we define \( T_e, \theta_e, \mathcal{R}_e, \eta_e \) using the following identities:

\[
\sigma_e = \exp(-(T_e + \sqrt{-1} \theta_e)),
\]

\[
\rho_e = \exp(-(\mathcal{R}_e + \sqrt{-1} \eta_e)).
\]

If \( e \) is a level 0 edge, then we define \( T_e \) using:

\[
\sigma_e = \exp(-T_e).
\]

We may also define \( T_e \) and \( \theta_e \) for a fine edge as in (8.31). Using \( \Phi \), we regard \( \mathcal{R}_e, \eta_e \) as functions of \( T_e', \theta_e' \) and \( \xi \). We again use the trivialization of the universal family to identify \( \Sigma_v^{-}(\bar{f}_0, \bar{\sigma}) \) (see (8.9)) for various choices of \( \bar{f}_0, \bar{\sigma} \). For the purpose of the next proposition, we also regard \( u'_{\bar{\sigma}, \xi, v}, U'_{\bar{\sigma}, \xi, v} \) as maps

\[
u'_{\bar{\sigma}, \xi, v}: \Sigma_v^{-}(\bar{\sigma}) \to X \setminus \mathcal{D}
\]

\[
U'_{\bar{\sigma}, \xi, v}: \Sigma_v^{-}(\bar{\sigma}) \to N_{\mathcal{D}}X \setminus \mathcal{D}.
\]

In particular, the domain of these maps are independent of \( T_e, \theta_e \) and \( \xi \).

**Proposition 8.14.** Let \( \ell \) be an arbitrary positive integer and \( k_e, k'_e \) be non-negative integers. Let \( v_e = 0 \) if \( k_e, k'_e = 0 \). Otherwise we define \( v_e = 1 \).

(1) We have the following exponential decay estimates:

\[
\prod_e \left| \frac{\partial^{k_e}}{\partial T_e^k} \frac{\partial^{k'_e}}{\partial \theta_e^k} u'_{\bar{\sigma}, \xi, v} \right|_{L^2(\Sigma_v^{-}(\bar{\sigma}))} \leq C \exp(-c \sum v_e T_e).
\]

(8.33)

\[
\prod_e \left| \frac{\partial^{k_e}}{\partial T_e^k} \frac{\partial^{k'_e}}{\partial \theta_e^k} U'_{\bar{\sigma}, \xi, v} \right|_{L^2(\Sigma_v^{-}(\bar{\sigma}))} \leq C \exp(-c \sum v_e T_e).
\]

Here \( C, c \) are positive constants depending on \( \ell, k_e, k'_e \). The same estimate holds for the \( \xi \) derivatives of \( u'_{v, \bar{\sigma}, \xi}, U'_{v, \bar{\sigma}, \xi} \).

(2) For any thick edge \( e \) which is not a level 0 edge, we also have the following exponential decay estimates:

\[
\prod_e \left| \frac{\partial^{k_e}}{\partial T_e^k} \frac{\partial^{k'_e}}{\partial \theta_e^k} (\mathcal{R}_{e_0} - m(e_0) T_{e_0}) \right| \leq C \exp(-c \sum v_e T_e)
\]

(8.34)

\[
\prod_e \left| \frac{\partial^{k_e}}{\partial T_e^k} \frac{\partial^{k'_e}}{\partial \theta_e^k} (\eta_{e_0} - m(e_0) \theta_{e_0}) \right| \leq C \exp(-c \sum v_e T_e).
\]
Here \( C, c \) are positive constants depending on \( \ell, k_e, k'_e \). The same estimate holds for the \( \xi \) derivatives of \( R_{v_0}, \eta_{v_0} \).

Similar to Proposition 5.5, Proposition 8.14 can be verified using the same argument as in the proof of [FOOO5, Section 6].

We now use Propositions 8.13 and 8.14 to produce a Kuranishi chart at \( u \). Let \( \eta = (\bar{\xi}, \bar{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \) be a representative of an element of \( \hat{U}(u, \Xi) \). Recall that we fix vector spaces \( E_v \) for each \( u \), and use target parallel transportation to obtain the vector spaces \( E_v(u'_v) \) and \( E_v(U'_v) \). We define:

\[
(8.35) \quad \mathcal{E}_{0,u}(\eta) = \bigoplus_{v \in C_0^{\text{int}}(\mathcal{R}), \lambda(v) = 0} E_v(u'_v) \bigoplus \bigoplus_{v \in C_0^{\text{int}}(\mathcal{R}), \lambda(v) > 0} E_v(U'_v).
\]

Using Proposition 8.14, it is easy to see that (8.35) defines a \( \Gamma_u \)-equivariant \( C^\ell \) vector bundle on \( \hat{U}(u, \Xi) \).

We define the other part of the obstruction bundle as follows. Let \( e \) be a thick edge which connects vertices \( v_1 \) and \( v_2 \) with \( 0 \leq \lambda(v_1) < \lambda(v_2) \). We fix the trivial line bundle \( \mathbb{C}_e \) on \( \hat{U}(u, \Xi) \). Let \( x_1 \sim x_2 \) and \( a_i \in \mathbb{C}_e \) \((i = 1, \ldots, |\lambda|)\) be as in Definition 8.8 (i)-(v). Then define an equivalence relation on \( \mathbb{C}_e \) where \((x_1, V_1) \sim (x_2, V_2)\) if:

\[
V_2 = a_{\lambda(v_1)+1} \cdots a_{\lambda(v_2)} V_1.
\]

We thus obtain a line bundle \( \mathcal{L}_e \) on \( \hat{U}(u, \Xi) \). The group \( \Gamma_u \) acts on \( \bigoplus_e \mathcal{L}_e \) in an obvious way. Our obstruction bundle \( \mathcal{E}_u \) on \( \hat{U}(u, \Xi) \) is defined to be:

\[
(8.36) \quad \mathcal{E}_u = \mathcal{E}_{0,u} \oplus \bigoplus_{e \in C_0^{\text{int}}(\mathcal{R}), \lambda(e) > 0} \mathcal{L}_e.
\]

It induces an orbi-bundle on \( \hat{U}(u, \Xi) \). By an abuse of notation, this orbibundle is also denoted by \( \mathcal{E}_u \).

Next, we define Kuranishi maps. If \( v \) is a vertex with \( \lambda(v) = 0 \), then we define:

\[
(8.37) \quad s_{u,v}(\bar{\xi}, \bar{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) = \bar{\sigma} u'_v \in E_v(u'_v).
\]

If \( v \) is a vertex with \( \lambda(v) > 0 \), then we define:

\[
(8.38) \quad s_{u,v}(\bar{\xi}, \bar{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) = \bar{\sigma} U'_v \in E_v(U'_v).
\]

If \( e \) is a thick edge connecting vertices \( v_1 \) and \( v_2 \) with \( 0 \leq \lambda(v_1) < \lambda(v_2) \), then we define:

\[
(8.39) \quad s_{u,e}(\bar{\xi}, \bar{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) = \rho_e - \rho_{\lambda(v_1)+1} \cdots \rho_{\lambda(v_2)}.
\]

We define:

\[
\mathcal{E}_u = ((s_{u,v} : v \in C_0^{\text{int}}(\mathcal{R})), (s_{u,e} : e \in C_0^{\text{int}}(\mathcal{R}), \lambda(e) > 0)).
\]

It is easy to see that \( \mathcal{E}_u \) induces a \( \Gamma_u \)-invariant section of \( \mathcal{E}_u \). Using Proposition 8.14, we can show that the section \( \mathcal{E}_u \) is smooth.
Theorem 8.15. which is a homeomorphism onto an open neighborhood of space \( M \) is pseudo holomorphic. Therefore, \((\Sigma(8.40) \psi\text{ change (8.47)}\) induces an element \((\vec{\psi}^L, \vec{\sigma})\) and marked points on \( \Sigma_\vec{f} \) determine an element of \( \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta) \). This element does not change if we change \((\vec{f}, \vec{\sigma}, (U'_v), (\rho_v), (\rho_i))\) by the \( \Gamma_u \)-action. We thus obtained:

\[
(8.40) \quad \psi_u : \mathfrak{s}_u^{-1}(0)/\Gamma_u \to \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta),
\]

which is a homeomorphism onto an open neighborhood of \( u \). We thus proved:

**Theorem 8.15.** \( \mathcal{U}_u = (\mathcal{U}(u), \mathcal{E}_u, \Gamma_u, \mathfrak{s}_u, \psi_u) \) is a Kuranishi chart of the moduli space \( \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta) \) at \( u \).

9. Construction of Kuranishi Structures

So far, we constructed a Kuranishi chart at each point of \( \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta) \). In this section we construct a global Kuranishi structure. We follow similar arguments as in [FOOO4, FOOO8]. However, there are certain points that our treatment is different. We discuss the construction emphasizing on those differences.

9.1. Compatible Trivialization and Stabilization Data. Throughout this subsection, we fix:

\[ u(j) = ((\Sigma(j), v, \vec{z}(j), v, u(j), v); v \in C^\text{int}_0(\tilde{R}(j))) \quad j = 1, 2 \]

an element of \( \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta) \) which is contained in the stratum corresponding to the very detailed DD-ribbon trees \( \tilde{R}(j) = (c(j), \alpha(j), m(j), \lambda(j)) \). We denote the union of the irreducible components \( \Sigma(j), v \) by \( \Sigma(j) \). The map \( u(j) \) are also defined similarly, and \( \vec{z}(j) \) is the set of boundary marked points of \( \Sigma(j) \). We use a similar convention several times in this section. We assume that \( u(2) \) belongs to a small neighborhood of \( u(1) \) in the RGW-topology. To be more precise, for \( u(j) \), let \( \Xi(j) = (\vec{w}(j), (\mathcal{N}(j), v, x), (\phi(j), v), (\varphi(j), v, e), \kappa(j)) \) be a fixed TSD. We assume that \( u(2) \) is represented by an element of the space \( \mathcal{U}(u(1), \Xi(1)) \).

This assumption implies that \( \tilde{R}(2) \) is obtained from \( \tilde{R}(1) \) by level shrinkings, level 0 edge shrinkings and fine edge shrinkings. In particular, we may regard:

\[ C^\text{int}_1(\tilde{R}(2)) \subseteq C^\text{int}_1(\tilde{R}(1)). \]

There also exists a surjective map \( \pi : \tilde{R}(1) \to \tilde{R}(2) \) inducing:

\[ \pi : C^\text{int}_0(\tilde{R}(1)) \to C^\text{int}_0(\tilde{R}(2)) \]

such that the irreducible component corresponding to \( v \in C^\text{int}_0(\tilde{R}(2)) \) is obtained by gluing the irreducible components corresponding to \( \tilde{v} \in \pi^{-1}(v) \subset C^\text{int}_0(\tilde{R}(1)) \). There also exists a surjective map:

\[ \nu : \{0, 1, \ldots, |\lambda(1)|\} \to \{0, 1, \ldots, |\lambda(2)|\} \]
such that $i \leq j$ implies $\nu(i) \leq \nu(j)$, and $\lambda_{(2)}(\pi(\hat{v})) = \nu(\lambda_{(1)}(\hat{v}))$ for inside vertices $\hat{v}$ of $\hat{R}_{(1)}$. The maps $\pi$ and $\nu$ are the analogue of treesh and levsh in [DF2, Lemma 5.23] defined for detailed trees.

To describe the coordinate change, it is convenient to start with the case that the TSDs $\Xi_{(1)}$ and $\Xi_{(2)}$ satisfy some compatibility conditions. In this subsection, we discuss these compatibility conditions and in Subsection 9.3, we explain how a coordinate change can be constructed assuming these conditions. In Subsection 9.4, we consider the case that $\Xi_{(2)}$ is isomorphic to $\Sigma_{(1)}$ determined by $u_{(1)}(\bar{f}_0, \bar{\sigma}_0)$ and $u_{(2)}$.

The assumption that $u_{(2)}$ belongs to a small neighborhood of $u_{(1)}$ implies that we can find:

$$\bar{\sigma}_0 = (\sigma_{0,e}; e \in C^1_1(\hat{R}_{(1)})) \in \prod_{e \in C^1_1(\hat{R}_{(1)})}\gamma^{\text{deform}}_{(1),e}$$

and

$$\bar{\nu}_0 = (\nu_{0,e}; v \in C^0_0(\hat{R}_{(1)})) \in \prod_{v \in C^0_0(\hat{R}_{(1)})}\gamma^{\text{source}}_{(1),v}$$

such that the inconsistent map:

$$(\Sigma_{(1)}(\bar{f}_0, \bar{\sigma}_0), \bar{z}_{(1)}(\bar{f}_0, \bar{\sigma}_0), u_{(1)}(\bar{f}_0, \bar{\sigma}_0))$$

is isomorphic to $(\Sigma_{(2)}, \bar{z}_{(2)}, u_{(2)})$. Although it is not clear from the notation, the map $u_{(1)}(\bar{f}_0, \bar{\sigma}_0)$ in (9.1) depends on $u_{(2)}$ and not just on $\bar{f}_0$ and $\bar{\sigma}_0$. We assume that the additional marked points and transversals in $\Xi_{(2)}$ satisfy the following conditions:

**Condition 9.1.** Since (9.1) is induced by an element of $U(u_{(1)}, \Xi_{(1)})$, there is a set of marked points $\bar{w}_{(1)}(\bar{f}_0, \bar{\sigma}_0) \subset \Sigma_{(1)}(\bar{f}_0, \bar{\sigma}_0)$ determined by $\bar{w}_{(1)}$, $\bar{f}_0$, and $\bar{\sigma}_0$. Then we require that the marked points $\bar{w}_{(2)}$ of $\Xi_{(2)}$ are chosen such that:

$$(\Sigma_{(1)}(\bar{f}_0, \bar{\sigma}_0), \bar{z}_{(1)}(\bar{f}_0, \bar{\sigma}_0) \cup \bar{w}_{(1)}(\bar{f}_0, \bar{\sigma}_0)) \cong (\Sigma_{(2)}, \bar{z}_{(2)} \cup \bar{w}_{(2)}).$$

Furthermore, if $w_{(2),v,i}$ corresponds to $w_{(1),\hat{v},i}$, then we require:

$$(9.3) \quad N_{(2),v',v} = N_{(1),v,i}.$$ 

Next, we impose some constraints on the choices of the maps $\phi_{(2),v}$ and $\varphi_{(2),v,e}$. Let $v$ be an interior vertex of $\hat{R}_{(2)}$. We consider the moduli space $\mathcal{M}^\text{source}_{v}$ of deformation of the irreducible component $\Sigma_{(2),v}$, $\bar{z}_{(2),v} \cup \bar{w}_{(2),v}$. We firstly fix a neighborhood $\gamma_{(2),v}$ of $\Sigma_{(2),v}$, $\bar{z}_{(2),v} \cup \bar{w}_{(2),v}$ in $\mathcal{M}^\text{source}_{v}$ as follows. The Riemann surface $\Sigma_{(2),v}$ is obtained by gluing spaces $\Sigma_{(1),\hat{v}}$ for

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29Here we use the correspondence between $\bar{w}_{(1)}$, $\bar{w}_{(2)}$ given by the identification in (9.2) and the correspondence between the elements of $w_{(1)}$ and $w_{(2)}$. 
\( \hat{v} \in \pi^{-1}(v) \). Here the complex structure on \( \Sigma_{(1)} \) is given by \( \mathfrak{f}_{0, \hat{v}} \) and the gluing parameters are \( \sigma_{(0), \hat{e}} \) for edges \( \hat{e} \) in \( \pi^{-1}(v) \). There is a neighborhood \( U^\text{source}_{(1), \hat{e}} \) of \( \mathfrak{f}_{0, \hat{e}} \) in \( \gamma^\text{source}_{(1), \hat{e}} \) and a neighborhood \( \gamma^\text{deform}_{(1),(2),e} \) of \( \sigma_{(0), \hat{e}} \in \gamma^\text{deform}_{(1), e} \) such the following map:

\[
\prod_{\hat{e} \in C^\text{int}_0(\hat{R}(1)), \pi(\hat{e}) = v} U^\text{source}_{(1), \hat{e}} \times \prod_{\hat{e} \in C^\text{int}_1(\hat{R}(1)), \pi(\hat{e}) = v} \gamma^\text{deform}_{(1),(2),e} \rightarrow \mathcal{M}^\text{source}_{v}.
\]

is an isomorphism onto an open neighborhood of the point determined by \( (\Sigma_{(2),v}, \hat{z}_2, v) \cup \hat{w}_2(2),v) \). Therefore, we may define:

\[
(9.4) \quad \gamma^\text{source}_{(2),v} := \prod_{\hat{e} \in C^\text{int}_0(\hat{R}(1)), \pi(\hat{e}) = v} U^\text{source}_{(1), \hat{e}} \times \prod_{\hat{e} \in C^\text{int}_1(\hat{R}(1)), \pi(\hat{e}) = v} \gamma^\text{deform}_{(1), e}.
\]

Let \( \mathfrak{f}_{2,v} = ((\mathfrak{f}_{1, \hat{e}}), (\sigma_{1, \hat{e}})) \) be an element of (9.4). Then \( \Sigma_{(2),v}(\mathfrak{f}_{2,v}) \), the Riemann surface \( \Sigma_{(2),v} \) with the complex structure induced by \( \mathfrak{f}_{2,v} \), has the following decomposition:

\[
(9.5) \quad \Sigma_{(2),v}(\mathfrak{f}_{2,v}) = \prod_{\hat{e} \in C^\text{int}_0(\hat{R}(1)), \pi(\hat{e}) = v} \Sigma_{(1), \hat{e}}(\mathfrak{f}_{1, \hat{e}}) \cup \prod_{e \in C^\text{int}_1(\hat{R}(2)), v \in \partial e} D^2 \cup \prod_{\hat{e} \in C^\text{int}_1(\hat{R}(1)), \pi(\hat{e}) = v} [-5T_{\hat{e}}, 5T_{\hat{e}}] \times S^1.
\]

The following comments about the above decomposition is in order. In (9.5), \( \Sigma_{(1), \hat{e}}(\mathfrak{f}_{1, \hat{e}}) \) denote the subspace of \( \Sigma_{(1), \hat{e}}(\mathfrak{f}_{1, \hat{e}}) \) given by the complements of the discs \( \varphi_{(1), \hat{e}, \hat{e}}(\mathfrak{f}_{1, \hat{e}}, D^2(1)) \) where \( \hat{e} \) runs among the edges of \( \hat{R}(1) \) which are connected to \( \hat{v} \). For each edge \( e \in C^\text{int}_1(\hat{R}(2)) \) which is incident to \( v \in C^\text{int}_0(\hat{R}(2)) \), there is a unique edge \( \hat{e} \in C^\text{int}_1(\hat{R}(1)) \) which is mapped to \( e \). In particular, one of the endpoints of \( \hat{e} \), denoted by \( \hat{v} \), is mapped to \( v \). The disc corresponding to \( e \) in (9.6) is given by the space \( \varphi_{(1), \hat{e}, \hat{e}}(\mathfrak{f}_{1, \hat{e}}, D^2(1)) \). Finally if an edge \( \hat{e} \in C^\text{int}_1(\hat{R}(1)) \) is mapped to a vertex \( v \in C^\text{int}_0(\hat{R}(2)) \) by \( \pi \), then the space in (9.7) is identified with the neck region associated to \( \hat{e} \). In particular, the positive number \( T_{\hat{e}} \) is determined by \( \sigma_{(1), \hat{e}} \). The union of the spaces in (9.5) and (9.6) is called the thick part of \( \Sigma_{(2),v}(\mathfrak{f}_{2,v}) \), and the spaces in (9.7) form the thin part of \( \Sigma_{(2),v}(\mathfrak{f}_{2,v}) \). The above decomposition can be used in an obvious way to define the map \( \varphi_{(2), v, e} \) on \( \gamma^\text{source}_{(2), v} \times \text{Int}(D^2) \).
We have the following decomposition of $\Sigma_{(2),v}$ as a special case of the above decomposition applied to the point $((x_0,\hat{v}), (\sigma_0,\hat{e}))$:

\begin{align}
\Sigma_{(2),v} &= \coprod_{\hat{v} \in C^\interior_0(\hat{R}_0), \pi(\hat{v}) = v} \Sigma_{(1),v}(x_0,\hat{v}) \nonumber \\
&\quad \cup \coprod_{\hat{e} \in C^\interior_1(\hat{R}_1), \pi(\hat{e}) = v} D^2 \nonumber \\
&\quad \cup \coprod_{\hat{e} \in C^\interior_1(\hat{R}_1), \pi(\hat{e}) = v} [-5T^e_{\hat{e}}, 5T^e_{\hat{e}}] \times S^1. \tag{9.10} \nonumber
\end{align}

The trivialization $\phi_{(2),v}$ that we intend to define is a family (parametrized by $x_2,v$) of diffeomorphisms from $\Sigma_{(2),v}$ to $\Sigma_{(2),v}(x_2,v)$. The trivialization $\phi_{(1),v}$ defines a diffeomorphism between the subspaces in (9.5) and (9.8). We then use the coordinate at nodal points, $\varphi_{(1),\hat{v},\hat{e}}$, to extend it to a diffeomorphism from the unions of the subspaces in (9.8) and (9.9) to the union of the subspaces in (9.5) and (9.6). Finally we extend this family of diffeomorphisms in an arbitrary way to the neck region to complete the construction of $\phi_{(2),v}$.

This construction of the maps $\varphi_{(2),v,e}$ and $\phi_{(2),v}$ is analogous to [FOOO8, Sublemma 10.15].

**Condition 9.2.** We require that the maps $\phi_{(2),v}$ and $\varphi_{(2),v,e}$ of the TSD $\Xi_{(2)}$ are obtained form the TSD $\Xi_{(1)}$ as above.

**Definition 9.3.** Let $u$ be an element of $M_{k+1}^{\text{RGW}}(L; \beta)$ and $\Xi$ be a TSD at $u$. An *inconsistent map near $u$ with respect to $\Xi$* is an object similar to the ones in Definition 8.8 where we relax the Cauchy-Riemann equations in (4) and (5).

This definition is almost a straightforward generalization of Definition 7.5 with the difference that we also include transversal constraints in (4.20) as one of the requirements for an inconsistent map near $u$.

Let $\Xi_{(2)}$ satisfy Conditions 9.1 and 9.2. For any:

\begin{align}
\tilde{\sigma}_{(2)} &= (\sigma_{2,e}; e \in C^\interior_1(\hat{R}_2)) \in \coprod_{e \in C^\interior_1(\hat{R}_2)} \nu^\text{deform}_{(2),v} \nonumber \\
\tilde{\mathbf{f}}_{(2)} &= (\mathbf{f}_{2,v}; v \in C^\interior_0(\hat{R}_2)) \in \coprod_{v \in C^\interior_0(\hat{R}_2)} \nu^\text{source}_{(2),v}, \tag{9.11} \nonumber
\end{align}

satisfying Definition 8.8 (1) with respect to $\Xi_{(2)}$, there exist:

\begin{align}
\tilde{\sigma}_{(1)} &= (\sigma_{1,e}; e \in C^\interior_1(\hat{R}_1)) \in \coprod_{e \in C^\interior_1(\hat{R}_1)} \nu^\text{deform}_{(1),v} \nonumber \\
\tilde{\mathbf{f}}_{(1)} &= (\mathbf{f}_{1,v}; v \in C^\interior_0(\hat{R}_1)) \in \coprod_{v \in C^\interior_0(\hat{R}_1)} \nu^\text{source}_{(1),v} \tag{9.12} \nonumber
\end{align}

satisfying Definition 8.8 (1) with respect to $\Xi_{(1)}$ such that:

\begin{equation}
(\Sigma_{(2)}(\tilde{\mathbf{f}}_{(2)}, \tilde{\sigma}_{(2)}), \tilde{z}_{(2)}(\tilde{\mathbf{f}}_{(2)}, \tilde{\sigma}_{(2)}) \cup \tilde{w}_{(2)}(\tilde{\mathbf{f}}_{(2)}, \tilde{\sigma}_{(2)})) \cong (9.13)
\end{equation}
(\Sigma(1)(\vec{f}(1), \vec{\sigma}(1)), \vec{z}(1)(\vec{f}(1), \vec{\sigma}(1)) \cup \vec{w}(1)(\vec{f}(1), \vec{\sigma}(1))).

Here \vec{f}(1), \vec{\sigma}(1) depend on \vec{f}(2), \vec{\sigma}(2). (See Figure 15.)

Next, let \eta(2) = (\vec{f}(2), \vec{\sigma}(2), (u'(2), \nu), (U'(2), \nu), (\rho(2), e), (\rho(2), i)) be an inconsistent map near \vec{u}(2) with respect to \Xi(2). Let \vec{f}(1) and \vec{\sigma}(1) be chosen as in the previous paragraph. Let \hat{v} be an interior vertex of \hat{R}(1). Identification in (9.13) and Condition 9.2 imply:

(9.14) \Sigma^+_{(1), \hat{v}}(\vec{f}(1), \vec{\sigma}(1)) \subseteq \Sigma^+_{(2), \pi(\hat{v})}(\vec{f}(2), \vec{\sigma}(2)).

We write \hat{I}_\hat{v} for this inclusion map. Define:

(9.15) U'_{(1), \hat{v}} = \begin{cases} u'_{(2), \pi(\hat{v})} \circ \hat{I}_\hat{v} & \text{if } \lambda(2)(\pi(\hat{v})) = 0 \\ U''_{(2), \pi(\hat{v})} \circ \hat{I}_\hat{v} & \text{if } \lambda(2)(\pi(\hat{v})) > 0 \end{cases}

u'_{(1), \hat{v}} = u'_{(2), \pi(\hat{v})} \circ \hat{I}_\hat{v}.

We also define:

(9.16) \rho_{(1), e} = \begin{cases} \rho(2), e & \text{if } e \in C^{\text{int}}(\hat{R}(2)) \subseteq C^{\text{int}}(\hat{R}(1)) \\ 1 & \text{and } e \text{ is not a level 0 edge,} \\ \rho(2), i & \text{otherwise.} \end{cases}

We next define:

(9.17) \rho_{(1), i} = \begin{cases} 1 & \text{if } \nu(i - 1) = \nu(i) \\ \rho(2), i & \text{otherwise.} \end{cases}

It is easy to check that \eta(1) = (\vec{f}(1), \vec{\sigma}(1), (u'_{(1), \hat{v}}), (U'_{(1), \hat{v}}), (\rho(1), e), (\rho(1), i)) is an inconsistent map near \vec{u}(1) with respect to \Xi(1). In fact, (8.20)-(8.23) for \eta(1) follow from the corresponding identities for \eta(2) and the definition. This discussion is summarized in the following lemma:

Lemma 9.4. Suppose \vec{u}(1) and \vec{u}(2) are as above and \Xi(j) is a TSD at \vec{u}(j) such that \Xi(1) and \Xi(2) satisfy Conditions 9.1 and 9.2. Then an inconsistent map near \vec{u}(2) with respect to \Xi(2) can be regarded as an inconsistent map near \vec{u}(1) with respect to \Xi(1).
Lemma 9.6. Suppose $u_{(1)}$, $u_{(2)}$, $\Xi_{(1)}$ and $\Xi_{(2)}$ are given as in Lemma 9.4. In particular, $u_{(2)}$ can be regarded as an inconsistent solution:

$$(\tilde{f}_0, \tilde{\sigma}_0, (u^0_{(0),\tilde{e}}), (U^0_{(0),\tilde{e}}), (\rho_0(0), \iota))$$

with respect to $\Xi_{(1)}$. An inconsistent map:

$$\eta_{(1)} = (\tilde{f}_1, \tilde{\sigma}_1, (u^1_{(1),\tilde{e}}), (U^1_{(1),\tilde{e}}), (\rho_1(1), \iota))$$

near $u_{(1)}$ with respect to $\Xi_{(1)}$ may be regarded as an inconsistent map near $u_{(2)}$ if and only if the following conditions hold:

1. For each vertex $\tilde{v} \in C^0_0(\tilde{R}_{(1)})$, the distance between $\tilde{f}_0(\tilde{e})$ and $\tilde{f}_1(\tilde{e})$ is less than $\kappa(2)$. (Recall that $\kappa(2)$ is the size of $\Xi_{(2)}$).
2. For each edge $e \in C^1_0(\tilde{R}_{(2)}) \subset C^1_0(\tilde{R}_{(1)})$, we have $|\sigma_{(1),e}| < \kappa(2)$.
3. If $e \in C^1_0(\tilde{R}_{(2)}) \subset C^1_0(\tilde{R}_{(1)})$ is (resp. is not) a level 0 edge, then the image of the restriction of $u^1_{(1),\tilde{e}}$ to $[-5T_e, 5T_e]_{r_e} \times [0, \pi]_{s_e}$ (resp. $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e}$) has a diameter less than $\kappa(2)$.
4. If $c(v) = d$ or $s$, then the $C^2$-distance between the restrictions of $u^0_{(0),\tilde{e}}$ and $u^1_{(1),\tilde{e}}$ to $\Sigma_v(\tilde{f}, \tilde{\sigma})$ is less than $\kappa(2)$. If $c(v) = D$, then we demand that the $C^2$-distance between the restrictions of $U^0_{(0),\tilde{e}}$ and $U^1_{(1),\tilde{e}}$ to $\Sigma_v(\tilde{f}, \tilde{\sigma})$ is less than $\kappa(2)$.
5. The consistency equation

$$\rho_{(1),e} = \rho_{(1),\lambda_{(1),e_1} + \cdots + \rho_{(1),\lambda_{(1),e_2}}$$

are satisfied for any edge $e \in C^1_{th}(\tilde{R}_{(1)}) \setminus C^1_{th}(\tilde{R}_{(2)})$ which is not a level 0 edge.

9.2. The Choice of Obstruction Spaces. In order to define an inconsistent solution, we need to fix obstruction spaces. For any inconsistent map $u$ and for any other inconsistent map $\eta$ which is close to $u$, we explained how to make a choice of $\mathcal{E}_{0,u}(\eta)$ in Section 8. We need to insure that we can arrange for such choices such that they satisfy some nice properties when we move $u$. More precisely, we need to pick them so that they are semi-continuous with respect to $u$. We will prove this property in the next subsection. In this subsection, we explain how we modify our choice of obstruction spaces.

For $j = 1, 2$, let $u_{(j)} = (\Sigma_{(j),u}, \tilde{\sigma}_{(j),u}, u_{(j),v}) \in C^1_{th}(\tilde{R}_{(j)})$ be representatives of two elements of $\mathcal{M}_{k+1}^{RGW}(L; \beta)$ in the strata corresponding to very detailed DD-ribbon trees $\tilde{R}_{(j)}$. We fix a TSD $\Xi_{(j)}$ at $u_{(j)}$. We do not assume that they are related as in Subsection 9.1. We also fix $E_{u_{(1),v}} \subset L_{m,\delta}^2(u_{(1),v})$ satisfying Condition 8.2. We wish to use $\{E_{u_{(1),v}}\}$ to define obstruction spaces for an inconsistent map with respect to $\Xi_{(2)}$, under the assumption
that \( u_2 \) is close to \( u_1 \). In particular, we assume that \( \bar{R}_2 \) is obtained from \( \bar{R}_1 \) by level shrinking, level 0 edge shrinking and fine edge shrinking. As in the previous subsection, we may define a surjective map \( \pi : \bar{R}_1 \to \bar{R}_2 \). Note that in Section 8, we studied the case \( u_2 = u_1 \).

The following lemma is a straightforward consequence of the implicit function theorem: (See [FOOO8, Lemma 9.9].)

\textbf{Lemma 9.7.} If \( u_2 \) is close enough to \( u_1 \) with respect to the \( C^1 \) distance, then for any \( \hat{v} \in C_0^{\text{int}}(\bar{R}_1) \), there exists a unique choice of \( \bar{w}_{(2),v} \subseteq \Sigma_{(2),v} \) with \( v \) being \( \pi(\hat{v}) \) such that:

1. \( (\Sigma_{(2),v}, \bar{z}_{(2),v} \cup \prod_{\pi(\hat{v})=v} \bar{w}_{(2),(1),\hat{v}}) \) is close to:

\[
\prod_{\pi(\hat{v})=v} (\Sigma_{(1),\hat{v}}, \bar{z}'_{(1),\hat{v}} \cup \bar{w}_{(1),\hat{v}})
\]

in the moduli space of stable curves with marked points. Here \( \bar{z}'_{(1),\hat{v}} \) is the subset of \( \bar{z}_{(1),\hat{v}} \) that is the set of all marked points on \( \Sigma_{(1),v} \) and nodal points on \( \Sigma_{(1),v} \) which correspond to the edges \( e \) incident to \( v \) with \( \pi(e) \neq v \).

2. \( u_2((u_{(2),v})_{\hat{v}},(v_{(1),v})_{\hat{v}}) \in \mathcal{N}_{(1),\hat{v}} \).

From now on, we assume that \( u_2 \) is close enough to \( u_1 \) such that the claim in Lemma 9.7 holds. Furthermore, Let also an inconsistent map:

\[ \eta = (\bar{\iota}, \bar{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \]

with respect to \( \Xi_{(2)} = (\bar{w}_{(2)}, (\mathcal{N}_{(2),v}), (\phi_{(2),v}), (\varphi_{(2),v}), (\kappa_{(2)})) \) be fixed. Suppose also \( (\Sigma_{(2),\bar{r}}, \bar{z}_{(2),\bar{r}}, \bar{w}_{(2)}, (\bar{r}, \bar{\sigma})) \) denotes the representative of \( \bar{r} \), as a part of the data of \( \eta \).

Let \( \hat{v} \) be a vertex of \( \bar{R}_1 \) and \( v = \pi(\hat{v}) \). Lemma 9.7 allows us to find \( u_{(2),v} \in \Sigma_{(2),v} \). If \( \Xi_{(2)} \) is small enough, then we can regard \( u_{(2),v} \) as an element of \( \Sigma_{(2),v} \), and hence an element of \( \Sigma_{(2),v}(\bar{r}, \bar{\sigma}) \). This implies that if we replace \( \bar{w}_{(2),v} \) with the points \( u_{(2),v}(\bar{r}, \bar{\sigma}) \) to obtain:

\[
(\Sigma_{(2),v}(\bar{r}, \bar{\sigma}), \bar{z}_{(2),v}(\bar{r}, \bar{\sigma}) \cup \prod_{\pi(\hat{v})=v} \bar{w}_{(2),(1),\hat{v}})
\]

then (9.18) is close to \( \prod_{\pi(\hat{v})=v}(\Sigma_{(1),\hat{v}}, \bar{z}'_{(1),\hat{v}} \cup \bar{w}_{(1),\hat{v}}) \).

We use this fact and the target space parallel transportation in the same way as in Section 4 to obtain the following map for any \( \hat{v} \in C_0^{\text{int}}(\bar{R}_1) \):

\[ \mathcal{P}_{\hat{v}} : E_{u(1),\hat{v}} \to L^2_{m,\delta}(\Sigma_{(2),v}(\bar{r}, \bar{\sigma})). \]

We then define for any \( v \in \bar{R}_2 \):

\[ E_{u(2),u(1),v}(U'_v) = \bigoplus_{\hat{v}, \pi(\hat{v})=v} \mathcal{P}_{\hat{v}}(E_{u(1),\hat{v}}) \]

for \( \lambda(v) = 0 \). We also define \( E_{u(2),u(1),v}(U'_v) \) for \( \lambda(v) > 0 \) by a similar formula.
Now we replace (8.35) by
\[ E_{u(2)},u(1),v(u') \]
\[ \bigoplus_{\mathcal{C}(0) \oplus \mathcal{C}(0), \lambda(v) > 0} E_{u(2),u(1),v(U')} . \]

Lemma 9.8. Suppose \( \Xi(1) \) is small enough such that we can apply the construction of Section 8 to \( \Xi(1) \) and the vector spaces \( \{E_{u(1),\hat{v}}\} \) to obtain a Kuranishi chart at \( u(1) \). Then for \( \Xi(2) \) small enough, applying the construction of Section 8 to \( \{E_{u(2),u(1),v}\} \) (instead of \( \{E_{u(2),\hat{v}}\} \)) gives rise to a Kuranishi chart at \( u(2) \).

This is immediate from the construction of Section 8. In fact, the choice of obstruction bundles we take here satisfies the ‘smoothness’ condition. See Definition 10.9 or [FOOO5, Definition 5.1 (2)]. (Smoothness here means smoothness with respect to \( y \).)

For each \( p \in M_{k+1}^{RGW}(L;\beta) \), let \( \Xi(p) = (\hat{w}, (\mathcal{N}_{p,v,i}), (\phi_{p,v,d}), (\varphi_{p,v,e}), \kappa_{p}) \) and \( E_{p,v} \) be a TSD and obstruction vector spaces at \( p \). We assume that \( \Xi(p) \) is small enough such that the assumption of Lemma 9.8 holds. Let \( \mathcal{U}(p) \) be a neighborhood of \( p \) in \( M_{k+1}^{RGW}(L;\beta) \) determined by the TSD \( \Xi(p) \). We also fix a compact neighborhood \( K(p) \) of \( p \) which is a subset of \( \mathcal{U}(p) \).

Compactness of \( M_{k+1}^{RGW}(L;\beta) \) implies that we can find a finite subset
\[ J = \{p_j : j = 1, \ldots, J\} \subset M_{k+1}^{RGW}(L;\beta) \]
such that
\[ M_{k+1}^{RGW}(L;\beta) \subseteq \bigcup_{j=1}^{J} \text{Int}(K(p_j)) . \]

For \( u \in M_{k+1}^{RGW}(L;\beta) \), we define:
\[ J(u) = \{p_j \mid u \in K(p_j)\} . \]

Lemma 9.9. Let \( \hat{R}_j \) be the very detailed tree associated to \( p_j \). We can perturb \( \{E_{p_j,v} \mid v \in C^0(\hat{R}_j)\} \) by an arbitrary small amount so that the following holds. For any \( u \in M_{k+1}^{RGW}(L;\beta) \), the vector spaces \( \mathcal{E}_{0,u,p_j}(u) \) for \( p_j \in J(u) \) are transversal, i.e., the sum of \( \mathcal{E}_{0,u,p_j}(u) \) for \( p_j \in J(u) \) is the direct sum:
\[ \bigoplus_{p_j \in J(u)} \mathcal{E}_{0,u,p_j}(u) . \]

Proof. The proof is the same as the proof of the analogous statement in the case of the stable map compactification. See the proof of [FOOO8, Lemma 11.7] in [FOOO8, Subsection 11.4].

Now we define a Kuranishi chart at each point \( u \in M_{k+1}^{RGW}(L;\beta) \) as follows:
Definition 9.10. Let $\Xi_u = (\tilde{w}_u, (\mathcal{N}_{u,v}, \kappa_u), (\phi_{u,v}), (\varphi_{u,v,e}), \kappa_u)$ be a TSD, which is small enough such that the conclusion of Lemma 9.8 holds for $u(1) = p_j$, $u(2) = u$, $\Xi(1) = \Xi_{p_j}$ and $\Xi(2) = \Xi_u$ with $p_j$ being an arbitrary element in $\mathfrak{J}(u)$. The Kuranishi neighborhood $U(u, \Xi_u)$ is the set of the equivalence classes of inconsistent maps $\eta = (\bar{f}, \bar{\sigma}, (u'_{(1)}), (U'_{(1)}), (\rho_{(1, e)}), (\rho_{(1, i)}))$ near $u$ such that

$$\partial_{j \epsilon} u'_{(1)} \in \bigoplus_{p_j \in \mathfrak{J}(u)} \mathcal{E}_{0, u, p_j}(\eta),$$

(9.24)

$$\partial_{j \epsilon} U'_{(1)} \in \bigoplus_{p_j \in \mathfrak{J}(u)} \mathcal{E}_{0, u, p_j}(\eta).$$

In other words, it is the set of inconsistent solutions (Definition 8.8) where (8.24) and (8.25) are replaced with (9.24).

The obstruction bundle $\mathcal{E}_u$ is defined in the same way as in (8.36) in the following way:

$$\mathcal{E}_{0, u}(\eta) = \bigoplus_{p_j \in \mathfrak{J}(u)} \mathcal{E}_{0, u, p_j}(\eta)$$

(9.25)

$$\mathcal{E}_{u, \Xi_u}(\eta) = \mathcal{E}_{0, u}(\eta) \oplus \bigoplus_{e \in C^*_{\epsilon}(\hat{R}), \lambda(e) > 0} \mathcal{L}_e,$$

(9.26)

where $\hat{R}$ is the very detailed tree associated to $u$. The Kuranishi map $s_u$ is defined in the same way as in (8.37), (8.38), (8.39), and the parametrization map $\psi_u$ is defined in the same way as in (8.40).

9.3. Construction of Coordinate Change I. In this and the next subsections, we construct coordinate changes. The next two lemmas state the semi-continuity of our obstruction spaces, a property that we hinted at the beginning of the last subsection.  

Lemma 9.11. For any $u(1) \in \mathcal{M}_{k+1}(L; \beta)$, there exists a neighborhood $U(u(1))$ of $u(1)$ in $\mathcal{M}_{k+1}(L; \beta)$ such that for any $u(2) \in U(u(1))$:

$$\mathfrak{J}(u(2)) \subseteq \mathfrak{J}(u(1)).$$

(9.27)

Proof. This is obvious because we pick the subspaces $\mathcal{K}(p_j)$ to be closed.  

Lemma 9.12. Let $u(2) \in U(u(1))$. For $j = 1, 2$, we choose a TSD $\Xi_{(j)} = (\bar{w}_{(j)}, (\mathcal{N}_{(j,v),x}), (\phi_{(j,v)}), (\varphi_{(j,v,e)}))$. We assume that $u(1), u(2), \Xi(1)$ and $\Xi(2)$ satisfy Conditions 9.1 and 9.2. Let:

$$\eta_{(2)} = (\bar{f}_{(2)}, \bar{\sigma}_{(2)}, (u'_{(2),v}), (U'_{(2),v}), (\rho_{(2,e)}), (\rho_{(2,i)}))$$

be an inconsistent map near $u(2)$ with respect to $\Xi(2)$ and

$$\eta_{(1)} = (\bar{f}_{(1)}, \bar{\sigma}_{(1)}, (u'_{(1),v}), (U'_{(1),v}), (\rho_{(1,e)}), (\rho_{(1,i)}))$$

30 Compare to [FOOO8, Definition 5.1 (4)] or Definition 10.5.
be the inconsistent map near $u(1)$ with respect to $\Xi(1)$, constructed by Lemma 9.4. Let $p_j \in J(u(2))$. Then we have an isomorphism:

$$(9.28) \quad \mathcal{E}_{0,u(2)}(p_j)(\eta(2)) \cong \mathcal{E}_{0,u(1)}(p_j)(\eta(1)).$$

**Proof.** Let $I'_\circ$ be the inclusion map (9.14). This is a holomorphic embedding. Then (9.15) induces the required isomorphism. The fact that transversality constraint (8.26) is preserved is an immediate consequence of (9.15) and our choice of transversals $N_{(j),\nu,i}$.

**Lemma 9.13.** Suppose $u(1), u(2), \Xi(1), \Xi(2), \eta(1)$ and $\eta(2)$ are given as in Lemma 9.12. If $\eta(2)$ is an element of $\hat{\mathcal{U}}(u(2), \Xi(2))$, then $\eta(1)$ is an element of $\hat{\mathcal{U}}(u(1), \Xi(1))$.

**Proof.** The isomorphism induced by $I'_\circ$ send $\bar{\partial}u'(2), \bar{\partial}u'(1)$ to $\bar{\partial}u'(1), \bar{\partial}u'(2)$. This is a consequence of (9.15). Therefore, if $\eta(2)$ satisfies (9.24) then $\eta(1)$ satisfies (9.24).

We thus constructed a $\Gamma_{u(2)}$-invariant map:

$$\varphi_{u(1), u(2)} : \hat{\mathcal{U}}(u(2), \Xi(2)) \to \hat{\mathcal{U}}(u(1), \Xi(1)).$$

It is clear from the construction that the above map can be lifted to a map $\tilde{\varphi}_{u(1), u(2)}$ from $\tilde{\mathcal{U}}(u(2), \Xi(2))$ to $\tilde{\mathcal{U}}(u(1), \Xi(1))$.

**Lemma 9.14.** The maps $\varphi_{u(1), u(2)}$ and $\tilde{\varphi}_{u(1), u(2)}$ are $C^k$ embeddings.

**Proof.** It follows from the definition of $\tilde{\varphi}_{u(1), u(2)}$ and the choices of $\Xi(1)$ that the following two diagrams commute:

$$\begin{array}{ccc}
\tilde{\mathcal{U}}(u(2), \Xi(2)) & \xrightarrow{F_1} & \prod_{v \in C^f_{0}(\hat{\Sigma}(2))} \nu^{source}_{(2,v)} \\
\downarrow {\tilde{\varphi}_{u(1), u(2)}} & & \downarrow R_1 \\
\tilde{\mathcal{U}}(u(1), \Xi(1)) & \xrightarrow{F_2} & \prod_{v \in C^f_{0}(\hat{\Sigma}(1))} \nu^{source}_{(1,v)} \\
\end{array}$$

$$(9.29) \quad \bar{\partial}u'(2), \bar{\partial}u'(1)$$

$$\begin{array}{ccc}
\tilde{\mathcal{U}}(u(2), \Xi(2)) & \xrightarrow{F_1} & \prod_{v \in C^f_{0}(\hat{\Sigma}(2))} \nu^{source}_{(2,v)} \\
\downarrow {\tilde{\varphi}_{u(1), u(2)}} & & \downarrow R_2 \\
\tilde{\mathcal{U}}(u(1), \Xi(1)) & \xrightarrow{F_2} & \prod_{v \in C^f_{0}(\hat{\Sigma}(1))} \nu^{source}_{(1,v)} \\
\end{array}$$

$$(9.30) \quad \bar{\partial}u'(2), \bar{\partial}u'(1)$$
Here the horizontal arrows $F_1$ and $F_2$ are as in (8.30). The right vertical arrow $R_1$ of Diagram (9.29) is obtained by requiring (9.13) and is a smooth embedding. Diagram (9.29) commutes since $\varphi_u(u(1)u(2))$ does not change the conformal structure of source (marked) curves. The right vertical arrow $R_2$ of Diagram (9.30) is obtained by restriction of domain and is a smooth map. Diagram (9.30) commutes because of Condition 9.2. Now the definitions of Diagram (9.30) is obtained by restriction of domain and is a smooth map. Lemma 9.15. Here the horizontal arrows of Diagram (9.29) is obtained by requiring (9.13) and is a smooth embedding. A similar argument applies to the map $\varphi_{u(1)u(2)}$. □

We next define a bundle map $\overline{\varphi}_{u(1)u(2)} : \mathcal{E}_u(2) \to \mathcal{E}_u(1)$ which lifts $\varphi_{u(1)u(2)}$. Using (9.27) and (9.28), we obtain a linear embedding:

\[(9.31) \bigoplus_{\mathcal{E}_{0,u(2),p_j}(\mathcal{E}(2))} \mathcal{E}_{0,u(1),p_j}(\mathcal{E}(1)) \to \bigoplus_{\mathcal{E}_{0,u(1),p_j}(\mathcal{E}(1))} \mathcal{E}_{0,u(1),p_j}(\mathcal{E}(1))\]

if $\eta(1) = \varphi_{u(1)u(2)}(\eta(2))$. The map:

\[(9.32) \bigoplus_{e \in C_{\text{int}}^h(R(2)) \neq 0} \mathcal{L}_e \to \bigoplus_{e \in C_{\text{int}}^h(R(1)) \neq 0} \mathcal{L}_e\]

is defined as identity on $e \in C_{\text{int}}^h(R(2)) \subset C_{\text{int}}^h(R(1))$ and is zero on the other factors. The bundle map $\overline{\varphi}_{u(1)u(2)}$ is defined using (9.31) and (9.32). Analogous to Lemma 9.14, we can prove that $\overline{\varphi}_{u(1)u(2)}$ is $C^\ell$.

**Lemma 9.15.**

\[\mathcal{S}(1) \circ \varphi_{u(1)u(2)} = \overline{\varphi}_{u(1)u(2)} \circ \mathcal{S}(2).\]

**Proof.** On the factor in (9.31) this is a consequence of the definitions of the map $\varphi_{u(1)u(2)}$ and (9.31). Namely, it follows from (9.15) and the fact that $I_\ell$ is bi-holomorphic. For the factor in (9.32), this is a consequence of (9.16). □

Compatibility of the parametrization map with $\varphi_{u(1)u(2)}$ is also an immediate consequence of the definitions. We thus proved that:

**Proposition 9.16.** Let $u(1), u(2), \Xi(1)$ and $\Xi(2)$ satisfy Conditions 9.1 and 9.2 and $u(2) \in U(u(1))$. Then the pair $(\varphi_{u(1)u(2)}, \overline{\varphi}_{u(1)u(2)})$ is a coordinate change of Kuranishi charts.

9.4. **Construction of Coordinate Change II.** Let $u = ((\Sigma_v, \Xi_v, u_v); v \in C^\text{int}_0(\mathbb{R})) \in \mathcal{M}_k\text{RGW}(L; \beta)$. We fix two TSDs:

\[\Xi(j) = (\Xi(j), (\mathcal{N}(j)_{v_1}), (\phi(j)_v), (\varphi(j)_v), (\kappa(j)))\]

at $u$ such that we can use Definition 9.10, to form Kuranishi charts:

\[\mathcal{U}_u,\Xi(j) = (\mathcal{U}(u, \Xi(j)), \mathcal{E}_u,\Xi(j), \Gamma_u, \mathcal{S}_u,\Xi(j), \psi_u,\Xi(j)).\]
These Kuranishi charts depend on the choices of the subset \{p_j\} of the moduli space \(M_{k+1}(L;\beta)\), the TSDs \(\Xi_{p_j}\), the vector spaces \(E_{p_j,v}\) and the open sets \(K(p_j)\). We assume that these choices agree with each other for the above two charts. In this subsection, we will construct a coordinate change from \(U_u,\xi(2)\) to \(U_u,\xi(1)\).

The TSD \(\Xi_{(j)}\) determines the subspace \(\Sigma_{(j),v}\) of \(\Sigma_v\) for each interior vertex \(v\) of \(\hat{R}\). We assume that \(\xi(2)\) is small enough such that \(\bar{w}(1) \cap \Sigma_v\) is a subset of \(\Sigma_{(2),v}\).

We pick an inconsistent map with respect to \(\Xi_{(2)}\) denoted by:

\[\eta(2) = (\bar{x}(2), \bar{\sigma}(2), (u'_j, v), (U_j, v), (\rho_2, v), (\rho_2, i))\]

Associated to \(\eta(2)\), we have \(\Sigma_{(2),v}(\bar{x}(2), \bar{\sigma}(2))\), which comes with marked points:

\[\bar{z}_{(2),v}(\bar{x}(2), \bar{\sigma}(2)) \cup \bar{w}_{(2),v}(\bar{x}(2), \bar{\sigma}(2)).\]

Here the elements of \(\bar{z}_{(2),v}(\bar{x}(2), \bar{\sigma}(2))\) are in correspondence with the boundary marked points of \(\Sigma_v\) and \(\bar{w}_{(2),v}(\bar{x}(2), \bar{\sigma}(2))\) are in correspondence with the additional marked points \(\bar{w}_{(2),v}\) given by \(\Xi_{(2)}\). We will write \(\bar{z}_{(2),v}(\bar{x}(2), \bar{\sigma}(2))\) for the union of all boundary marked points of \(\Sigma_{(2)}(\bar{x}(2), \bar{\sigma}(2))\). The following lemma is the analogue of Lemma 9.7:

**Lemma 9.17.** There exists \(\bar{w}_{(2),v}(\eta(2)) \subset \Sigma_{(2),v}(\bar{x}(2), \bar{\sigma}(2))\) such that:

1. \(\bar{w}_{(2),v,i}(\eta(2))\) is close to \(\bar{w}_{(1),v,i}\). Here we identify \(\Sigma_{(2),v}(\bar{x}(2), \bar{\sigma}(2))\) and \(\Sigma_{(1),v}\) using \(\Xi_{(2)}\).
2. \(u'_{(2),v}(\bar{w}_{(2),v,i}(\eta(2))) \in \mathcal{N}_{(1),v,i}\).

We define:

\[\bar{w}_{(2),v}(\bar{x}(2), \bar{\sigma}(2)) = \bigcup_{v \in C^{(2)}_{int}(\hat{R})} \bar{w}_{(2),v}(\bar{x}(2), \bar{\sigma}(2)).\]

Then \((\Sigma_{(2)}(\bar{x}(2), \bar{\sigma}(2)), \bar{z}_{(2),v}(\bar{x}(2), \bar{\sigma}(2)) \cup \bar{w}_{(2),v}(\eta(2)))\) is close to \((\Sigma, \bar{z} \cup \bar{w}_{(1)})\) in the moduli space of bordered nodal curves. Therefore, there exists \(\bar{x}(1), \bar{\sigma}(1)\) such that:

\[\text{(9.33)} \quad (\Sigma_{(2)}(\bar{x}(2), \bar{\sigma}(2)), \bar{z}_{(2),v}(\bar{x}(2), \bar{\sigma}(2)) \cup \bar{w}_{(2),v}(\eta(2))) \cong (\Sigma_{(1)}(\bar{x}(1), \bar{\sigma}(1)), \bar{z}_{(1)}(\bar{x}(1), \bar{\sigma}(1)) \cup \bar{w}_{(1)}(\eta(1), \Omega(1))).\]

Here we use \(\Xi_{(1)}\) to define the right hand side. Let \(I\) be an isomorphism from the right hand side of (9.33) to the left hand side. Note that the choices of \(\bar{x}(1), \bar{\sigma}(1)\) and \(I\) are unique up to an element of \(\Gamma_{u}\).

---

\(^{31}\)In fact, these two coordinate charts are isomorphic after possibly shrinking \(U(u, \Xi_{(j)})\) into appropriate open subspaces.
We consider decompositions:

\[
\Sigma_{(j)}(\mathbf{\bar{r}}(j), \mathbf{\bar{\sigma}}(j)) = \bigcup_{v \in C^0_{\text{int}}(\mathcal{R})} \Sigma^-_{(j),v}(\mathbf{\bar{r}}(j), \mathbf{\bar{\sigma}}(j)) \\
\cup \bigcup_{e \in C^1_{\text{int}}(\mathcal{R})} [-5T_{(j),e}, 5T_{(j),e}] \times S^1.
\]  

(9.34)

for \( j = 1, 2 \). In the above identity, we define \( T_{(j),e} \) by requiring \( e^{-10T_{(j),e}} = |\sigma_{(j),e}| \). Here for simplicity, we assume that \( \sigma_{(2),e} \) is non-zero for all interior edges \( e \) of \( \mathcal{R} \). A similar discussion applies to the case that \( \sigma_{(2),e} = 0 \) with minor modifications. For example, in (9.34) we need to include two half cylinders for each \( e \) that \( \sigma_{(2),e} = 0 \). We also have:

\[
\Sigma_{(j)}(\mathbf{\bar{r}}(j), \mathbf{\bar{\sigma}}(j)) = \bigcup_{v \in C^0_{\text{int}}(\mathcal{R})} \Sigma^+_{(j),v}(\mathbf{\bar{r}}(j), \mathbf{\bar{\sigma}}(j)).
\]

(9.35)

Although \( \mathcal{I} \) in (9.33) is an isomorphism, it does not respect the decompositions in (9.34) or (9.35) for \( j = 1, 2 \). This is because \( \Xi_{(1)} \neq \Xi_{(2)} \).

Nevertheless, one can easily prove:

**Lemma 9.18.** If \( \Xi_{(2)} \) is small enough, then \( \mathcal{I} \) can be chosen such that the following holds. Let \( v \in C^0_{\text{int}}(\mathcal{R}) \) and \( \mathbf{\bar{z}} \in \Sigma^+_{(1),v}(\mathbf{\bar{r}}(1), \mathbf{\bar{\sigma}}(1)) \). Then at least one of the following conditions holds:

1. \( I(\mathbf{\bar{z}}) \in \Sigma^+_{(2),v}(\mathbf{\bar{r}}(2), \mathbf{\bar{\sigma}}(2)) \).
2. There exists \( e \in C^1_{\text{int}}(\mathcal{R}) \) with \( \partial e = \{v, v'\} \) such that \( I(\mathbf{\bar{z}}) \in \Sigma^+_{(2),v',\mathbf{\bar{r}}(2),v'} \).

This is the consequence of the fact that the decomposition (9.35) is ‘mostly preserved’ by \( \mathcal{I} \). Now we define \( u'_{(1),v}, U'_{(1),v} \) as follows. If \( \lambda(v) = 0 \), we have:

\[
u'_{(1),v}(\mathbf{\bar{z}}) = \begin{cases} u'_{(2),v}(\mathbf{\bar{z}}) & \text{if (I) holds,} \\
U'_{(2),v'}(\mathbf{\bar{z}}) & \text{if (II) holds,} \\
u'_{(2),v'}(\mathbf{\bar{z}}) & \text{if (II) holds,} \end{cases}
\]

and if \( \lambda(v) > 0 \), we have:

\[
U'_{(1),v}(\mathbf{\bar{z}}) = \begin{cases} U'_{(2),v}(\mathbf{\bar{z}}) & \text{if (I) holds,} \\
U'_{(2),v'}(\mathbf{\bar{z}}) & \text{if (II) holds,} \\
\text{Dil}_{\rho_{(2),e}} \circ U'_{(2),v}(\mathbf{\bar{z}}) & \text{if (II) holds,} \\
\text{Dil}_{1/\rho_{(2),e}} \circ U'_{(2),v'}(\mathbf{\bar{z}}) & \text{if (II) holds,} \\
\text{Dil}_{1/\rho_{(2),e}} \circ u'_{(2),v'}(\mathbf{\bar{z}}) & \text{if (II) holds,} \end{cases}
\]

(9.37)

Using the fact that \( \eta_{(2)} \) satisfies (8.20), (8.21), (8.22), we can easily check that in the case that (I) and (II) are both satisfied the right hand sides coincide.

---

\(^{32}\)Conditions 9.1 and 9.2 are used in Subsection 9.3 to show the compatibility of the similar decompositions. We do not assume them here.
We also define
\[ \rho(1), e = \rho(2), e, \quad \rho(1), i = \rho(2), i. \]

**Lemma 9.19.** The 6-tuple
\[ \eta(1) = (\vec{x}, \vec{y}, (U'_{(1),v}), (\rho(1), e), (\rho(1), i)) \]
is an inconsistent solution near \( u \) with respect to \( \Xi(1) \).

**Proof.** Definition 8.8 (1), (2), (3) are obvious. (4)-(8), (11) and (12) follow from the definition of \( \eta(1) \) and the corresponding conditions for \( \eta(2) \). (9) and (10) hold by shrinking the size of \( \Xi(2) \) if necessary. (13) is a consequence of Lemma 9.17. \( \square \)

Thus after shrinking the size of \( \Xi(2) \) if necessary, we may define:

\[ \hat{\varphi}(u, \Xi(1))(u, \Xi(2)) : \hat{\mathcal{U}}(u, \Xi(2)) \to \hat{\mathcal{U}}(u, \Xi(1)) \]

by:

\[ \hat{\varphi}(u, \Xi(1))(u, \Xi(2))(\eta(2)) = \eta(1). \]

Similarly, we can define \( \varphi(u, \Xi(1))(u, \Xi(2)) \) and \( \varphi(u, \Xi(1))(u, \Xi(2)) \):

\[ \mathcal{U}(u, \Xi(2)) \to \mathcal{U}(u, \Xi(1)) \]

**Lemma 9.20.** The maps \( \hat{\varphi}(u, \Xi(1))(u, \Xi(2)) \) and \( \varphi(u, \Xi(1))(u, \Xi(2)) \) are \( C^d \) diffeomorphisms into their images.

**Proof.** We cannot apply the same proof as in Lemma 9.14. In fact, Diagram (9.30) does not commute anymore because our TSDs \( \Xi(1) \) and \( \Xi(2) \) are not compatible in the sense of Conditions 9.1 and 9.2. In order to resolve this issue, we need to modify the definition of right vertical arrow in Diagram (9.30).

Assuming \( \Xi(2) \) is small enough, we define a map:

\[ \mathcal{J}_{v_0} : \prod_{v \in C^0_{\text{int}}(\tilde{R})} \mathcal{V}_{\text{source}}(2), v \times \prod_{e \in C^1_{\text{int}}(\tilde{R})} \mathcal{V}_{\text{deform}}(2), e \to \Sigma^-_{(1),v_0} \to \Sigma^-_{(2),v_0} \]

for any interior vertex \( v_0 \) of \( \tilde{R} \) as follows. Fix an element \( (\vec{x}, \vec{y}) \) of \( \prod_{v \in C^0_{\text{int}}(\tilde{R})} \mathcal{V}_{\text{source}}(2), v \times \prod_{e \in C^1_{\text{int}}(\tilde{R})} \mathcal{V}_{\text{deform}}(2), e \), and let \( (\vec{x}, \vec{y}) \) be the element of \( \prod_{v \in C^0_{\text{int}}(\tilde{R})} \mathcal{V}_{\text{source}}(1), v \times \prod_{e \in C^1_{\text{int}}(\tilde{R})} \mathcal{V}_{\text{deform}}(1), e \) that satisfies (9.33). By taking \( \Xi(2) \) small enough, we can form the following composition:

\[ \Sigma^-_{(1),v_0} \to \Sigma^-_{(1),v_0} (\vec{x}, \vec{y}) \to \Sigma^-_{(2),v_0} (\vec{x}, \vec{y}) \to \Sigma^-_{(2),v_0} \]

Here the first map is defined using \( \Xi(1) \), the second map is induced by the isomorphism (9.33), and the last map is defined using \( \Xi(2) \). For \( \hat{j} \in \Sigma^-_{(1),v_0} \), we define \( \mathcal{J}_{v_0}(\vec{x}, \vec{y}, \hat{j}) \) to be the image of \( \hat{j} \) by the map (9.40). It is clear that \( \mathcal{J}_{v_0} \) is a smooth map.
For a vertex $v_0$ with $\lambda(v_0) = 0$, define:

\[
\mathcal{J}^*_{v_0} : \prod_{v \in C^0\text{int}(R)} \gamma^\text{source}(2)_v \times \prod_{e \in C^1\text{int}(R)} \gamma^\text{deform}(2)_e \\
\times L^2_{m+\ell+1}(\Sigma^-(2), v_0, X \setminus \mathcal{D}) \to L^2_{m+1}(\Sigma^-(1), v_0, X \setminus \mathcal{D})
\]

as follows:

\[
\mathcal{J}^*_{v_0}(\bar{\mathcal{F}}(2), \bar{\sigma}(2), u') = u'(\mathcal{J}^*_{v_0}(\bar{\mathcal{F}}(2), \bar{\sigma}(2), \delta)).
\]

Note that we pick different Sobolev exponents for the Sobolev spaces on the domain and the target of $\mathcal{J}^*_{v_0}$. This allows us to obtain a $C^\ell$ map $\mathcal{J}^*_{v_0}$.

Similarly, for a vertex $v_0$ with $\lambda(v_0) > 0$, we can define a $C^\ell$ map:

\[
\mathcal{J}^*_{v_0} : \prod_{v \in C^0\text{int}(R)} \gamma^\text{source}(2)_v \times \prod_{e \in C^1\text{int}(R)} \gamma^\text{deform}(2)_e \\
\times L^2_{m+\ell+1}(\Sigma^-(2), v_0, \mathcal{N}_D X \setminus \mathcal{D}) \to L^2_{m+1}(\Sigma^-(1), v_0, \mathcal{N}_D X \setminus \mathcal{D})
\]

Now we replace Diagram (9.30) with the following:

\[
\mathcal{U}(u, \mathcal{X}(2)) \xrightarrow{\mathcal{J}^*_{v_0}} \prod_{v \in C^0\text{int}(R)} \gamma^\text{source}(2)_v \times \prod_{e \in C^1\text{int}(R)} \gamma^\text{deform}(2)_e \\
\times L^2_{m+\ell+1}(\Sigma^-(2), v_0, \mathcal{N}_D X \setminus \mathcal{D}) \to L^2_{m+1}(\Sigma^-(1), v_0, \mathcal{N}_D X \setminus \mathcal{D})
\]

Here horizontal arrows are defined as in (8.30). Commutativity of (9.41) is immediate from the definition. We can also form a diagram similar to Diagram (9.29), which is commutative by the same reason as in Lemma 9.14. Commutativity of these two diagrams and the fact that $\mathcal{J}^*_{v_0}$ is $C^\ell$ implies that $\tilde{\varphi}(u, \Xi(1))(u, \Xi(2))$ is also $C^\ell$. A similar argument applies to $\varphi(u, \Xi(1))(u, \Xi(2))$.

By changing the role of $\Xi(1)$ and $\Xi(2)$, we can similarly obtain $C^\ell$ maps in different directions. To be more precise we can define maps $\tilde{\varphi}(u, \Xi(2))(u, \Xi'(1))$.

\[
\tilde{\varphi}(u, \Xi(1))(u, \Xi(2)) \xrightarrow{\mathcal{J}^*_{v_0}} \prod_{v \in C^0\text{int}(R), \lambda(v) > 0} \gamma^\text{source}(2)_v \times \prod_{e \in C^1\text{int}(R), \lambda(e) > 0} \gamma^\text{deform}(2)_e \\
\times L^2_{m+\ell+1}(\Sigma^-(2), v_0, \mathcal{N}_D X \setminus \mathcal{D}) \to L^2_{m+1}(\Sigma^-(1), v_0, \mathcal{N}_D X \setminus \mathcal{D})
\]

\[
\tilde{\varphi}(u, \Xi(2))(u, \Xi'(1)) \xrightarrow{\mathcal{J}^*_{v_0}} \prod_{v \in C^0\text{int}(R), \lambda(v) > 0} \gamma^\text{source}(2)_v \times \prod_{e \in C^1\text{int}(R), \lambda(e) > 0} \gamma^\text{deform}(2)_e \\
\times L^2_{m+\ell+1}(\Sigma^-(1), v_0, \mathcal{N}_D X \setminus \mathcal{D}) \to L^2_{m+1}(\Sigma^-(1), v_0, \mathcal{N}_D X \setminus \mathcal{D})
\]
and \( \varphi(u, \Xi_{(1)}) (u, \Xi'_{(1)}) \) where \( \Xi'_{(1)} \) is given by a small enough shrinking of \( \Xi_{(1)} \). The compositions:
\[
\tilde{\varphi}(u, \Xi_{(1)})(u, \Xi_{(2)}) \circ \varphi(u, \Xi_{(1)})(u, \Xi'_{(1)}) = \varphi(u, \Xi_{(1)})(u, \Xi_{(2)}) \circ \varphi(u, \Xi_{(1)})(u, \Xi'_{(1)})
\]
are equal to the identity map. Moreover, the compositions
\[
\tilde{\varphi}(u, \Xi_{(2)})(u, \Xi'_{(1)}) \circ \varphi(u, \Xi_{(1)})(u, \Xi_{(2)}) = \varphi(u, \Xi_{(1)})(u, \Xi'_{(1)}) \circ \varphi(u, \Xi_{(1)})(u, \Xi_{(2)})
\]
are also equal to the identity map, wherever they are defined. This implies that \( \tilde{\varphi}(u, \Xi_{(1)})(u, \Xi_{(2)}) \) and \( \varphi(u, \Xi_{(1)})(u, \Xi_{(2)}) \) are diffeomorphisms, after possibly shrinking \( \Xi_{(2)} \).

\[\square\]

**Remark 9.21.** By following an argument similar to the case of stable maps, one can show that \( \tilde{\varphi}(u, \Xi_{(1)})(u, \Xi_{(2)}) \) and \( \varphi(u, \Xi_{(1)})(u, \Xi_{(2)}) \) are \( C^\infty \) using the above \( C^\ell \) property for all values of \( \ell \). We omit this argument and refer the reader to [FOOO5, Section 12] for details of the proof. (See also Remark 10.10.)

We thus constructed a \( C^\ell \) embedding \( \varphi(u, \Xi_{(1)})(u, \Xi_{(2)}) \). One can easily define a lift \( \tilde{\varphi}(u, \Xi_{(1)})(u, \Xi_{(2)}) \) of \( \varphi(u, \Xi_{(1)})(u, \Xi_{(2)}) \) and obtain embedding of obstruction bundles. The compatibility of the Kuranishi maps and the parametrization maps with the maps \( \varphi(u, \Xi_{(1)})(u, \Xi_{(2)}) \) and \( \tilde{\varphi}(u, \Xi_{(1)})(u, \Xi_{(2)}) \) are immediate from the construction. In summary, we have coordinate change:
\[
\Phi(u, \Xi_{(1)})(u, \Xi_{(2)}) = (\varphi(u, \Xi_{(1)})(u, \Xi_{(2)}), \tilde{\varphi}(u, \Xi_{(1)})(u, \Xi_{(2)})): \mathcal{U}_u \Xi_{(2)} \to \mathcal{U}_u \Xi_{(1)}.
\]

### 9.5. Co-cycle Condition for Coordinate Changes

For \( j = 1, 2 \), let \( u(j) \in \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta) \), and \( \Xi(j) \) be a TSD at \( u(j) \). We assume that \( \Xi(j) \) is small enough such that we can form the Kuranishi chart \( \mathcal{U}_{u(j)} \Xi_{j} \) as in Definition 9.10. We also assume that \( u(2) \) is sufficiently close to \( u(1) \) in the sense that it belongs to the open subset of \( \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta) \) determined by \( \mathcal{U}_{u(1)} \Xi_{(1)} \). Therefore, we may use the constructions of Subsection 9.1 to obtain a TSD \( \Xi_{(2)}(1) \) at \( u(2) \) which is compatible with \( \Xi_{(1)} \), namely, it satisfies Conditions 9.1 and 9.2. Finally by shrinking \( \Xi_{(2)} \), we can assume that we can define the coordinate change \( \Phi(u(2), \Xi_{(2)}; (1))(u(2), \Xi_{(1)}) \) following the construction of the previous subsection. Now we define:

**Definition 9.22.** We define the coordinate change:
\[
\Phi(u(1), \Xi_{(1)})(u(2), \Xi_{(2)}): \mathcal{U}_{u(2)} \Xi_{(2)} \to \mathcal{U}_{u(1)} \Xi_{(1)},
\]
as the composition
\[
\Phi(u(1), \Xi_{(1)})(u(2), \Xi_{(2)}) = \Phi(u(1), \Xi_{(1)})(u(2), \Xi_{(2)}; (1)) \circ \Phi(u(2), \Xi_{(2)}; (1))(u(2), \Xi_{(2)}).
\]
Here
\[
\Phi(u(1), \Xi_{(1)})(u(2), \Xi_{(2)}; (1)): \mathcal{U}_{u(1)} \Xi_{(2)}; (1) \to \mathcal{U}_{u(1)} \Xi_{(1)}
\]
is defined in Subsection 9.3 and
\[
\Phi(u(2), \Xi_{(2)}; (1))(u(2), \Xi_{(1)}): \mathcal{U}_{u(2)} \Xi_{(2)}; (1) \to \mathcal{U}_{u(2)} \Xi_{(2)}; (1)
\]
is defined in Subsection 9.4.
To complete the construction of the Kuranishi structure on $\mathcal{M}^\text{RGW}_{k+1}(L; \beta)$, we need to prove the next lemma.

**Lemma 9.23.** For $j = 1, 2, 3$, let $u_{(j)} \in \mathcal{M}^\text{RGW}_{k+1}(L; \beta)$, and $\Xi_{(j)}$ be a TSD at $u_{(j)}$ such that we can use Definition 9.22, to define the coordinate changes $\Phi_{(u_{(j)}; \Xi_{(1)})(u_{(j)}; \Xi_{(2)})}$, $\Phi_{(u_{(j)}; \Xi_{(1)})(u_{(j)}; \Xi_{(3)})}$. Then we have:

\[
\Phi_{(u_{(1)}; \Xi_{(1)})(u_{(2)}; \Xi_{(2)})} \circ \Phi_{(u_{(2)}; \Xi_{(2)})(u_{(3)}; \Xi_{(3)})} = \Phi_{(u_{(1)}; \Xi_{(1)})(u_{(3)}; \Xi_{(3)})}.
\]

**Proof.** We use the constructions of Subsection 9.1 to find TSDs $\Xi_{(3);(1)}$, $\Xi_{(3);(2)}$ at $u_{(3)}$ such that the pairs $(\Xi_{(1)}; \Xi_{(3);(1)})$ $(\Xi_{(2)}; \Xi_{(3);(2)})$ both satisfy Conditions 9.1 and 9.2. We similarly choose the TSD $\Xi_{(2);(1)}$ at $u_{(2)}$. We can easily check the following three formulas:

\[
\begin{align*}
\Phi_{(u_{(3)}; \Xi_{(3);(1)})(u_{(3)}; \Xi_{(3);(2)})} \circ \Phi_{(u_{(3)}; \Xi_{(3);(2)})(u_{(3)}; \Xi_{(3)})} &= \Phi_{(u_{(3)}; \Xi_{(3);(1)})(u_{(3)}; \Xi_{(3)})} \\
\Phi_{(u_{(1)}; \Xi_{(1)})(u_{(2)}; \Xi_{(2);(1)})} \circ \Phi_{(u_{(2)}; \Xi_{(2);(1)})(u_{(3)}; \Xi_{(3);(1)})} &= \Phi_{(u_{(1)}; \Xi_{(1)})(u_{(3)}; \Xi_{(3);(1)})} \\
\Phi_{(u_{(2)}; \Xi_{(2);(1)})(u_{(2)}; \Xi_{(3);(2)})} \circ \Phi_{(u_{(2)}; \Xi_{(3);(2)})(u_{(3)}; \Xi_{(3);(2)})} &= \Phi_{(u_{(2)}; \Xi_{(2);(1)})(u_{(3)}; \Xi_{(3);(2)})}.
\end{align*}
\]

Then (9.43) is a consequence of these three formulas and Definition 9.22. See the diagram below. In this diagram, the notation $\Phi_{\Xi_{(3);(1)}; \Xi_{(3);(2)}}$ is simplified to $\Phi_{\Xi_{(3);(1)}; \Xi_{(3);(2)}}$. Similar notations for other coordinate changes are used.

This lemma completes the proof of the following result:

**Theorem 9.24.** The space $\mathcal{M}^\text{RGW}_{k+1}(L; \beta)$ carries a Kuranishi structure.
10. Construction of a System of Kuranishi Structures

10.1. Statement. In Section 9, we completed the construction of Kuranishi structure for each moduli space $\mathcal{M}^{RGW}_{k+1}(L; \beta)$. In this section, we study how these Kuranishi structures are related to each other at their boundaries and corners. More specifically, we prove the disk moduli version of [DF2, Lemma 3.67], stated as Theorem 10.1. The notation $\hat{\times}_L$ is discussed in [DF2, Subsection 3.7]. Recall also from [DF2, Subsection 3.7] that $\mathcal{M}^{RGW}_{k+1}(L; \beta_1)$ is the union of the strata of $\mathcal{M}^{RGW}_{k+1}(L; \beta)$ which are described by DD-ribbon trees with at least one positive level. The proof of [DF2, Lemma 3.67] is entirely similar to one of Theorem 10.1. So we only focus on the proof of Theorem 10.1.

**Theorem 10.1.** Suppose $E$ is a positive real number and $N$ is a positive integer. There is a system of Kuranishi structures on the moduli spaces $\{\mathcal{M}^{RGW}_{k+1}(L; \beta)\}_{k, \beta}$ with $\omega \cap \beta \leq E$ and $k \leq N$ such that if $\beta_1 + \beta_2 = \beta$, $k_1 + k_2 = k$, then the space

$$\mathcal{M}^{RGW}_{k_1+1}(L; \beta_1) \hat{\times}_L \mathcal{M}^{RGW}_{k_2+1}(L; \beta_2)$$

is a codimension one stratum of $\mathcal{M}^{RGW}_{k+1}(L; \beta)$ with the following properties. There exists a continuous map:

$$\Pi : \mathcal{M}^{RGW}_{k_1+1}(L; \beta_1) \hat{\times}_L \mathcal{M}^{RGW}_{k_2+1}(L; \beta_2) \to \mathcal{M}^{RGW}_{k_1+1}(L; \beta_1) \times_L \mathcal{M}^{RGW}_{k_2+1}(L; \beta_2)$$

with the following properties.

1. On the inverse image of the complement of

$$(\mathcal{M}^{RGW}_{k_1+1}(L; \beta_1)^{(1)} \times_L \mathcal{M}^{RGW}_{k_2+1}(L; \beta_2))$$

and

$$\cup (\mathcal{M}^{RGW}_{k_2+1}(L; \beta_2) \times_L \mathcal{M}^{RGW}_{k_1+1}(L; \beta_1)^{(1)})$$

$\Pi$ is induced by an isomorphism of Kuranishi structures.

2. Let $p \in \mathcal{M}^{RGW}_{k_1+1}(L; \beta_1) \hat{\times}_L \mathcal{M}^{RGW}_{k_2+1}(L; \beta_2)$ and $\Pi(p) = \bar{p} \in \mathcal{M}^{RGW}_{k_1+1}(L; \beta_1) \times_L \mathcal{M}^{RGW}_{k_2+1}(L; \beta_2)$.

Let $\mathcal{U}_p = (U_p, E_p, s_p, \psi_p)$ and $\mathcal{U}_{\bar{p}} = (U_{\bar{p}}, E_{\bar{p}}, s_{\bar{p}}, \psi_{\bar{p}})$ be the Kuranishi neighborhoods of $p$ and $\bar{p}$ assigned by our Kuranishi structures. Let also $U_p = V_p/\Gamma_p$ and $U_{\bar{p}} = V_{\bar{p}}/\Gamma_{\bar{p}}$. Then we have:

(a) There exists an injective homomorphism $\phi_p : \Gamma_p \to \Gamma_{\bar{p}}$.

(b) There exists a $\Gamma_{\bar{p}}$-equivariant map

$$F_p : V_p \to V_{\bar{p}}$$

that is a strata-wise smooth submersion.
(c) $E_p$ is isomorphic to the pullback of $E_{\mathcal{F}}$ by $F_p$. In other words, there exists fiberwise isomorphic lift

$$\tilde{F}_p : E_p \to E_{\mathcal{F}}$$

of $F_p$, which is $\Gamma_p$-equivariant.

(d) $\tilde{F}_p \circ s_p = s_{\mathcal{F}} \circ F_p$.

(e) $\psi_{\mathcal{F}} \circ F_p = \Pi \circ \psi_p$ on $s_p^{-1}(0)$.

(3) $\tilde{F}_p$, $F_p$ are compatible with the coordinate changes.

For elaboration on item (3) of the above theorem, see the discussion proceeding [DF2, Subsection 3.7].

**Remark 10.2.** In order to prove Theorem 10.1, we need to slightly modify our choices of obstruction bundles used in the proof of Theorem 9.24.

**Remark 10.3.** Theorem 10.1 concerns only the behavior of Kuranishi structures at codimension one boundary components. In fact, there is a similar statement for the behavior of our Kuranishi structures at higher co-dimension corners. This generalization to higher co-dimensional corners are counterparts of [FOOO7, Condition 16.1 XI,XII and Condition 21.7 X,XI] in the context of the stable map compactification. The main difference is that we again need to replace $\times_L$ with $\hat{\times}_L$. To work out the whole construction of simultaneous perturbations, we need the generalization of Theorem 10.1 to the higher co-dimensional corners.

In Subsection 10.2, we will formulate a condition (Definition 10.6) for the obstruction spaces, which implies the consistency of Kuranishi structures at the corners of arbitrary codimension. Since the proof and the statement for the case of corners is a straightforward generalization of the case of boundary (but cumbersome to write in detail), we focus on the case of codimension one boundary components.

10.2. Disk component-wise-ness of the Obstruction Bundle Data. A disk splitting tree is defined to be a very detailed DD-ribbon tree $S$ such that the color of all vertices is $d$. We say a detailed DD-ribbon tree $\tilde{R}$ belongs to a disk splitting tree $S$ if $S$ is obtained from $\tilde{R}$ by level shrinking and fine edge shrinking. In other words, geometrical objects with combinatorial type $\tilde{R}$ are limits of objects with type $S$ such that new disc bubble does not occur. However, it is possible to have sphere bubbles.

Let $u \in \mathcal{M}^{R\mathcal{GW}}_{k+1}(L; \beta)$ and $\tilde{R}$ be the associated very detailed tree. Suppose $S$ is a disk splitting tree such that $\tilde{R}$ belongs to $S$. Let also $\lambda$ be the level function assigned to $\tilde{R}$. For each interior vertex $v$ of $S$, let $\tilde{R}_v$ be the subtree of $\tilde{R}$ given by the connected component of:

$$\tilde{R} \setminus \bigcup_{e \in C^{\text{int}}(\tilde{R}), \lambda(e)=0} e$$

\footnote{In [FOOO3], the corresponding statement is called corner compatibility conditions.}
which contains the vertex $v$. Let $\mathcal{S}$ be a disk splitting tree obtained from $\mathcal{S}$ by a sequence of shrinking of level 0 edges [DF2, Definition 3.57]. Let $\pi : \mathcal{S} \to \mathcal{S}$ be the associated contraction map. For each $w \in C^\text{int}_0(\mathcal{S})$, let $\check{R}(w)$ be the very detailed DD-ribbon tree defined as:

$$
(10.2) \quad \check{R}(w) = \bigcup_{\pi(v)=w} \check{R}_v \cup \bigcup_{e \in C^\text{int}_1(\check{R}), \lambda(e)=0, \pi(e) \text{ is adjacent to } w} e.
$$

Clearly we have $C^\text{int}_0(\check{R}(w)) \subseteq C^\text{int}_0(\check{R})$ and $C^\text{int}_1(\check{R}(w)) \subseteq C^\text{int}_1(\check{R})$.

The restriction of the quasi-order\(^{34}\) of $C^\text{int}_0(\check{R})$ to the set $C^\text{int}_0(\check{R}(w))$ determines\(^{35}\) a level function $\lambda_w$ for $\check{R}(w)$. The tree $\check{R}(w)$ also inherits a multiplicity function, a homology class assigned to each interior vertex and a color function from $\check{R}$, which turn $\check{R}(w)$ into a very detailed tree associated to a detailed DD-ribbon tree $\mathcal{R}(w)$. There is a map

$$
(10.3) \quad \pi_w : \{1, \ldots, |\lambda|\} \to \{1, \ldots, |\lambda_w|\}
$$

such that $i \leq j$ implies $\pi_w(i) \leq \pi_w(j)$ and for any $v \in C^\text{int}_0(\mathcal{R}(w)) \subseteq C^\text{int}_0(\mathcal{R})$

$$
(10.4) \quad \lambda_w(v) = \pi_w(\lambda(v)).
$$

Let $\Sigma_u$ be the source curve of $u$, and $\Sigma_{u,v}$ denote the irreducible component of $\Sigma_u$ corresponding to an interior vertex $v$ of $\check{R}$. For any $w \in C^\text{int}_0(\mathcal{S})$, we define $\Sigma_{u,w}$ to be the union of irreducible components $\Sigma_{u,v}$ where $v \in C^\text{int}_0(\check{R}(w))$. A boundary marked point of $\Sigma_{u,w}$ is either a boundary marked point of a disc component $\Sigma_{u,v}$ in $\Sigma_{u,w}$ or a boundary nodal point of $\Sigma_u$ which joins an irreducible component of $\Sigma_{u,w}$ to an irreducible component of $\Sigma_u$, which is not in $\Sigma_{u,w}$. The 0-th boundary marked point $z_0,w$ of $\Sigma_{u,w}$ is defined as follows. If the 0-th boundary marked point $z_0$ of $\Sigma_u$ is contained in $\Sigma_{u,w}$ then $z_0,w = z_0$. If not, $z_0,w$ is the boundary nodal point such that $z_0$ and $\Sigma_{u,w} \setminus \{z_0,w\}$ are contained in the different connected component of $\Sigma_u \setminus \{z_0\}$.

The restriction of $u_d : (\Sigma_u, \partial \Sigma_u) \to (X, L)$ to $\Sigma_{u,w}$ defines a map $u_{u,w} : (\Sigma_{u,w}, \partial \Sigma_{u,w}) \to (X, L)$. The bordered nodal curve $\Sigma_{u,w}$ together with the boundary marked points described above, the choice of the 0-th boundary marked point $z_0,w$ and the map $u_{u,w}$ determines an element of the moduli space $\mathcal{M}_k^{\text{RGW}}(L; \beta(w))$ where $\beta(w) = \sum_{v \in C^\text{int}_0(\check{R}(w))} \alpha(v)$ and $k + 1$ is the number of the boundary marked points of $\Sigma_{u,w}$. We denote this element by $u_{u,w}$.

Let $\Xi_u = (\vec{w}_u, (\lambda_{u,v}), (\phi_{u,v}), (\varphi_{u,v,e}))$ be a TSD for $u$. This induces a TSD $\Xi_{u,w}$ for $u_{u,w}$ in an obvious way. Let

$$
(10.5) \quad \eta = (\vec{f}, \vec{g}, (u'_v), (U'_e), (\rho_e), (\rho_i))
$$

be an inconsistent map with respect to $\Xi_u$. Let $\mathcal{S}'$ be a disc splitting tree such that the very detailed tree of $\eta$ belongs to $\mathcal{S}'$. We assume that $\mathcal{S}$ is obtained

\(^{34}\)See [DF2, Subsection 3.4] for the definition of a quasi-order.

\(^{35}\)See [DF2, Lemma 3.34].
from $S'$ by a sequence of shrinking of level 0 edges. Given $w \in C^\text{int}_0(\mathfrak{S})$, let $\sigma_e = 0$ for any level 0 edge $e \in C^\text{int}_0(\mathfrak{R})$ that corresponds to an exterior edge of $\mathfrak{R}(w)$. Then we can define an inconsistent map $\eta(w)$ with respect to $\Xi_{uw}$ in the following way. Since $C^\text{int}_0(\mathfrak{R}(w)) \subseteq C^\text{int}_0(\mathfrak{R})$, $C^\text{int}_1(\mathfrak{R}(w)) \subseteq C^\text{int}_1(\mathfrak{R})$, the restriction of the data of $\eta$ determine $\tilde{r}_w$, $\tilde{\sigma}_w$, $(u'_v, w)$, $(U'_v, w)$ and $(\rho_e, w)$. We also define:

$$\rho_{w,i} = \prod_{\hat{i}: \pi_w(\hat{i}) = i} \rho_{\hat{i}}$$

where $\pi_w$ is given in (10.3).

**Lemma 10.4.** The following element is an inconsistent map with respect to $\Xi_{uw}$:

$$(10.6) \quad \eta(w) = (\tilde{r}_w, \tilde{\sigma}_w, (u'_v, w), (U'_v, w), (\rho_e, w), (\rho_{w,i})).$$

Next, we shall formulate a condition on the obstruction spaces so that the resulting system of Kuranishi structures satisfy the claims in Theorem 10.1. For this purpose, we firstly introduce the notion of *obstruction bundle data*.

**Definition 10.5.** Suppose we are given vector spaces $\{E_{u, \Xi}(\eta)\}$ for any $u \in \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)$, any small enough TSD $\Xi$ at $u$, and an inconsistent map $\eta$ with respect to $\Xi$. This data is called an *obstruction bundle data* for $\mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)$ if the following holds:

1. We have:
   $$E_{u, \Xi}(\eta) = \bigoplus_{v \in C^\text{int}_0(\mathfrak{R})} E_{u, \Xi,v}(\eta)$$
   where $E_{u, \Xi,v}(\eta) \subset L^2_{m,\delta}(\Sigma_{u,v}, \Lambda^{0,1} \otimes T)$.
2. $E_{u, \Xi,v}(\eta)$ is a finite dimensional subspace. The supports of its elements are subsets of $\Sigma_{u,v}$ and are away from the boundary.
3. $E_{u, \Xi,v}(\eta)$ is independent of $\Xi$ in the sense of Definition 10.7.
4. $E_{u, \Xi,v}(\eta)$ is semi-continuous with respect to $u$ in the sense of Definition 10.8.
5. $E_{u, \Xi,v}(\eta)$ is smooth with respect to $\eta$ in the sense of Definition 10.9.
6. The linearization of the Cauchy-Riemann equation is transversal to $E_{u, \Xi,v}(\eta)$ in the sense of Definition 10.11.
7. $E_{u, \Xi,v}(\eta)$ is invariant under the $\Gamma_u$-action. (See [FOOO8, Definition 5.1 (5)].)

Definition 10.5 is the RGW counterpart of [FOOO8, Definition 5.1] for the stable map compactification. Before discussing the precise meaning of (3), (4), (5) and (6), we define *disk-component-wise-ness* of a system of obstruction bundle data. This is the analogue of [FOOO3, Definition 4.2.2] for the stable map compactification:

**Definition 10.6.** Suppose $E$ is a positive real number and $N$ is a positive integer. Suppose $\{E_{u, \Xi}(\eta)\}$ is a system of obstruction bundle data for the
spaces $\{\mathcal{M}^{RGW}_{k+1}(L; \beta)\}_{k, \beta}$ where $k = 0, 1, 2, \ldots, N$, $\beta \in H_2(X, L)$ and $\beta \cap [D] = 0$ with $\omega \cap \beta \leq E$. We say this system is disk-component-wise if we always have the identification:

\[(10.7)\quad E_{u, \Xi}(\eta) = \bigoplus_{w \in C^\text{int}_0(\mathcal{S})} E_{u_w, \Xi_w}(\eta(w)).\]

where $\mathcal{S}$ is a detailed DD-ribbon tree as in the beginning of the subsection and $\eta(w)$ is as in (10.6).

**Explanation of Definition 10.5 (3).** We pick two TSDs at $u$ denoted by $\Xi(u) = (\hat{w}(j), (N(j), \nu), (\phi(j), \nu), (\varphi(j), \nu), \kappa(j))$. If $\Xi(2)$ is small enough in comparison to $\Xi(1)$, then as in (9.35) and (9.37), we can assign to any inconsistent map:

$\eta(2) = (\bar{\eta}(2), \bar{\sigma}(2), (u'(2), \nu), (U'_2, \nu), (\rho(2), \nu), (\rho(2), \nu))$

with respect to $\Xi(2)$ an inconsistent map

$\eta(1) = (\bar{\eta}(1), \bar{\sigma}(1), (u'(1), \nu), (U'_1, \nu), (\rho(1), \nu), (\rho(1), \nu))$

with respect to $\Xi(1)$. In particular, there is a bi-holomorphic embedding:

$I_{v; \Xi(2)\Xi(1)} : \Sigma^-(\bar{\eta}(1), \bar{\sigma}(1)) \to \Sigma^-(\bar{\eta}(2), \bar{\sigma}(2))$

as in (9.40) such that:

$u'(2), \nu \circ I_{v; \Xi(2)\Xi(1)} = u'(1), \nu$ if $\lambda(\nu) = 0$

$U'_2, \nu \circ I_{v; \Xi(2)\Xi(1)} = U'_1, \nu$ if $\lambda(\nu) > 0$

It induces a map

$I_{v; \Xi(2)\Xi(1)} : L^2_m(\Sigma^-(\bar{\eta}(2), \bar{\sigma}(2)), T \otimes L^0, 1) \to L^2_m(\Sigma^-(\bar{\eta}(1), \bar{\sigma}(1)), T \otimes L^0, 1)$.

**Definition 10.7.** We say the system $\{E_{u, \Xi}(\eta)\}$ is independent of $\Xi$, if we always have:

\[(10.8)\quad I_{v; \Xi(2)\Xi(1)}(E_{u, \Xi(2)}(\eta(2))) = E_{u, \Xi(1)}(\eta(1)).\]

The choices of obstruction bundles that we made in the previous section have this property. In fact, this property was used in the proof of Lemma 9.19.

**Explanation of Definition 10.5 (4).** Let $u(1) \in \mathcal{M}^{RGW}_{k+1}(L; \beta)$ and $\Xi(1)$ be a small enough TDS at $u(1)$. Let also $u(2) \in \mathcal{M}^{RGW}_{k+1}(L; \beta)$ be in a neighborhood of $u(1)$ determined by $\Xi(1)$ and $\Xi(2)$ be a TSD at $u(2)$. We assume that $\Xi(1), \Xi(2)$ satisfy Conditions 9.1 and 9.2. Let $\hat{R}(j)$ be the very detailed tree associated to $u(j)$. Our assumption implies that there is a map $\pi : \hat{R}(1) \to \hat{R}(2)$. Let $\eta(2)$ be an inconsistent map with respect to $\Xi(2)$. Lemma 9.4 associates an inconsistent map $\eta(1)$ with respect to $\Xi(1)$. In particular, for any $\hat{v} \in C^\text{int}_0(\hat{R}(1))$ with $v := \pi(\hat{v})$, we have a bi-holomorphic isomorphism:

$I_{\hat{v}} : \Sigma^-(\hat{v}, \hat{\sigma}) \to \Sigma^-(v, \sigma)$
such that
\[ u'_{(2),v} \circ I_\delta = u'_{(1),\hat{\delta}} \quad \text{if } \lambda(v) = 0, \]
\[ U'_{(2),v} \circ I_\delta = U'_{(1),\hat{\delta}} \quad \text{if } \lambda(v) > 0. \]
It induces a map:
\[ \mathcal{J}_{\mathfrak{v};\eta(1)\eta(2)} : L^2_m(\Sigma^-(2),\lambda^0,1) \otimes T) \to \bigoplus_{\pi(\hat{\delta}) = v} L^2_m(\Sigma^-_{(1),\hat{\delta}},\lambda^0,1 \otimes T). \]

**Definition 10.8.** We say that \( \{E_{u,\Xi}(\mathfrak{v})\} \) is semi-continuous with respect to \( u \) if the following condition holds. If \( u_{(1)}, u_{(2)}, \eta_{(1)}, \eta_{(2)}, \Xi_{(1)} \) and \( \Xi_{(2)} \) are as above, then we have:

\begin{equation}
\mathcal{J}_{\mathfrak{v};\eta(1)\eta(2)}(E_{u_{(2)},\Xi_{(2)}}(\eta_{(2)})) \subseteq E_{u_{(1)},\Xi_{(1)}}(\eta_{(1)}).
\end{equation}

Lemmas 9.11 and 9.12 imply that our choices of obstruction bundles in Section 9 satisfy the above property.

**Explanation of Definition 10.5 (5).** Let \( u \in \mathcal{M}^{RGW}_{k+1}(L,\beta) \), \( \hat{R} \) be the very detailed tree associated to \( u \), and \( \Xi \) be a choice of TSD at \( u \). Let also \( \eta = (\bar{\mathfrak{v}},\bar{\mathfrak{s}},(u'_v), (U'_v), (p_\mathfrak{v}), (p_v)) \) be an inconsistent map with respect to \( \Xi \). For \( v \in C^0_{\text{int}}(\hat{R}) \), the TSD \( \Xi \) determines an isomorphism \( I_{\mathfrak{v},v} : \Sigma^-_v(\bar{\mathfrak{v}},\bar{\mathfrak{s}}) \to \Sigma^-_u(\bar{\mathfrak{v}},\bar{\mathfrak{s}}) \). Here \( \bar{\mathfrak{s}} \) is a vector with zero entries. If \( \Xi \) is small enough, then \( u_v \circ I_{\mathfrak{v},v} \) (resp. \( U_v \circ I_{\mathfrak{v},v} \) in the case \( c(v) = D \)) is \( C^2 \)-close to \( u'_{\mathfrak{v},v} \) (resp. \( U'_{\mathfrak{v},v} \)). Therefore, we obtain:

\[ \mathcal{J}_{\mathfrak{v},v} : L^2_m(\Sigma^-_{\mathfrak{v},v}(\bar{\mathfrak{v}},\bar{\mathfrak{s}}); \lambda^0,1 \otimes T) \to L^2_m(\Sigma^-_{\mathfrak{v},v}(\bar{\mathfrak{v}},\bar{\mathfrak{s}}); \lambda^0,1 \otimes T). \]

Let \( \mathcal{L}^2_{m+\ell}(u,v) \) be a small neighborhood of \( u_v|_{\Sigma^-_v} \) or \( U_v|_{\Sigma^-_v} \) with respect to the \( L^2_{m+\ell} \)-norm.

**Definition 10.9.** We say \( \{E_{u,\Xi}(\mathfrak{v})\} \) is in \( C^\ell \) with respect to \( \eta \), if there exists a \( C^\ell \) map:

\[ \epsilon_i : \prod_{v \in C^0_{\text{int}}(S)} \gamma^\text{deform}_{i} \times \prod_{v \in C^0_{\text{int}}(S)} \gamma^\text{source}_{i} \times \mathcal{L}^2_{m+\ell+1}(u,v) \to L^2_m(\Sigma^-_v; \lambda^0,1 \otimes T) \]

for \( i = 1, \ldots, \dim(E_{u,\Xi}(\mathfrak{v})) \) with the following properties. For the inconsistent map \( \eta \) with respect to \( \Xi \) and \( v \in C^0_{\text{int}}(\hat{R}) \), let \( \eta(v) \in \mathcal{L}^2_{m+\ell+1}(u,v) \) be the map \( u'_{\mathfrak{v},v} \circ I_{\mathfrak{v},v} \) or \( U'_{\mathfrak{v},v} \circ I_{\mathfrak{v},v} \). Then the set of elements:

\[ \mathcal{J}_{\mathfrak{v},v} \circ \epsilon_i(\bar{\mathfrak{v}},\bar{\mathfrak{s}},\eta(v)) \]

for \( i = 1, \ldots, \dim(E_{u,\Xi}(\mathfrak{v})) \) forms a basis for \( E_{u,\Xi}(\mathfrak{v}) \).

This condition is mostly the analogue of [FOOO5, Definition 8.7] in the context of the stable map compactifications and we refer the reader to the discussion there for a more detailed explanation. If this condition is satisfied, then the gluing analysis in the previous sections gives rise to \( C^\ell \)-Kuranishi charts and \( C^\ell \)-coordinate changes. The proof of the fact that the choices of obstruction data in the previous section and Subsection 10.3 satisfy this condition is similar to [FOOO5, Subsection 11.4] and hence is omitted.
Remark 10.10. We discussed the notion of $C^\ell$-obstruction data. There is also the notion of smooth obstruction data which is slightly stronger. This is related to [FOOO5, Definition 8.7] Item (3), and we do not discuss this point in this paper. This condition is necessary to construct smooth Kuranishi structures rather than $C^\ell$-Kuranishi structures. The Kuranishi structure of class $C^\ell$ would suffice for our purposes in [DF2]. Using smooth Kuranishi structures would be essential to study the Bott-Morse case and/or construct filtered $A_\infty$-category based on de-Rham model.

Explanation of Definition 10.5 (6). We consider $u \in \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta)$ and $\Xi$. A system $\{E_{u,\Xi}(\eta)\}$ determines the vector spaces $E_{u,\Xi}(u)$ in the case that $\eta = u$.

Definition 10.11. We say the linearization of the Cauchy-Riemann equation is transversal to $E_{u,\Xi}(u)$, if $L_{m,\delta}^2(u; T \otimes \Lambda^{0,1})$ is generated by the image of the operator $D_u\mathfrak{F}$ in (8.2) and $E_{u,\Xi}(u)$.

From Disk-component-wise-ness to Theorem 10.1. The construction of the last section implies that we can use an obstruction bundle data to construct a Kuranishi structure. The next lemma shows that to prove Theorem 10.1 it suffices to find a system of obstruction bundle data which is disk-component-wise:

Lemma 10.12. If a system of obstruction bundle data is disk-component-wise, then the Kuranishi structures constructed in the last section on moduli spaces $\mathcal{M}_{k+1}^{\text{RGW}}(L; \beta)$ satisfy the claims in Theorem 10.1.

Proof. This is in fact true by tautology. For the sake of completeness, we give the proof below. Let $u \in \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta)$. $R$ be the very detailed DD-ribbon tree associated to $u$, and $S$ be the disk splitting tree such that $R$ belongs to $S$. We assume that $u$ is a boundary point, i.e., there are $k_1$, $k_2$, $\beta_1$ and $\beta_2$ such that $u$ is contained in $\mathcal{M}_{k_1+1}^{\text{RGW}}(L; \beta_1) \times_L \mathcal{M}_{k_2+1}^{\text{RGW}}(L; \beta_2)$. In particular, the disk splitting tree $\mathfrak{S}$ in Figure 16 is obtained from $S$ by shrinking of level 0 edges. We also have a map $\pi : S \to \mathfrak{S}$. The construction of the beginning of this subsection allows us to from $u_{m_1} \in \mathcal{M}_{k_1+1}^{\text{RGW}}(L; \beta_1)$ and $u_{m_2} \in \mathcal{M}_{k_2+1}^{\text{RGW}}(L; \beta_2)$ from $u$. Here $m_1$, $m_2$ are the two interior vertices of $\mathfrak{S}$. (See Figure 16.) The map $\Pi$ in (10.1) is given by $\Pi(u) = (u_{m_1}, u_{m_2})$. Let $\mathfrak{u} = (u_{m_1}, u_{m_2})$.

A Kuranishi neighborhood of $u$ in $\mathcal{M}_{k_1+1}^{\text{RGW}}(L; \beta_1) \times_L \mathcal{M}_{k_2+1}^{\text{RGW}}(L; \beta_2)$ coincides with a Kuranishi neighborhood of $u$ in a normalized boundary of $\mathcal{M}_{k+1}^{\text{RGW}}(L; \beta)$. It contains inconsistent solutions $\eta = (\bar{\varphi}, \mathfrak{S}, (u_p'), (U_p'), (\rho_p), (\rho_p))$ with respect to $\Xi$ such that $\sigma_{e_0} = 0$. Here $e_0$ is the unique interior edge of level 0 of $\mathfrak{S}$. We may regard $e_0$ as an edge of $S$ and $\mathfrak{R}$, too. We denote this set by $\partial e_0 U(\mathfrak{u}; \Xi)$.

The TSD $\Xi$ induces the TSD $\Xi_j$ on $u_{m_j}$ for $j = 1, 2$. Then we obtain a Kuranishi neighborhood of $U(u_{m_j}; \Xi_j)$ of $u_{m_j}$ in $\mathcal{M}_{k_j+1}^{\text{RGW}}(L; \beta_j)$, for $j = 1, 2$. 


We can define evaluation maps $\text{ev}_{j,i} : \mathcal{U}(u_{m_j}; \Xi_j) \to L$ for $i = 0, \ldots, k_j$ and define
\begin{equation}
\mathcal{U}(u_{m_1}; \Xi_1)_{\text{ev}_{1,i}} \times_{\text{ev}_{2,0}} \mathcal{U}(u_{m_2}; \Xi_2).
\end{equation}
Here $i$ is determined so that the edge $e_0$ is the $i$-th edge of $w_1$. (10.10) is a Kuranishi neighborhood of $(u_{m_1}, u_{m_2})$ in the fiber product Kuranishi structure of $\mathcal{M}^{RGW}_{k_1+1} (L; \beta_1) \times_L \mathcal{M}^{RGW}_{k_2+1} (L; \beta_2)$.

We next define a map
\[
F_u : \partial_{e_0} \mathcal{U}(u; \Xi) \to \mathcal{U}(u_{m_1}; \Xi_1)_{\text{ev}_{1,i}} \times_{\text{ev}_{2,0}} \mathcal{U}(u_{m_2}; \Xi_2).
\]
For $j = 1, 2$, let $\tilde{R}(w_j)$ be the very detailed DD-ribbon tree associated to $w_j$, defined in the beginning of this subsection. Given an inconsistent solution $\eta \in \partial_{e_0} \mathcal{U}(u; \Xi)$, we can define $\eta_{(j)} = (\tilde{\Gamma}_{(j)}, \tilde{\sigma}_{(j)}, (u'_{(j), i}), (U'_{(j), i}), (\rho_{(j), e}), (\rho_{(j), i}))$, an inconsistent solution with respect to $\Xi_j$, as in (10.6). Identity (10.7) implies that $\eta_{(j)}$ satisfies (8.24) and (8.25), the thickened non-linear Cauchy-Riemann equations. Thus $\eta_{(j)}$ is an inconsistent solution with respect to $\Xi_j$ for $j = 1, 2$. Since $\eta$ is an inconsistent solution with $\sigma_{e_0} = 0$, we also have: (See Definition 8.8 Items (10).)
\[
\text{ev}_{1,i}(\eta_{(1)}) = \text{ev}_{2,0}(\eta_{(2)}).
\]
We define $F_u(\eta) = (\eta_{(1)}, \eta_{(2)})$. We have:
\begin{equation}
\text{Aut}(u) \subseteq \text{Aut}(u_{m_1}) \times \text{Aut}(u_{m_2}).
\end{equation}
because the restriction of all automorphisms to disk components are identity maps. Thus any $\gamma \in \text{Aut}(u)$ maps the sources curves of $u_{m_1}$ and $u_{m_2}$ to themselves. Consequently, $\gamma$ induces $(\gamma_1, \gamma_2) \in \text{Aut}(u_{m_1}) \times \text{Aut}(u_{m_2})$ which determines $\gamma$ uniquely.\textsuperscript{30} It is then easy to see that $F_u$ is $\text{Aut}(u)$-invariant.

By (10.7) we have:
\[
\mathcal{E}_{0,u,\Xi}(\eta) \cong \bigoplus_{j=1,2} \mathcal{E}_{0,u_{m_j},\Xi_j}(\eta_{(j)})
\]
\textsuperscript{30}However, any $(\gamma_1, \gamma_2) \in \text{Aut}(u_{m_1}) \times \text{Aut}(u_{m_2})$ does not necessarily determine an element of $\text{Aut}(u)$. For example, we could have two vertices $v_1$ and $v_2$ with the same positive levels such that $v_i$ belongs to $C^{\text{int},}_{\omega} (\tilde{R}(w_i))$. Then there is $c_i \in \mathbb{C}_+ \text{ such that } U_{v_i} \circ \gamma_i = c_i \cdot U_{v_i}(v_i)$. In the case that $c_1 \neq c_2$, we cannot produce an automorphism of $u$ using $\gamma_1, \gamma_2$.\textsuperscript{30}
We also have:

\[
\bigoplus_{e \in C^{\mathrm{int}}_{\ell h}(R), \lambda(e) > 0} \mathcal{L}_e \cong \bigoplus_{j=1,2} \bigoplus_{e \in C^{\mathrm{int}}(\tilde{R}(w_j)), \lambda(e) > 0} \mathcal{L}_e.
\]

This is because the set of the edges of positive level of \( \tilde{R} \) is the union of the set of the edges of positive level of \( \tilde{R}(w_1) \) and \( \tilde{R}(w_2) \). Therefore, we obtain a bundle map:

\[
\tilde{F}_u : \mathcal{E}_{u, \Xi} \to \mathcal{E}_{u w_1, \Xi_1} \oplus \mathcal{E}_{u w_2, \Xi_2}
\]

which is a lift of \( F_u \). The bundle map \( \tilde{F}_u \) is an isomorphism on each fiber. Therefore, we proved (a), (b) and (c) of Theorem 10.1 (2). Parts (d), compatibility with Kuranishi maps, and (e), compatibility with the parametrization maps, are obvious from the construction. Item (3), compatibility with the coordinate change, is also an immediate consequence of the definitions.

It remains to prove that \( F_u \) is an isomorphism outside the strata of codimension 2. For this purpose, it suffices to consider the cases where \( \tilde{R}(w_1) \) and \( \tilde{R}(w_2) \) have no vertex of positive level. Note that if we ignore the parameter \( \rho_i \), then the map \( F_u \) is a bijection. In the present case where there is no vertex of positive level, there is no parameter \( \rho_i \). This completes the proof of Lemma 10.12. \( \square \)

10.3. Existence of disk-component-wise Obstruction Bundle Data.

The main goal of this subsection is to prove:

**Proposition 10.13.** There exists a system of obstruction bundle data which is disk-component-wise.

The proof is divided into 5 parts. In (Part 1-Part 3) we define various objects (OBI)-(OBIII) and formulate certain conditions we require them to satisfy. We then show that we can use them to obtain a system of obstruction bundle data which is disk-component-wise (Part 4). Finally, in Part 5, we show the existence of objects satisfying the required conditions.

**Disk-component-wise Obstruction Bundle Data: Part 1.** Suppose \( E \) is a positive real number and \( N \) is a positive integer. Let \( \mathcal{P} \) be the set of all pairs \( (k, \beta) \) such that \( \mathcal{M}^{\mathrm{RW}}_{k+1}(L; \beta) \neq \emptyset \), \( \omega \cap \beta \leq E \) and \( k \leq N \). Let \( (k, \beta), (k', \beta') \in \mathcal{P} \) we say \( (k', \beta') < (k, \beta) \) if \( \beta' \cap \omega < \beta \cap \omega \) or \( \beta' \cap \omega = \beta \cap \omega, k' < k \). We also say \( (k', \beta') \leq (k, \beta) \) if \( (k', \beta') < (k, \beta) \) or \( (k', \beta') = (k, \beta) \). By Gromov compactness theorem, for each \( (k, \beta) \in \mathcal{P} \) the set \{ \( (k', \beta') \in \mathcal{P} | (k', \beta') < (k, \beta) \) \} is a finite set.

(OBI): For \( (k, \beta) \in \mathcal{P}, \mathcal{P}(k+1, \beta) \) is a finite subset of \( \text{Int}(\mathcal{M}^{\mathrm{RGW}}_{k+1}(L; \beta)) \), the interior of the moduli space \( \mathcal{M}^{\mathrm{RGW}}_{k+1}(L; \beta) \). To be more specific, the space \( \text{Int}(\mathcal{M}^{\mathrm{RGW}}_{k+1}(L; \beta)) \) consists of elements that their source curves have only one disc component.

Let \( p \in \mathcal{P}(k+1, \beta) \). We write \( \Sigma_p \) for the source curve of \( p \) and \( u_p : (\Sigma_p, \partial \Sigma_p) \to (X, L) \) for the map part of \( p \). Let \( \tilde{R}_p \) be the very detailed tree...
describing the combinatorial type of \( p \). For \( v \in C_{0}^{\text{int}}(\tilde{R}_{p}) \), we denote the corresponding component of \( \Sigma_{p} \) by \( \Sigma_{p,v} \) and the restriction of \( p \) to \( \Sigma_{p,v} \) by \( p_{v} \).

(OBII): For any \( v \in C_{0}^{\text{int}}(\tilde{R}_{p}) \), we take a finite dimensional subspace:

\[
E_{p_{v}} \subseteq \begin{cases} C^{\infty}(\Sigma_{p_{v}}; T_{X} \otimes \Lambda^{0,1}) & \text{if } c(v) = d \text{ or } s, \\ C^{\infty}(\Sigma_{p_{v}}; T_{D} \otimes \Lambda^{0,1}) & \text{if } c(v) = D. \end{cases}
\]

whose support is away from nodal and marked points and the boundary of \( \Sigma_{p_{v}} \).

We require:

**Condition 10.14.** The restriction of \( u_{p} \) to a neighborhood of the support of \( \text{Supp}(E_{p_{v}}) \) is a smooth embedding. In particular, if \( \text{Supp}(E_{p_{v}}) \) is nonzero, \( u_{p_{v}} \) is non-constant.

**Disk-component-wise Obstruction Bundle Data: Part 2.** We fix an element \( u = (\Sigma, u_{v}, u_{p,v}, v) \in C_{0}^{\text{int}}(\tilde{R}_{u}) \) of \( \mathcal{M}_{k+1}^{RGW}(L; \beta) \), where \( \tilde{R}_{u} \) is the very detailed tree assigned to \( u \).

There is a forgetful map from the moduli space \( \mathcal{M}_{k+1}^{RGW}(L; \beta) \) to the moduli space of stable discs \( \mathcal{M}_{k+1}^{d} \) where for any \( u \in \mathcal{M}_{k+1}^{RGW}(L; \beta) \) we firstly forget all the data of \( u \) except the source curve \( \Sigma_{u} \), and then shrink the unstable components. There is a metric space \( C_{k+1}^{d} \), called the universal family, with a map \( \pi : C_{k+1}^{d} \to \mathcal{M}_{k+1}^{d} \) such that \( \pi^{-1}(\zeta) \), for \( \zeta \in \mathcal{M}_{k+1}^{d} \), is a representative for \( \zeta \) (see, for example, [FOOO8, Section 2] or [DF2, Subsection 5.1].) We pull-back \( C_{k+1}^{d}(L; \beta) \) to \( \mathcal{M}_{k+1}^{RGW}(L; \beta) \) via the forgetful map to obtain the space \( C_{k+1}^{RGW}(L; \beta) \) with the projection map \( \pi_{RGW} : C_{k+1}^{RGW}(L; \beta) \to \mathcal{M}_{k+1}^{RGW}(L; \beta) \). The pull-back of the metric on \( C_{k+1}^{d}(L; \beta) \) to \( C_{k+1}^{RGW}(L; \beta) \) defines a quasi metric\(^{37} \) on \( C_{k+1}^{RGW}(L; \beta) \). Here we obtain a quasi-metric because the forgetful map from \( \mathcal{M}_{k+1}^{RGW}(L; \beta) \) to \( \mathcal{M}_{k+1}^{d} \) is not injective. Note that this quasi metric is in fact a metric in each fiber \( \pi_{RGW}^{-1}(u) \). The fiber \( \pi_{RGW}^{-1}(u) \) can be identified with a quotient of \( \Sigma_{u} \). Thus by pulling back the metric on each fiber \( \pi_{RGW}^{-1}(u) \), we define a quasi metric on the source curve \( \Sigma_{u} \) of \( u \).

**Lemma 10.15.** For each \( \beta \) and \( k \), there is a positive constant \( \delta(k, \beta) \) with the following property. If \( u \in \mathcal{M}_{k+1}^{RGW}(L; \beta) \) and \( v \in C_{0}^{\text{int}}(\tilde{R}_{u}) \) is a vertex with \( u_{u,v} : \Sigma_{u,v} \to X \) being a non-constant map, then there is \( x \in \Sigma_{u,v} \) such that the distance between \( x \) and any nodal point and boundary point of \( \Sigma_{u} \) is greater than \( \delta(k, \beta) \). Moreover, if \( x' \in \Sigma_{u} \) is chosen such that \( u_{u}(x) = u_{u}(x') \), then the distance between \( x \) and \( x' \) is greater than \( \delta(k, \beta) \).

**Proof.** Given any \( u \), there is a constant \( \delta(k, \beta, u) \) such that the lemma holds for any non-constant irreducible component \( u_{u,v} \) of \( u_{u} \). In fact, there is a neighborhood \( \mathcal{U}(u) \) such that the lemma holds for the constant \( \delta(k, \beta, u) \)

\(^{37}\)A quasi-metric is a distance function which satisfies the reflexive property and triangle inequality. But we allow for two distinct points to have distance zero.
and any \( u' \in U(u) \). Now we can conclude the lemma using compactness of \( M_{k+1}^{RGW}(L; \beta) \).

In the following definition, \( \epsilon(k', \beta') \) is a constant which shall be fixed later.

**Definition 10.16.** A triple \((u, p, \phi)\) is said to be a *local approximation* to \( u \), if we have:

1. There is \((k', \beta') \leq (k, \beta)\) such that \( p \in \Psi(k' + 1, \beta') \).
2. \( \phi \) is a smooth embedding from a neighborhood of \( \bigcup v \text{Supp}(E_{p_v}) \) to \( \Sigma_u \). If \( x \) belongs to the image of \( \phi \), then its distance to the nodal points in \( \Sigma_u \) is greater than \( \delta(k', \beta') \). For each \( v \in C_{0}^{\text{int}}(R_{u}) \), there is \( v' \in C_{0}^{\text{int}}(R_{u}) \) such that \( \phi \) maps \( \text{Supp}(E_{p_v}) \) to \( \Sigma_{u,v'} \). Furthermore, if \( x' \) is another point in the source curve of \( u \) such that \( u_{u}(x) = u_{u}(x') \), then the distance between \( x \) and \( x' \) is greater than \( \delta(k', \beta') \).
3. For each \( v \), we require:
   \[ d_{C^{2}; \text{Supp}(E_{p_v})}(u_{u} \circ \phi, u_p) < \epsilon(k', \beta') \]
4. \( \phi \) satisfies the following point-wise inequality:
   \[ |\partial \phi| < |\partial \phi|/100 \]

The next definition is similar to Condition 4.5.

**Definition 10.17.** Let \((u, p, \phi)\) be a local approximation to \( u \). We say a map \( \hat{\phi} \) from a neighborhood of \( \bigcup v \text{Supp}(E_{p_v}) \) to \( \Sigma_u \) is a *normalization* of \( \phi \) if the following holds:

1. If \( x \) belongs to the image of \( \hat{\phi} \), then its distance to the nodal points in \( \Sigma_u \) is greater than \( \frac{1}{3} \delta(k', \beta') \). Furthermore, if \( x' \) is another point in the source curve of \( u \) such that \( u_{u}(x) = u_{u}(x') \), then the distance between \( x \) and \( x' \) is greater than \( \frac{1}{3} \delta(k', \beta') \).
2. For each \( v \), we require:
   \[ d_{C^{2}; \text{Supp}(E_{p_v})}(u_{u} \circ \hat{\phi}, u_p) < 2\epsilon(k', \beta') \]
   and:
   \[ d_{C^{0}; \text{Supp}(E_{p_v})}(\hat{\phi}, \phi) < \frac{\delta(k', \beta')}{3} \]
3. Let \( z \) be in a neighborhood of \( \text{Supp}(E_{p_v}) \):
   (a) Suppose \( z \) is in a component with color d or s. We take the unique minimal geodesic \( \gamma \) in \( X \setminus D \) (with respect to the metric \( g \)), which joins \( u_p(z) \) to \( (u_{u} \circ \hat{\phi})(z) \). Then:
   \[ \frac{d\gamma}{dt}(0) \perp T_{u_p(z)}u_p(\Sigma_p) \]
   (b) Suppose \( z \) and \( \hat{\phi}(z) \) are in a component with color D. We take the unique minimal geodesic \( \gamma \) in \( D \) (with respect to the metric \( g' \)), which joins \( u_p(z) \) to \( (u_{u} \circ \hat{\phi})(z) \). Then:
   \[ \frac{d\gamma}{dt}(0) \perp T_{u_p(z)}u_p(\Sigma_p) \]
(c) Suppose $z$ is in a component with color $D$ and $\hat{\phi}(z)$ is in a component with color $d$ or $s$. We take the unique minimal geodesic $\gamma$ in $D$ (with respect to the metric $g'$), which joins $u_p(z)$ to $(\pi \circ u_u \circ \hat{\phi})(z)$. Then:

$$\frac{d\gamma}{dt}(0) \perp T_{u_p(z)}u_p(\Sigma_p).$$

**Lemma 10.18.** If the constant $\epsilon(k', \beta')$ is small enough, then for any local approximation $(u, p, \phi)$ to $u \in \mathcal{M}_{k+1}^{\text{RGW}}(L; \beta)$, there exists a normalization $\hat{\phi}$ of $\phi$ and for any other normalization $\hat{\psi}$ of $\phi$, we have:

$$\hat{\phi}|_{\bigcup_v \text{Supp}(E_{pv})} = \hat{\psi}|_{\bigcup_v \text{Supp}(E_{pv})}.$$  

From now on, we assume that the constant $\epsilon(k', \beta')$ satisfies the assumption of this lemma.

**Proof.** This is a consequence of the implicit function theorem and compactness of $\mathcal{M}_{k+1}^{\text{RGW}}(L; \beta)$. \qed

**Definition 10.19.** For $j = 1, 2$, suppose $(u, p_j, \phi_j)$ is a local approximation to $u$. We say these two approximations are equivalent if $p_1 = p_2$ and  

$$\hat{\phi}_1|_{\bigcup_v \text{Supp}(E_{pv})} = \hat{\phi}_2|_{\bigcup_v \text{Supp}(E_{pv})}.$$  

This is obviously an equivalence relation. Each equivalence class is called a quasi component (of $u$). See Figure 17 for the schematic picture of a quasi-component.

![Figure 17](image-url)

**Figure 17.** $[u, p, \phi]$ is a quasi-component of $u$

We next define obstruction spaces $E_{u,p,\Xi}(\eta)$ where $p$ is a quasi component of $u$ and $\eta = (\xi, \sigma, (u'_v), (U'_v), (\rho_e), (\rho_i))$ is an inconsistent map with respect to a TSD $\Xi$ at $u$. The definition is similar to the corresponding definitions in Subsection 9.2. The TSD $\Xi$ induces a holomorphic embedding:

$$\psi_{\eta,u} : \bigcup_{v \in C_{0,0}^{\text{RGW}}(R_u)} \Sigma_{u,v}^{-} \rightarrow \Sigma(\xi, \sigma).$$
By taking $\Xi$ to be small enough, we may assume the image of $\phi$ is contained in the domain of $\psi_{\eta,u}$. Define:

$$\phi_{\eta,p} = \psi_{\eta,u} \circ \phi.$$  

Using Implicit function theorem, we can modify $\phi_{\eta,p}$ to obtain $\hat{\phi}_{\eta,p}$ from a neighborhood of $\bigcup_v \text{Supp}(E_{p,v})$ to $\Sigma_{\eta}$ such that the analogue of Definition 10.17 is satisfied. This map is clearly independent of the representative of $p$ and also independent of $\Xi$ if this TSD is sufficiently small.

By replacing $I(t, s(x))$, $I(t, s(x))$ and $I(t, d(x))$ with $\hat{\phi}_{\eta,p}$ and using the vector spaces $E_{p,v}$, we obtain:

$$E_{u,p,\Xi}(\eta) \subset \bigoplus_{v \in C^\eta(R_{\eta})} L^2_m(\Sigma_{\eta,x}; T \otimes \Lambda^{0,1})$$

in the same way as in (4.5). Here $\hat{R}_{\eta}$ is the very detailed tree describing the combinatorial type of $\eta$.

**Disk-component-wise Obstruction Bundle Data: Part 3.** Our obstruction bundle data $E_{u,\Xi}(\eta)$ is a direct sum of $E_{u,p,\Xi}(\eta)$ for an appropriate set of quasi components $p$ of $u$. Our next task is to find a way to choose this set of quasi components.

**Definition 10.20.** For $u \in M_{k+1}^{RGW}(L; \beta)$, we denote by $\mathcal{D}(k, \beta)(u)$ the set of all quasi components of $u$. Let:

$$\mathcal{D}(k, \beta) := \bigcup_{u \in M_{k+1}(L; \beta)} \mathcal{D}(k, \beta)(u).$$

The map

$$\Pi : \mathcal{D}(k, \beta) \to M_{k+1}^{RGW}(L; \beta)$$

is the obvious projection.

**Lemma 10.21.** If the constant $\epsilon(k', \beta')$ is small enough, then $\mathcal{D}(k, \beta)(u)$ is a finite set.

**Proof.** By Gromov compactness, there is only a finite number of $(k', \beta') \leq (k, \beta)$ such that $M_{k'+1}^{RGW}(L; \beta') \neq \emptyset$. Let $(k', \beta')$ be such a pair and $p$ be an element of the finite set $\mathcal{P}(k'+1, \beta')$. Assume $y$ is an element of the source curve of $p$. There is a neighborhood $U_y$ of $y$ in the source curve of $y$ such that if $\epsilon'(k', \beta')$ is small enough, then the following holds. Let $[u, p, \phi]$ and $[u, p, \psi]$ be two quasi components of an element $u \in M_{k+1}^{RGW}(L; \beta)$ with $\hat{\phi}$ and $\hat{\psi}$ being the normalizations of $\phi$ and $\psi$. If $\hat{\phi}(y) \neq \hat{\psi}(y)$, then $\hat{\phi}|_{U_y}$ and $\hat{\psi}|_{U_y}$ are disjoint. This would imply that given the element $u$, there are only finitely many possibilities for the restriction of the normalization map to $U_y$. Therefore, there are finitely many quasi components of the form $[u, p, \phi]$ for $u$. Finiteness of the sets $\mathcal{P}(k'+1, \beta')$ completes the proof.
We next define a topology on \( \mathcal{QC}(k, \beta) \). Let \( u \in \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) be a TSD at \( u \) and \( \mathcal{U}(u, \Xi) \) be the associated set of inconsistent solutions. We construct a map
\[
\mathcal{I} : \mathcal{U}(u, \Xi) \cap \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \rightarrow \mathcal{D}(k, \beta)
\]
such that \( \Pi \circ \mathcal{I} = \text{id} \) assuming \( \Xi \) is small enough. The TSD \( \Xi \) induces a map:
\[
\psi_{u', u} : \bigcup_{v \in C_0^{\text{int}}(\tilde{R}_u)} \Sigma^{-}_{u, v} \rightarrow \Sigma_{u'}
\]
for \( u' \in \mathcal{U}(u, \Xi) \). Let \((u, p, \phi)\) be a local approximation to \( u \). If \( \Xi \) is sufficiently small, then \((u, p, \psi_{u', u} \circ \phi)\) is a local approximation to \( u' \). Using Implicit function theorem, it is easy to see that the equivalence class of \((u, p, \psi_{u', u} \circ \phi)\) depends only on the equivalence class of \((u, p, \phi)\). We thus obtain the map \( \mathcal{I} \). This map in a small neighborhood of \( u \) is independent of the choice of \( \Xi \).

The map \( \mathcal{I} \) is also injective.

**Definition 10.22.** A neighborhood system of a quasi component \( p = [u, p, \phi] \) of \( u \) in \( \mathcal{QC}(k, \beta) \) is given by mapping the neighborhood system of \( u \) in \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) via the map \( \mathcal{I} \).

It is straightforward to see from the above definition that:

**Lemma 10.23.** \( \mathcal{D}(k, \beta) \) is Hausdorff and metrizable with respect to this topology. For each quasi component \( p \) of \( u \) in \( \mathcal{D}(k, \beta) \) there exists a neighborhood of \( p \) in \( \mathcal{D}(k, \beta) \) such that the restriction of \( \Pi \) to this neighborhood is a homeomorphism onto an open subset.

Let \( \mathcal{F} \) be a subset of \( \mathcal{D}(k, \beta) \). For \( u \in \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \), we define:
\[
\mathcal{F}(u) = \Pi^{-1}(u) \cap \mathcal{F}.
\]

It is a map which assigns to \( u \) a finite set of quasi components of \( u \). Justified by this, we call \( \mathcal{F} \) a quasi component choice map.

**Definition 10.24.** A quasi component choice map \( \mathcal{F} \) is open (resp. closed) if it is an open (resp. closed) subset of \( \mathcal{D}(k, \beta) \) (with respect to the topology of Definition 10.22). We say \( \mathcal{F} \) is proper if the restriction of \( \Pi \) to \( \mathcal{F} \) is proper.

**(OBIII):** For each \((k, \beta) \in \mathcal{TP}\), we take quasi component choice maps \( \mathcal{F}_{k, \beta} \) and \( \mathcal{F}_{k, \beta}^\circ \).

We are mainly concerned with the objects as in (OBIII) which satisfy the following condition:

**Condition 10.25.** The quasi component choice map \( \mathcal{F}_{k, \beta}^\circ \) is open and is a subset of \( \mathcal{F}_{k, \beta} \). The quasi component map \( \mathcal{F}_{k, \beta} \) is proper.

The next condition is related to the disk-component-wise-ness. Let \( u \in \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) and \( \tilde{R} \) be the very detailed tree associated to \( u \). Let \( \tilde{R} \) belong to the disk splitting tree \( S \). Let \( w \) be an interior vertex of \( S \). Following
the discussion of the beginning of Subsection 10.2, we obtain an element
$u \in M_{k+1}^{\text{RGW}}(L; \beta(w))$ for each interior vertex $w$ of $S$. Define:
\begin{equation}
\mathcal{I}_w : \mathcal{D}(k, \beta(w))(u_w) \rightarrow \mathcal{D}(k, \beta)(u)
\end{equation}
to be the map given by:
\[ \mathcal{I}_w([u_w, p, \phi]) = [u, p, \phi]. \]

The target of $\phi$ on the left hand side is $\Sigma_{u_w}$, the source curve of $u_w$, which is
a subset of the source curve $\Sigma_u$ of $u$. So $\phi$ on the right hand side is regarded
as a map to $\Sigma_u$. It is clear that $\mathcal{I}_w$ maps equivalent objects to equivalent
ones and hence the above definition is well-defined.

**Lemma 10.26.** The map $\mathcal{I}_w$ is injective. If $w \neq w'$, then the image of $\mathcal{I}_w$
is disjoint from the image of $\mathcal{I}_{w'}$.

**Proof.** The injectivity is obvious from the definition. If $[u, p, \phi]$ is in the
image of $\mathcal{I}_w$, then the image of $\phi$ is contained in the component $\Sigma_{u_w}$ corre-
sponding to $w$. Therefore, for $w \neq w'$, the image of the maps $\mathcal{I}_w$ and $\mathcal{I}_{w'}$
are disjoint. \(\square\)

**Condition 10.27.** Let $u \in M_{k+1}^{\text{RGW}}(L; \beta)$ and $\check{R}$ and $S$ be given as above.
Then we require:
\begin{equation}
\mathcal{F}_{k, \beta}(u) = \bigcup_w \mathcal{I}_w \left( \mathcal{F}_{k, \beta}(u_w) \right),
\end{equation}
\begin{equation}
\mathcal{F}_{k, \beta}^\circ(u) = \bigcup_w \mathcal{I}_w \left( \mathcal{F}_{k, \beta}^\circ(u_w) \right).
\end{equation}

We need the following definition to state the next condition:

**Definition 10.28.** Let $u \in M_{k+1}^{\text{RGW}}(L; \beta)$ and $\eta$ be an inconsistent map with
respect to a TSD $\Xi$ at $u$. We assume $\Xi$ is sufficiently small such that the
vector spaces $E_{u,p,\Xi}(\eta)$ in (10.12) are well-defined. We define:
\begin{equation}
E_{u,\check{F},\Xi}(\eta) := \sum_{[p] \in \check{F}_{k,\beta}(u)} E_{u,p,\Xi}(\eta) \subset \bigoplus_{v \in C_{\text{int}}(\check{R})} L_m^2(\Sigma_{\eta,v}; T \otimes \Lambda^{0,1}).
\end{equation}
where $\Sigma$ denotes the sum of vector subspaces of a vector space. Similarly, we define:
\begin{equation}
E_{u,\check{F}^\circ,\Xi}(\eta) := \sum_{[p] \in \check{F}_{k,\beta}^\circ(u)} E_{u,p,\Xi}(\eta) \subset \bigoplus_{v \in C_{\text{int}}(\check{R})} L_m^2(\Sigma_{\eta,v}; T \otimes \Lambda^{0,1}).
\end{equation}

Note that $E_{u,\check{F},\Xi}(\eta) \subseteq E_{u,\check{F}^\circ,\Xi}(\eta)$.

**Condition 10.29.** We require that the sum in (10.16) is a direct sum for
$\eta = u$. Namely,
\begin{equation}
E_{u,\check{F},\Xi}(u) = \bigoplus_{[p] \in \check{F}_{k,\beta}(u)} E_{u,p,\Xi}(\eta).
\end{equation}

Note that the above condition implies that the sum in (10.17) for $\eta = u$
is also a direct sum.
Definition 10.30. We say the linearization of the non-linear Cauchy-Riemann equation is \textit{transversal} to $E_u, F ◦, Ξ(u)$ if the sum of the images of the operator $D_u\overline{\partial}$ in (8.2) and $E_u, F ◦, Ξ(u)$ is $L^2_{m,δ}(u; T ⊗ Λ^{0,1})$.

Definition 10.31. Consider the operator:

$\mathcal{E}V_{z_0} : W^2_{m+1,δ}(u; T) → T_{u(z_0)}L$

given by evaluation at the point $z_0$, the 0-th boundary marked point of the source of $u$. Recall that the Hilbert space $W^2_{m+1,δ}(u; T)$ is the domain of the operator $D_u\overline{\partial}$ in (8.2). We say $E_u, F ◦, Ξ(u)$ satisfies the \textit{mapping transversality} property, if the restriction:

$\mathcal{E}V_{z_0} |_{D_u\overline{\partial} - 1(E_u, F ◦, Ξ(u))} : D_u\overline{\partial}^{-1}(E_u, F ◦, Ξ(u)) → T_{u(z_0)}L,$

is surjective.

Condition 10.32. We require that the linearization of the non-linear Cauchy-Riemann equation is transversal to $E_u, F ◦, Ξ(u)$ and $E_u, F ◦, Ξ(u)$ satisfies the mapping transversality property.

Let $\text{Aut}(u)$ be the group of automorphisms of $u$. If $γ ∈ \text{Aut}(u)$ and $[u, p, φ]$ is a quasi component of $u$, then $[u, p, γ ◦ φ]$ is also a quasi component of $u$. Thus $\text{Aut}(u)$ acts on $\mathcal{E}S(k, β)(u)$

Condition 10.33. We require that $F_{k, β}(u), F_{k, β}(u)$ are invariant with respect to the action of $\text{Aut}(u)$.

\textit{Disk-component-wise Obstruction Bundle Data: Part 4.} Given the above objects satisfying the mentioned conditions, we can construct the desired obstruction bundle data:

Lemma 10.34. Suppose we are given the objects in (OBI)-(OBIII), which satisfy Conditions 10.25, 10.27, 10.29, 10.32, 10.33. Then $\{E_{u, F ◦, Ξ(η)}\}$ is a disk-component-wise system of obstruction bundle data.

\textit{Proof.} The system $\{E_{u, F ◦, Ξ(η)}\}$ satisfies Definition 10.5 (1), (2), (3) and (5) as immediate consequences of the construction. Definition 10.5 (3) is a consequence of the properness of $F_{k, β}$. (Compare to Lemma 9.11.) Definition 10.5 (5) is a consequence of Condition 10.32. Definition 10.5 (6) is a consequence of Condition 10.33. Disc-component-wise-ness is an immediate consequence of Condition 10.27. \hfill □

\textit{Disk-component-wise Obstruction Bundle Data: Part 5.} To complete the proof of Proposition 10.13, it suffices to prove the next lemma.

Lemma 10.35. There exist objects (OBI)-(OBIII) which satisfy Conditions 10.25, 10.27, 10.29, 10.32, 10.33.

\textit{Proof.} The proof is by induction on $(k, β)$ and is given in several steps.

\textbf{Step 1 (The base of induction):} We assume that $(k, β)$ is minimal in this order $<$. In this case, the moduli space $\mathcal{M}^{RGW}_{k+1}(L; β)$ has no boundary. We
follow a similar argument as in Subsection 9.2. For each \( p \in M_{k+1}^{RGW}(L; \beta) \), we fix a vector space \( E_{p,v} \) for \( v \in C^0_{\text{int}}(R_p) \) as in (ORII). We require that the linearization of the non-linear Cauchy-Riemann equation is transversal to

\[
\bigoplus_{v \in C^0_{\text{int}}(R_p)} E_{p,v}
\]

at \( p \) (Definition 10.30) and (10.19) has the mapping transversality property at \( p \) (Definition 10.31). Using Lemma 10.15, we may assume that the distance of any point \( x \) in the support of the elements of \( E_{p,v} \) to nodal points of \( \Sigma_p \) is at least \( \delta(k, \beta) \). Moreover, if \( x' \) is another point in the source curve of \( p \) such that \( u_p(x) = u_p(x') \), then the distance between \( x \) and \( x' \) is greater than \( \delta(k', \beta') \).

For each \( p \), we also pick a TSD \( \Xi_p \) and a compact neighborhood \( K(p) \) of \( p \) in \( M_{k+1}^{RGW}(L; \beta) \), which satisfy the following conditions. Firstly we require that the compact neighborhood \( K(p) \) is included in the set of inconsistent maps determined by \( \Xi_p \). Thus for any \( u \in K(p) \), there is a holomorphic embedding \( \phi_{u,p} \) from a neighborhood of \( \bigcup_v \text{Supp}(E_{p,v}) \) to \( \Sigma_u \), assuming that \( \Xi_p \) is small enough. We may also assume that we can normalize \( \phi_{u,p} \) as in Definition 10.17 to obtain:

\[
\hat{\phi}_{u,p} : \bigcup_v \text{Supp}(E_{p,v}) \to \Sigma_u.
\]

We use \( \hat{\phi}_{u,p} \) to transport \( E_{p,v} \) to the point \( u \) and obtain \( E_{p,v}(u) \). We may choose \( \Xi_p \) small enough such that the linearization of the non-linear Cauchy-Riemann equation is transversal to

\[
E_{p,v}(u) = \bigoplus_{v \in C^0_{\text{int}}(R_p)} E_{p,v}(u)
\]

at \( u \) and (10.20) satisfies the mapping transversality property at \( u \), for any \( u \in K(p) \).

Now we take a finite set \( \mathfrak{P}(k+1, \beta) \subset M_{k+1}^{RGW}(L; \beta) \) such that

\[
\bigcup_{p \in \mathfrak{P}(k+1, \beta)} \text{Int } K(p) = M_{k+1}^{RGW}(L; \beta).
\]

We define:

\[
\mathcal{F}_{k,\beta}(u) = \{ [u, p, \phi_{u,p}] \mid p \in \mathfrak{P}(k+1, \beta), \ u \in K(p) \}
\]

\[
\mathcal{F}^0_{k,\beta}(u) = \{ [u, p, \phi_{u,p}] \mid p \in \mathfrak{P}(k+1, \beta), \ u \in \text{Int } K(p) \}
\]

Condition 10.25 is immediate from the definition. Condition 10.27 is void in this case. We can perturb \( E_{p,v} \) arbitrarily small in \( C^2 \) topology so that Condition 10.29 holds. (See Lemma 9.9.) Condition 10.32 follows from the choice of \( E_{p,v} \) and (10.21). We can take \( E_{p,v} \) to be invariant under the action of \( \text{Aut}(p) \) and hence Condition 10.33 holds. Thus we complete the first step of the induction.
Next, we suppose that the required objects in (OBI)-(OBIII) are defined for \((k', \beta') < (k, \beta)\). We use Condition 10.27 to define \(\mathcal{F}_{k, \beta}(u), \mathcal{F}_{k, \beta}^0(u)\) for \(u \in \partial M_{k+1}(L; \beta)\).

**Step 2:** The set:

\[
\bigcup_{u \in \partial M_{k+1}(L; \beta)} \mathcal{F}_{k, \beta}^0(u)
\]

is an open subset of \(\Pi^{-1}(\partial M_{k+1}(L; \beta))\), where \(\Pi\) is the map in (10.13).

Let \(p_j \in \mathcal{D}(k, \beta)\) with \(u_j = \Pi(p_j) \in \partial M_{k+1}(L; \beta)\). Suppose also \(\lim_{j \to \infty} p_j = p \in \mathcal{F}_{k, \beta}(u)\) with \(u = \lim_{j \to \infty} u_j\). We need to show that \(p_j \in \mathcal{F}_{k, \beta}^0(u_j)\) for sufficiently large values of \(j\). Let the combinatorial type of \(u\) be given by a very detailed DD-ribbon tree \(\tilde{R}\) which belongs to the disc splitting tree \(S\). We may assume that the very detailed tree associated to \(u_j\) is independent of \(j\) because there are finitely many very detailed tree obtained by level shrinking, level 0 edge shrinking and fine edge shrinking. We denote this very detailed DD-ribbon tree by \(\tilde{R}'\). We also assume that \(\tilde{R}'\) belongs to the disc splitting tree \(S'\). Since \(S'\) is obtained from \(S\) by shrinking level 0 edges, there is a standard shrinking map \(\pi : S \to S'\). Note that \(S\) and \(S'\) have at least two interior vertices.

By Condition 10.27, there exists \(w \in C^0_{\text{int}}(S)\) and \(p_w \in \mathcal{F}_{k, \beta}^0(u_w)\) such that

\[
p = \mathcal{I}_w(p_w).
\]

Let \(w' = \pi(w)\), which is an interior vertex of \(S'\). We also define \(S(w') := \pi^{-1}(w')\), which is a subtree of \(S\). Let \(u_{S(w')} \in \mathcal{M}_{k+1}(L; \beta_{w'})\) which is obtained from \(u\) and \(S(w')\) in the same way as in the beginning of Subsection 10.2. \(\lim_{j \to \infty} u_j = u\) implies

\[
\lim_{j \to \infty} u_{j, w'} = u_{S(w')}
\]

by the definition of the RGW topology.

Since \(w\) is a vertex of \(\mathcal{I}(w')\), there exists:

\[
\mathcal{I}'_{w} : \mathcal{D}(k, \beta, w)(u_w) \to \mathcal{D}(k, \beta, w')(u_{S(w')}).
\]

We define \(p_{w'} = \mathcal{I}'_{w}(p_w)\). Using the definition of the topology of \(\mathcal{D}(k, \beta)\) and of \(\mathcal{I}_w\), it is easy to see that there exists \(p_{j, w'} \in \mathcal{D}(k, \beta, w')\) such that

\[
\lim_{j \to \infty} p_{j, w'} = p_{w'}
\]

in \(\mathcal{D}(k, \beta, w')\) and

\[
\mathcal{I}'_{w}(p_{j, w'}) = p_j.
\]

Now by induction hypothesis:

\[
p_{j, w'} \in \mathcal{F}_{k, \beta}^0(u_{j, w'})
\]

for sufficiently large \(j\). Condition 10.27 implies \(p_j \in \mathcal{F}_{k, \beta}^0(u_j)\) for large \(j\), as required.
Step 3: The restriction of $\Pi$ to
$$\bigcup_{u \in \partial \mathcal{M}_{k+1}(L;\beta)} \mathcal{F}_{k,\beta}(u)$$
is a proper map to $\partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)$.

Let $p_j \in \mathcal{F}_{k,\beta}'(u_j)$ with $u_j \in \partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)$. Suppose $\lim_{j \to \infty} u_j = u \in \partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)$. It suffices to find a subsequence of $p_j$ which converges to an element of $\mathcal{F}_{k,\beta}(u)$. Let the combinatorial type of $u$ be given by a very detailed DD-ribbon tree $\hat{R}$ which belongs to the disc splitting tree $S$. After passing to a subsequence, we may assume that the very detailed tree associated to $u_j$ is independent of $j$. We denote this tree by $\hat{R}'$ which belongs to the disc splitting tree $S'$. Let $\pi : S \to S'$ be defined as in the previous step.

By Condition 10.27 and after passing to a subsequence, we may assume that there exist $w \in C_0^0(S')$ and $p_{j,w} \in \mathcal{F}_{k,w,\beta,w}(u_{j,w})$ such that
$$\mathcal{F}_w(p_{j,w}) = p_j.$$Let $S_w$ be the subtree $\pi^{-1}(w)$ of $S$. We obtain $u_{\partial w}$ from $u$ in the same way as in the beginning of Subsection 10.2. Convergence of $u_j$ to $u$ implies that:
$$\lim_{j \to \infty} u_{j,w} = u_{\partial w},$$by the definition of the RGW topology. Now we use the induction hypothesis to find a subsequence such that $p_{j,w} \in \mathcal{F}_{k,w,\beta,w}(u_{j,w})$ converges to $p_w \in \mathcal{F}_{k,w,\beta,w}(u_{\partial w})$. Therefore:
$$\lim_{j \to \infty} p_j = \mathcal{F}_w(p_w) \in \mathcal{F}_{k,\beta}'(u).$$This completes the proof of this step.

Step 4: (Extension to a neighborhood of the boundary) In the previous steps, we defined $\mathcal{F}_{k,\beta}'$ and $\mathcal{F}_{k,\beta}^0$ on the boundary. Next, we extend these quasi component choice maps to a neighborhood of the boundary. We fix $\rho > 0$ sufficiently small such that if $d(u,u') < 5\rho$, $u' \in \partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)$ and $[u',p,\phi] \in \mathcal{F}_{k,\beta}'(u')$, then $[u,p,\psi_{u,u'} \circ \phi]$ is a quasi component. Then for $u \in \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)$ with $d(u,\partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)) < 2\rho$, we define:

1. $\mathcal{F}_{k,\beta}'(u)$ is the set of $[u,p,\phi]$ such that there are $u' \in \partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)$ and $[u',p,\phi'] \in \mathcal{F}_{k,\beta}'(u')$ with the following properties:
   - (a) $d(u,u') \leq 2d(u,\partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)) \leq \rho$.
   - (b) $(u,p,\phi)$ is equivalent to $(u,p,\psi_{u,u'} \circ \phi')$.

2. $\mathcal{F}_{k,\beta}^0(u)$ is the set of $[u,p,\phi]$ such that there are $u' \in \partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)$ and $[u',p,\phi'] \in \mathcal{F}_{k,\beta}^0(u')$ with the following properties:
   - (a) $u = u'$ or $d(u,u') < 2d(u,\partial \mathcal{M}_{k+1}^{\text{RGW}}(L;\beta)) < \rho$.
   - (b) $(u,p,\phi)$ is equivalent to $(u,p,\psi_{u,u'} \circ \phi')$.

We put
$$\mathcal{F}_{k,\beta}^0 = \bigcup_u \mathcal{F}_{k,\beta}^0(u), \quad \mathcal{F}_{k,\beta}' = \bigcup_u \mathcal{F}_{k,\beta}'(u).$$
It follows easily from Step 2 that $F'\circ k,\beta$ is open. It follows easily from Step 3 that the restriction of $\Pi$ to $F'\circ k,\beta$ is proper.

Items (b) in the above definition imply that $F'_k,\beta(u)$ and $F'\circ k,\beta(u)$ coincide with the previously defined spaces for $u \in \partial M_{RGW}^{k+1}(L; \beta)$. Therefore, Condition 10.27 holds. Thus we have constructed objects for $(k, \beta)$ which satisfy all the required conditions except Condition 10.32. By taking a smaller value of $\rho$ if necessary, we can guarantee that Condition 10.32 is also satisfied.

**Step 5:** (Extension to the rest of the moduli space $M_{RGW}^{k+1}(L; \beta)$) The rest of the proof is similar to Step 1. For each $p \in \text{Int}(M_{RGW}^{k+1}(L; \beta))$ we choose $\Xi_p$, $E_{p,v}$ and $K(p)$ as in the first step of the induction. We take a finite set $\mathcal{P}(k+1, \beta)$ such that:

\begin{equation}
\bigcup_{p \in \mathcal{P}(k+1, \beta)} \text{Int} K(p) = M_{RGW}^{k+1}(L; \beta) \setminus B_\rho(\partial M_{RGW}^{k+1}(L; \beta)).
\end{equation}

![](Equation 10.23)

is satisfied instead of (10.21). Now we define:

\begin{align}
F_{k, \beta}(u) &= F'_k,\beta(u) \cup \{[u, p, \phi_u, p] \mid p \in \mathcal{P}(k, \beta), u \in \mathcal{K}(p)\} \\
F'\circ k,\beta(u) &= F'\circ k,\beta(u) \cup \{[u, p, \phi_u, p] \mid p \in \mathcal{P}(k, \beta), u \in \text{Int} \mathcal{K}(p)\}.
\end{align}

They satisfy all the required conditions including Condition 10.32. We thus have completed the inductive step. \hfill \square

We verified Proposition 10.13 and hence Theorem 10.1. This completes the construction of a system of Kuranishi structures on $M_{RGW}^{k+1}(L; \beta)$ which are compatible at the boundary components and corners. If $L_0$ and $L_1$ are monotone Lagrangians in $X\setminus D$ with minimal Maslov number 2, then we also need compatibility of Kuranishi structures of $M_{RGW}^{k+1}(L_1, L_0; p, q; \beta)$ with the forgetful maps of boundary marked points to define Lagrangian Floer homology. (See [DF2, Lemma 3.67] for the precise statement of this compatibility at the boundary. The proof in the case of strips is similar to the case of disks and we omit it here.

**Remark 10.36.** The proof in this subsection is different from the approach in [FOOO9, Section 8], where the case of stable map compactification is treated. In this subsection, we use target space parallel transportation. On the other hand, in [FOOO9, Section 8] extra marked points are added to $p \in \mathcal{P}(k+1, \beta)$ and are used to fix a diffeomorphism between open subsets of the source domains. Both methods work in both situations.

11. **Compatibility of Kuranishi Structures with Forgetful Maps**

So far, we have constructed a system of Kuranishi structures on the moduli spaces $\{M_{RGW}^{k+1}(L_1, L_0; p, q; \beta)\}_{\omega \cap \beta < E}$ which are compatible over the boundary components and corners. If $L_0$ and $L_1$ are monotone Lagrangians in $X\setminus D$ with minimal Maslov number 2, then we also need compatibility of Kuranishi structures of $M_{RGW}^{k+1}(L_1, L_0; p, q; \beta)$ with the forgetful maps of boundary marked points to define Lagrangian Floer homology. (See [DF2,
Section 4.) This compatibility requirement is discussed in this subsection. As in previous sections, we focus mainly on the analogous results for the case of discs. The proof in the case of moduli space of strips is similar. If the reader is only interested in the case of Lagrangian Floer homology for Lagrangians with minimal Maslov number greater than two, then this section can be skipped. This turns out to be the case for Lagrangians coming from Yang-Mills gauge theory [DF1].

Definition 11.1. Let \( u \in \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \). Let \( R \) be the very detailed tree describing the combinatorial type of \( u \). We fix a TSD \( \Xi \) at \( u \). Let \( \eta = (\vec{r}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) \) be an inconsistent map with respect to \( \Xi \).

1. Remove all the edges \( e \) of \( R \) with \( \sigma_e = 0 \), and let \( R_0 \) be one of the connected components of the resulting graph. The union of all the spaces \( \Sigma_{\eta,v} \), where \( v \) belongs to \( R_0 \), is called an irreducible component of \( \eta \). If it does not make any confusion, the union of all the interior vertices \( v \) of \( R \), which belong to \( R_0 \), is also called an irreducible component.

2. An irreducible component of \( \eta \) is called a trivial component if the following holds:
   (a) All the vertices in this component have color \( d \).
   (b) All the homology classes assigned to the vertices in this component are 0.

3. We say \( \eta \) preserves triviality if for any interior vertex \( v \) in a trivial component, the map \( u'_v \) is constant.

Lemma 11.2. Given any element \( u \in \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \), the Kuranishi neighborhood of \( u \), constructed in Subsection 10.3, is contained in the set of inconsistent maps which preserve triviality.

Proof. Suppose \( \Xi \) is a small enough TSD such that we can form the obstruction bundle \( E_{u,\vec{r},\Xi}(\eta) \) over inconsistent maps \( \eta \) with respect to \( \Xi \). We assume that \( \eta \) is chosen such that it represents an element of \( \tilde{U}(u, \Xi) \). If \( [u, p, \phi] \) is a quasi component of \( u \), then the image of \( \phi \) is away from the components of \( u \) with trivial homology classes. Consequently, restriction of the obstruction bundle \( E_{u,\vec{r},\Xi}(\eta) \) to any trivial component of \( \eta \) is trivial. Therefore, the restriction of \( u' \) to any such component has trivial homology class and satisfies the Cauchy-Riemann equation with a trivial obstruction bundle, and hence it is a constant map. \( \square \)

Suppose \( \eta \) is an inconsistent map with respect to \( \Xi \). Let \( \Xi' \) be another TSD at the same point \( u \). If \( \Xi' \) is small enough, then we obtain a corresponding inconsistent map \( \eta' \) with respect to \( \Xi' \). (See the discussion preceding Lemma 9.19.) It is clear that \( \eta \) preserves triviality if and only if \( \eta' \) preserves triviality.

We form the forgetful map:

\[
\text{fgg} : \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \rightarrow \mathcal{M}^{\text{RGW}}_1(L; \beta)
\]
by forgetting the boundary marked points other than the 0-th one. (See [DF2, Definition 3.72 and Lemma 3.74] for the definition of the forgetful maps of one boundary marked point. The forgetful map in (11.1) can be defined in a similar way.)

**Lemma 11.3.** Let \( u \in \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) and \( u' = f_{\mathbb{gg}}(u) \in \mathcal{M}^{\text{RGW}}_{1}(L; \beta) \). For any TSD \( \Xi' = (\vec{w}_{u'}, (N_{u',v,e}), (\phi_{u',v}), (\varphi_{u',v,e}), \delta') \) at \( u' \) there is a TSD \( \Xi = (\vec{w}_u, (N_{u,v,e}), (\phi_{u,v}), (\varphi_{u,v,e}), \delta) \) at \( u \) such that any inconsistent map \( \eta \) with respect to \( \Xi \) which preserves triviality induces an inconsistent map \( \eta' \) with respect to \( u' \).

**Proof.** Let \( \hat{R} \) (resp. \( \hat{R}' \)) be the very detailed DD-ribbon tree describing the combinatorial type of \( u \) (resp. \( u' \)). By construction we observe that the vertices of \( \hat{R} \) with color \( s \) or \( D \) are in one to one correspondence with the vertices of the same color in \( \hat{R}' \). Also the set of vertices of \( \hat{R}' \) with color \( d \) is a subset of the vertices of \( \hat{R} \) with color \( d \). The difference \( C^\text{int}_0(\hat{R}) \setminus C^\text{int}_0(\hat{R}') \) consists of vertices \( v \) such that the map \( u_v \) is constant on it.\(^{38}\) In particular, for any such vertex \( v \), the component \( \Sigma_{u,v} \) together with the marked points and nodal points is already source stable. Therefore, we can require that the additional marked points \( \vec{w}_u \) of the TSD \( \Xi \) do not belong to such irreducible components of \( \Sigma_u \). Thus we may find marked points \( \vec{w}_u \) on \( \Sigma_{u'} \) such that they are identified with \( \vec{w}_{u'} \) using the map \( \Sigma_u \to \Sigma_{u'} \) which collapses the components associated to \( C^\text{int}_0(\hat{R}) \setminus C^\text{int}_0(\hat{R}') \). We define \( \vec{w}_u \) to be the set of the additional marked points of \( \Xi \). We also assume that the set of transversals of \( \Xi \) are identified with that of \( \Xi' \) in an obvious way.

We also require that the trivialization of the universal families of irreducible components associated to \( \Xi \) and \( \Xi' \) to be related to each other as follows. For any sphere component we assume that the associated trivializations coincide with each other. For \( v \in C^\text{int}_0(\hat{R}') \subset C^\text{int}_0(\hat{R}) \) with level 0, let \( \mathcal{M}^\text{source}_{u,v}, \mathcal{M}^\text{source}_{u',v} \) be the corresponding moduli spaces of marked disks and \( \mathcal{C}^\text{source}_{u,v}, \mathcal{C}^\text{source}_{u',v} \) be the universal families. We take a trivialization of \( \mathcal{C}^\text{source}_{u,v} \) over a sufficiently small neighborhood \( \mathcal{V}^\text{source}_{u,v} \) of \( \Sigma_{u,v} \) so that the next diagram commutes:

\[
\begin{array}{ccc}
\mathcal{V}^\text{source}_{u,v} \times \Sigma_{u,v} & \xrightarrow{\phi_{u,v}} & \mathcal{C}^\text{source}_{u,v} \\
\downarrow & & \downarrow \\
\mathcal{V}^\text{source}_{u',v} \times \Sigma_{u',v} & \xrightarrow{\phi_{u',v}} & \mathcal{C}^\text{source}_{u',v}
\end{array}
\]

where the vertical arrows are obvious forgetful maps. For the trivializations of the universal families of disk components corresponding to \( v \in C^\text{int}_0(\hat{R}') \setminus C^\text{int}_0(\hat{R}) \), that are parts of \( \Xi \), we take an arbitrary choice.

For \( v \in C^\text{int}_0(\hat{R}') \subset C^\text{int}_0(\hat{R}) \) and an edge \( \epsilon \) incident to \( v \), we pick \( \varphi_{u,v,e} \) to be the analytic family induced by \( \varphi_{u',v,e} \). In the case that \( v \in C^\text{int}_0(\hat{R}) \setminus C^\text{int}_0(\hat{R}') \),

\(^{38}\)This is not a necessary and sufficient condition.
the corresponding component $\Sigma_{u,v}$ in addition to boundary marked points has at most two boundary nodes. If there are two boundary nodes inducing edges $e_+$ and $e_-$ incident to $v$, then we can identify $\Sigma_{u,v}$ with the strip $[0,1] \times \mathbb{R}$ where the boundary node associated to $e_{\pm}$ is in correspondence with the point at $\pm\infty$ on the boundary of $[0,1] \times \mathbb{R}$. We fix one such identification and let $[0,1] \times [T, \infty)$ and $[0,1] \times (-\infty, -T]$, for a large value of $T$, induce the analytic families of coordinates $\varphi_{u,v,\pm e}$. In the case that there is only one interior edge incident to $v$, we follow a similar strategy with the difference that we only need to use the half strip $[0,1] \times [T, \infty)$ to define the corresponding analytic family of coordinates. We also let $\delta = \delta'$. 

Now, let

$$\eta = (\tilde{x}_y, \tilde{\sigma}_y, (u'_{0,v}), (U'_{0,v}), (\rho_{0,e}), (\rho_{0,i}))$$

be an inconsistent map with respect to $\Xi$ which preserves triviality. We wish to define

$$\eta' = (\tilde{x}_{y'}, \tilde{\sigma}_{y'}, (u'_{0',v'}), (U'_{0',v'}), (\rho_{0',e}), (\rho_{0',i}))$$

an inconsistent map with respect to $\Xi'$. It is clear from the definition of $\Xi$ that there are $\tilde{x}_{y'}, \tilde{\sigma}_{y'}$ such that:

$$\Sigma_u(\tilde{x}_y, \tilde{\sigma}_y) \cong \Sigma_{u'}(\tilde{x}_{y'}, \tilde{\sigma}_{y'}).$$

We take $\rho_{0,e} = \rho_{0',e}$, $\rho_{0,i} = \rho_{0',i}$. Moreover, $U'_{0,v} = U'_{0',v'}$ and $u'_{0,v} = u'_{0',v'}$ if the color of $v$ is $s$.

We consider a disk component $\Sigma_{y',v'}$. There exists a unique irreducible component (in the sense of Definition 11.1, where we use $\tilde{\sigma}_{y'}$) which contains this component. We denote by $\Sigma_{u,v}(\tilde{x}_{y'}, \tilde{\sigma}_{y'})$ the union of the disk components contained in this irreducible component. We take the irreducible components of $\eta$ which correspond to it and define $\Sigma_{u,v}\Sigma(\tilde{x}_y, \tilde{\sigma}_y)$ in the same way. By (11.3) we have an isomorphism

$$\Sigma_{u,v}(\tilde{x}_y, \tilde{\sigma}_y) \cong \Sigma_{u',v'}(\tilde{x}_{y'}, \tilde{\sigma}_{y'}).$$

The maps $u'_{0,v}$ for various $v$ in this irreducible component induces a map

$$\Sigma_{u,v}(\tilde{x}_y, \tilde{\sigma}_y) \to (X, L).$$

This map is smooth. (Since $\Sigma_{u,v}(\tilde{x}_y, \tilde{\sigma}_y)$ is obtained by gluing along the components associated to the level 0 edges, the maps $u'_{0,v}$ are consistent on overlaps.) We use (11.4) and (11.5) to define $u'_{y',v'}$. Using the fact that $\eta$ is an consistent map preserving triviality, it is easy to see that $u'_{y',v'}$ for various $v$ are consistent at the nodal points corresponding to the level 0 edges $e$ with $\sigma_{y,e} = 0$, and $\eta' = (\tilde{x}_{y'}, \tilde{\sigma}_{y'}, (u'_{0',v'}), (U'_{0',v'}), (\rho_{0',e}), (\rho_{0',i}))$ is an inconsistent map with respect to $\Xi'$.

\[\square\]

**Remark 11.4.** The notion of preserving triviality plays an important role in the proof. The other important point is that we do not put any obstruction bundle on the components where the maps are constant.

---

\[39\text{See (8.9) for the meaning of the symbol +.}\]
Let \( u, \eta, u' \) and \( \eta' \) be as in Lemma 11.2 and \( \bar{R}, \bar{R}' \) be the very detailed tree describing the combinatorial types of \( u, u' \), respectively. We define:

\[
L^2_{m,\delta,\text{nontri}}(\eta, u) = \bigoplus_{v \in C^\text{nontri}_0(R); c(v) = s} L^2_{m,\delta}(\Sigma^+, \eta, v; (u_{0,v}')^*TX \otimes \Lambda^{0,1}) \bigoplus_{v \in C^\text{nontri}_0(R); c(v) = D} L^2_{m,\delta}(\Sigma^+, \eta, v; (\pi \circ U'_{0,v})^*TD \otimes \Lambda^{0,1}) \bigoplus_{v \in C^\text{nontri}_0(R); c(v) = d, u'_{0,v} \text{ is not constant}} L^2_{m,\delta}(\Sigma^+, \eta, v; (u'_{0,v})^*TX \otimes \Lambda^{0,1}).
\]

\[
L^2_{m,\delta,\text{nontri}}(\eta', u') = \bigoplus_{v \in C^\text{nontri}_0(R'); c(v) = s} L^2_{m,\delta}(\Sigma^-, \eta', v; (u'_{0,v}')^*TX \otimes \Lambda^{0,1}) \bigoplus_{v \in C^\text{nontri}_0(R'); c(v) = D} L^2_{m,\delta}(\Sigma^-, \eta', v; (\pi \circ U'_{0,v})^*TD \otimes \Lambda^{0,1}) \bigoplus_{v \in C^\text{nontri}_0(R'); c(v) = d, u'_{0,v} \text{ is not constant}} L^2_{m,\delta}(\Sigma^-, \eta', v; (u'_{0,v}')^*TX \otimes \Lambda^{0,1}).
\]

There are canonical identification between the components appearing in the above two formulas. Therefore, there exists a canonical map:

\[
I_{\eta\eta'} : L^2_{m,\delta,\text{nontri}}(\eta', u') \to L^2_{m,\delta,\text{nontri}}(\eta, u).
\]

**Definition 11.5.** Let \( \{E_u, \Xi(\eta)\} \) and \( \{E_u', \Xi(\eta')\} \) be obstruction bundle data for \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) and \( \mathcal{M}^{\text{RGW}}_1(L; \beta) \), respectively. We say that they are compatible with the forgetful map if

\[
I_{\eta\eta'}(E_{u'}, \Xi(\eta')) = E_u, \Xi(\eta)
\]

when \( u' \), \( \Xi' \), \( \eta' \) are related to \( u \), \( \Xi \), \( \eta \) as in Lemma 11.2.

**Definition 11.6.** A system of obstruction bundle data for moduli spaces \( \{\mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)\}_{\omega \cap \beta \leq E} \) is said to be compatible with the forgetful map if Definition 11.5 holds for each of spaces \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) and \( \mathcal{M}^{\text{RGW}}_1(L; \beta) \) with \( \omega \cap \beta \leq E \).

Suppose \( \{E_u, \Xi(\eta)\} \) is a system of obstruction bundle data for moduli spaces \( \{\mathcal{M}^{\text{RGW}}_{k+1}(L; \beta)\}_{\omega \cap \beta \leq E} \), which is disk-component-wise and is compatible with forgetful map. Let \( u \) be an element of \( \mathcal{M}^{\text{RGW}}_{k+1}(L; \beta) \) and \( u' = \text{fgg}(u) \). Suppose \( \Xi, \Xi' \) are TSDs at \( u, u' \) which are related to each other as in Lemma 11.3. Using Lemmas 11.2 and 11.3 and consistency of obstruction bundle data with the forgetful map, we can define a map:

\[
F_u : \bar{U}(u; \Xi) \to \bar{U}(u'; \Xi').
\]

In the process of forgetting boundary marked points and passing from \( u \) to \( u' \), we only might collapse disc components. Since the elements of \( \Gamma_u \) and
Γ_u' act as identity on disc components, the isotropy groups Γ_u and Γ_u' are isomorphic. The map F_u is also Γ_u equivariant.

It is straightforward to lift the map F_u to a Γ_u-equivariant map:

\[ \tilde{F}_u : \mathcal{E}_u \to \mathcal{E}_u' \]

such that:

\[ \tilde{F}_u \circ s_u = s_u' \circ F_u \]

and for any \( y \in s_u^{-1}(0)/\Gamma_u \) we have:

\[ \psi_{u'} \circ F_u(y) = \tilde{f} \tilde{g} \circ \psi_u(y) \]

The maps \( F_u \) and \( \tilde{F}_u \) are also compatible with coordinate changes. We can summarize this discussion as follows:

**Theorem 11.7.** Suppose a system of obstruction bundle data \( \{ E_u, \Xi(u) \} \) for moduli spaces \( \{ M_{k+1}^{RGW}(L; \beta) \}_{\omega \cap \beta \leq E} \) is disk-component-wise and is compatible with forgetful map. Then the resulting system of Kuranishi structures is compatible at the boundary components and corners (in the sense of Theorem 11.1) and compatible with the forgetful map (in the sense of [DF2, Lemma 3.75]).

**Proof.** Compatibility of the forgetful map with Kuranishi structures of moduli spaces \( M_{k+1}^{RGW}(L; \beta) \) is equivalent to the existence of the maps \( F_u \) and \( F_u' \) with the above properties. (See [DF2, Lemma 3.75] for more details.) We just need to point our that in [DF2, Lemma 3.75] we consider the map

\[ f_{g}^{\partial} : M_{k+1}^{RGW}(L; \beta) \to M_{k}^{RGW}(L; \beta), \]

given by forgetting the \( j \)-th marked point. The proof of a similar result for the map \( f_{g}^{\partial} \) is essentially the same. Let \( u_1 \in M_{k+1}^{RGW}(L; \beta) \), \( u_2 = f_{g}^{\partial}(u_1) \) and . Starting with a TSD \( \Xi_2 \) at \( u_2 \), we can follow the proof of Lemma 11.3 to define a TSD \( \Xi_1 \) at \( u_1 \), and form a map from \( \tilde{U}(u_1, \Xi_1) \) to \( \tilde{U}(u_2, \Xi_2) \). The remaining properties can be verified in a similar way. \( \square \)

**Remark 11.8.** In general, one needs to be careful about the differentiability of \( F_u \). The strata-wise smoothness is easy to show by elliptic regularity. The issue of differentiability where strata changes is discussed in [FOOO2, page 778]. This issue is relevant to the application of [DF2, Lemma 3.75], when we want to pull-back a multi-valued perturbation by the forgetful map.

There are two ways to resolve this issue. First we can consider multi-sections which have exponential decay in the gluing parameter \( T \). (We use \( T, \theta \) where \( \sigma = \exp(-(T + \theta \sqrt{-1})) \).) Even though the forgetful map \( F_u \) may not be smooth, the pull back of a multi-section with exponential decay is a multi-section which is not only smooth but also has an exponential decay. (See also [FOOO2, page 778])

In our situation discussed in the next subsection, we can use a simpler method to resolve this issue. For the purpose of this paper, we need to pull back a never vanishing multi-section. Thus pulling back the multi-section in \( C^0 \) sense is enough. (See the proof of Proposition 12.5.) This is because
we need differentiability of the multi-section only in a neighborhood of its zero set.

To complete our construction of a system of Kuranishi structures which is compatible with the forgetful map, it remains to prove the following result:

**Lemma 11.9.** There exists a system of obstruction bundle data which is disk component wise and is compatible with forgetful map.

**Proof.** The proof is essentially the same as the proof of Proposition 10.13. As before, we construct the system of obstruction bundle data by induction on $\beta \cap [\omega]$. In each step of the induction, we firstly construct an obstruction bundle data on $M_{1 \text{RGW}}^L(\beta)$. This system automatically induces an obstruction bundle data on $M_{k+1}^L(\beta)$ by requiring Condition (11.7).

To be more detailed, we fix a finite subset $\mathfrak{P}(\beta)$ of $M_{1 \text{RGW}}^L(\beta)$ as in (OBI) and a vector space $E_p$ for $p \in \mathfrak{P}(\beta)$ as in (OBII). We also fix spaces $\mathcal{F}_{\beta}$, $\mathcal{F}_{\beta}^0$ which fixes a set of quasi components for each $u \in M_{k+1}^L(\beta)$. We require these objects satisfy Conditions 10.25, 10.27, 10.29, 10.32, 10.33. If $u \in M_{k+1}^L(\beta)$, then we define $\mathcal{F}_{\beta}(u)$, $\mathcal{F}_{\beta}^0(u)$ to be $\mathcal{F}_{\beta}(u')$, $\mathcal{F}_{\beta}^0(u')$ where $u' = \text{fgg}(u)$. Since the obstruction bundle data for $M_{1 \text{RGW}}^L(\beta)$ satisfies Conditions 10.25, 10.27, 10.29, 10.32, 10.33, the induced obstruction bundle for $M_{k+1}^L(\beta)$ satisfies the corresponding conditions. \qed

12. Construction of a System of Multi-sections

The purpose of this section is to construct a system of multivalued perturbations used in [DF2, Section 4] to prove the main theorems there. As in [DF2, Section 4], the proof in the case that minimal Maslov numbers are greater than 2 is simpler than the case that the minimal Maslov numbers could be 2. In fact, only in the case of minimal Maslov number 2, we need to use the results of the last section and perturb the moduli spaces in a way that are compatible with the forgetful map.

In this section, we focus on the case of strips rather than discs because the description of the boundary in the case of discs is different and the moduli spaces of holomorphic strips are used to prove the main results of [DF2]. Here we do not prove the existence of a system of transversal multi-sections in complete generality and restrict ourselves to the cases which suffices for the proof of our main theorems. In fact, we perturb $M_{1 \text{RGW}}^L(L; \beta)$ only in the case that the Maslov index of $\beta$ is 2. In other words, we do not prove [DF2, Proposition 4.7] in the generality that it is stated.\(^{40}\)

Let $L_0, L_1 \subset X \setminus D$ be a pair of compact Lagrangian submanifolds. We assume that they are monotone and their intersection is transversal. For $p, q \in L_0 \cap L_1$, we defined $\Pi_2(X; L_1, L_0; p, q)$, the set of homology classes of strips

\(^{40}\)The construction of such system of multisectons in the general case can be carried out in the same way as in [FOOO7, Section 20]. In this paper we only perturb moduli space of virtual dimension 1 or less. Because of the monotonicity assumption in [DF2], it suffices to study only such moduli spaces to prove the main results of [DF2].
asymptotic to \( p \) and \( q \), in [DF2, Definition 2.1]. Let \( \mathcal{M}^{\text{RGW}}_{k_1,k_0}(L_1,L_0;p,q;\beta) \) be the compactified moduli space of pseudo-holomorphic strips of the homology class \( \beta \in \Pi_2(X;L_1,L_0;p,q) \) with \( k_1 \) and \( k_0 \) boundary marked points. See [DF2, Sections 3 and 5] for the definition of this compactification.

12.1. Lagrangians with Minimal Maslov Number greater than 2.
In this subsection, we prove the existence of a system of multivalued perturbations that is used in [DF2, Section 4] in the case that the minimal Maslov numbers are greater than 2. More precisely we prove the following proposition:

**Proposition 12.1.** Let \( L_0,L_1 \subset X \setminus D \) be a pair of compact Lagrangian submanifolds. We assume that they are monotone and their minimal Maslov numbers are strictly greater than 2. We assume that \( L_0 \) is transversal to \( L_1 \). Let \( E \) be a positive number. Then there exists a system of multivalued perturbations \( \{s_n\} \) on the moduli spaces \( \mathcal{M}^{\text{RGW}}_{k_1,k_0}(L_1,L_0;p,q;\beta) \) of virtual dimension at most 1 and \( \omega(\beta) \leq E \) such that:

1. The multi-section \( s_n \) is \( C^0 \) and is \( C^1 \) in a neighborhood of \( s_n^{-1}(0) \). The multi-sections \( \{s_n\} \) are transversal to 0. The sequence of multi-sections \( \{s_n\} \) converges to the Kuranishi map in \( C^0 \). Moreover, this convergence is in \( C^1 \) in a neighborhood of the zero locus of the Kuranishi map.

2. The multivalued perturbations \( \{s_n\} \) are compatible with the description of the boundary given by [DF2, Lemmata 3.67, 3.70].

3. Suppose that the (virtual) dimension of \( \mathcal{M}^{\text{RGW}}_{k_1,k_0}(L_1,L_0;p,q;\beta) \) is not greater than 1. Then the multisection \( s_n \) does not vanish on the codimension 2 stratum \( \mathcal{M}^{\text{RGW}}_{k_1,k_0}(L_1,L_0;p,q;\beta)^{(1)} \) described by [DF2, Proposition 3.63].

This proposition is a weaker version of [DF2, Proposition 4.7]. Note that we do not claim compatibility with the forgetful map ([DF2, Proposition 4.7 (3)]) and multivalued perturbations are given only on the moduli spaces of virtual dimension at most 1. We do not need to perturb moduli spaces of pseudo holomorphic disks \( \mathcal{M}^{\text{RGW}}_{k_1}(L;\beta) \) in Proposition 12.1.

**Proof.** The proof is by induction on \( \omega \cap \beta \) and \((k_0,k_1)\). In this inductive process we construct multi-valued perturbations for all moduli spaces \( \omega \cap \beta \leq E \) and \( k_0 + k_1 \leq N \) for some constants \( E \) and \( N \). In particular, we may construct perturbations for the moduli spaces with dimension greater than 1. But the conditions (1) and (3) hold only for moduli spaces with dimension at most 1. We assume that we already constructed multivalued perturbations for the moduli spaces of type \((\beta;k_0,k_1;p,q)\) such that \( \omega \cap \beta < \omega \cap \alpha \) or \( \omega \cap \beta = \omega \cap \alpha \) and \( k_1 + k_1 < j_1 + j_2 \).

We can use the induction hypothesis to define a continuous multi-section on \( \partial \mathcal{M}^{\text{RGW}}_{j_1,j_0}(L_0;p,q;\alpha) \). For example, part of the boundary of the moduli space \( \mathcal{M}^{\text{RGW}}_{j_1,j_0}(L_1,L_0;p,q;\alpha) \) is described by the moduli spaces of the
following form:

\[(12.1) \quad \mathcal{M}^\text{RGW}_{j',j_0}(L_1, L_0; p, q; \beta_1) \times_{L_1} \mathcal{M}^\text{RGW}_{j''+1}(L_1; \beta_2)\]

where \(j' + j'' = j_1 + 1\), \(\beta_1 \cap D = 0\) and \(\beta_1 \neq \beta_2 = \alpha\). Assuming \(\mathcal{M}^\text{RGW}_{j''+1}(L_1; \beta_2)\) is non-empty, we have \(\omega \cap \beta_1 < \omega \cap \alpha\) or \(j'_1 < j_1\) and \(\omega \cap \beta_1 = \omega \cap \alpha\). Therefore, the induction hypothesis implies that we already fix a multi-valued perturbation for \(\mathcal{M}^\text{RGW}_{j'_1,j_0}(L_1, L_0; p, q; \beta_1)\). We use the fiber product of this perturbation and the trivial perturbation for \(\mathcal{M}^\text{RGW}_{j''+1}(L_1; \beta_2)\) to define a multi-valued perturbation for:

\[\mathcal{M}^\text{RGW}_{j',j_0}(L_1, L_0; p, q; \beta_1) \times_{L_1} \mathcal{M}^\text{RGW}_{j''+1}(L_1; \beta_2)\]

Now we use the analogue of the map \(\Pi\) in Theorem 10.1 to pull-back this perturbation to the space in (12.1). More generally, we can use the already constructed perturbations for the moduli spaces of strips and trivial perturbations for the moduli spaces of discs to define a multi-valued perturbation for any boundary component and corner of \(\mathcal{M}^\text{RGW}_{j_1,j_0}(L_1, L_0; p, q; \alpha)\). The induction hypothesis implies that these perturbations are compatible.

In the case that the virtual dimension of \(\mathcal{M}^\text{RGW}_{j_1,j_0}(L_1, L_0; p, q; \alpha)\) is greater than 1, we extend the chosen multi-valued perturbation on the boundary into a \(C^0\) perturbation defined over the whole moduli space. In the case that the virtual dimension is at most 1, we need to choose this extension such that the conditions in (1) and (3) of Proposition 12.1 are satisfied. To achieve this goal, we analyze the vanishing locus of the multi-valued perturbation over the boundary of \(\mathcal{M}^\text{RGW}_{j_1,j_0}(L_1, L_0; p, q; \alpha)\).

Let \(\mathcal{M}^\text{RGW}_{j_1,j_0}(L_1, L_0; p, q; \alpha)\) have virtual dimension not greater than 1. On the stratum of the boundary where there exists at least one disk bubble on which the map is non-constant, the assumption implies that the Maslov number of the disk bubble is at least 4. This implies that there is at least one irreducible component, which is a strip with homology class \(\beta_1\) and is contained in a moduli space with negative virtual dimension. (See also the proof of [DF2, Lemma 4.12].) Therefore, our multivalued perturbation on this boundary component does not vanish.

The rest of the proof is divided into two parts. We firstly consider the case where \(\dim(\mathcal{M}^\text{RGW}_{j_1,j_0}(L_1, L_0; p, q; \alpha))\) is non-positive. The part of the boundary corresponding to splitting into two or more strips has a strip component which has negative virtual dimension. Therefore, our multi-valued perturbation does not vanish on this part of the boundary, too. As a consequence, we can extend the perturbation in the \(C^0\) sense to a neighborhood of the boundary such that it is still non-vanishing in this neighborhood. We approximate the perturbation by a section which is \(C^1\) outside a smaller neighborhood of the boundary. Now we can extend this multi-section in a way which is

\[41\text{See [FOOO7, Lemma-Definition 20.16] for the definition of fiber product of multi-valued perturbations.}\]
transversal to 0 on $\mathcal{M}_{k_1,k_0}^{\text{RGW}}(L_1, L_0; p, q; \beta)$ and $\mathcal{M}_{k_1,k_0}^{\text{RGW}}(L_1, L_0; p, q; \beta)^{(1)}$ using the existence theorem of multivalued perturbations. (See, for example, [FOOO6, Proposition 13.29].) If the virtual dimension is 0, there exists finitely many zero’s which do not belong to $\mathcal{M}_{k_1,k_0}^{\text{RGW}}(L_1, L_0; p, q; \beta)^{(1)}$. If the virtual dimension is negative, the multivalued perturbation does not vanish. This completes the proof in the case that $\dim(\mathcal{M}_{j_1,j_0}^{\text{RGW}}(L_1, L_0; p, q; \alpha)) \leq 0$.

Next, we consider the case that $\mathcal{M}_{j_1,j_0}^{\text{RGW}}(L_1, L_0; p, q; \alpha)$ is 1-dimensional. The constructed multivalued perturbation on the boundary does not vanish except the boundary components of the forms

\begin{equation}
\mathcal{M}_{j_1,j_0}^{\text{RGW}}(L_1, L_0; r; \beta_1) \times \mathcal{M}_{j_1',j_0'}^{\text{RGW}}(L_1, L_0; r, q; \beta_2)
\end{equation}

or

\begin{equation}
\mathcal{M}_{j_1-1,j_0}^{\text{RGW}}(L_1, L_0; p, q; \alpha) \times \mathcal{M}_3^{\text{RGW}}(L_1; 0),
\end{equation}

\begin{equation}
\mathcal{M}_{j_1,j_0-1}^{\text{RGW}}(L_1, L_0; p, q; \alpha) \times \mathcal{M}_3^{\text{RGW}}(L_0; 0).
\end{equation}

In (12.2), both of the factors have virtual dimension 0. Therefore, we may assume that the zero sets of the multivalued perturbation we have produced on those factors do not lie in the strata of codimension at least 2. Since the multi-sections on the two factors have finitely many zeros and the map $\Pi$ in [DF2, Lemma 3.67] is an isomorphism, the first factor gives rise to finitely many points in the boundary. In the case of (12.3), the first factor has virtual dimension 0 and the multi-section there vanishes only at finitely many points. The second factor is identified with $L_1$ or $L_0$. Therefore, the fiber product is identified with the first factor. In summary, the multi-valued perturbation has finitely many zeros on the boundary and item (3) holds.

We fix Kuranishi charts at the finitely many zeros on the boundary. Since these are boundary points, we can easily extend our multivalued perturbation to the interior of the chosen Kuranishi charts so that it is transversal to 0. Now we extend the multi-valued perturbation further to a neighborhood of the boundary of $\mathcal{M}_{j_1,j_0}^{\text{RGW}}(L_1, L_0; p, q; \alpha)$ in the $C^0$ sense such that the multivalued perturbation does not vanish except at those finitely many charts. We may assume that the multi-valued perturbation is $C^1$ outside a smaller neighborhood of the boundary. We use again the existence theorem of multi-sections that are transversal to zero everywhere to complete the construction of the multi-valued perturbation on $\mathcal{M}_{j_1,j_0}^{\text{RGW}}(L_1, L_0; p, q; \alpha)$.

**Remark 12.2.** This proof never uses the smoothness of the coordinate change with respect to the gluing parameters $\sigma_e$ when $\sigma_e = 0$. In most part of the proof, we extend the multi-section at the boundary to the interior only in the $C^0$ sense. When we extend the multi-section near a point of the boundary where it vanishes, we fix a chart there and extend it on that chart. We use other charts to extend the multi-section in $C^0$ sense to a neighborhood of the boundary. (Recall that we only need the differentiability of the multivalued perturbation in a neighborhood of its vanishing set to

\[42\] See also Theorem 10.1.
define virtual fundamental chain.) The key point here is that the multivalued perturbation on the boundary has only isolated zeros.

**Remark 12.3.** Even though we do not perturb the moduli spaces of disks in the proof of Proposition 12.1, we used the fact that these moduli spaces admit Kuranishi structures.

**Remark 12.4.** Proposition 12.1 is one of the main inputs in the definition of Lagrangian Floer homology\(^{43}\) for the pair \(L_0, L_1\). Similar results also play a key role in showing that Floer homology is invariant of Hamiltonian perturbations compactly supported on \(X \setminus \mathcal{D}\). See [DF2, Section 4] for more details.

12.2. **Lagrangians with Minimal Maslov Number 2.** Now we turn our attention to the case that the Maslov numbers of our Lagrangians in \(X \setminus \mathcal{D}\) could be 2. Let \(L\) be a compact and monotone Lagrangian submanifold in \(X \setminus \mathcal{D}\) such that its minimal Maslov number is 2. Let the Maslov index of \(\alpha \in H_2(X; L)\) be 2 and \(\alpha \cap \mathcal{D} = 0\), and form the moduli space \(M_{RGW}^1(L; \alpha)\). This moduli space has a Kuranishi structure without boundary, which has virtual dimension \(n = \dim L\). For \(p \in L\), we form the fiber product:

\[
M_{RGW}^1(L; \alpha; p) = M_{RGW}^1(L; \alpha) \times_L \{p\}.
\]

of virtual dimension 0. We fix a multi-valued perturbation on \(M_{RGW}^1(L; \alpha)\) such that the induced multivalued perturbation on \(M_{RGW}^1(L; \alpha; p)\) is transversal to 0. More generally, we can assume that this transversality assumption holds when \(p\) is an element of a finite set. Thus the multi-section on \(M_{RGW}^1(L; \alpha; p)\) has only finitely many zeros. The size of this zero set counted with multiplicity and weight is defined to be \(\Phi L\).

**Proposition 12.5.** Let \(L_0, L_1\) be a pair of transversal compact monotone Lagrangians in \(X \setminus \mathcal{D}\) such that their minimal Maslov numbers are 2. For a positive number \(E\), there exists a system of multivalued perturbations \(\{s_n\}\) on the moduli spaces \(M_{RGW}^{k_1, k_0}(L_1, L_0; p, q; \beta)\) of virtual dimension \(\leq 1\) and \(\omega \cap \beta \leq E\) such that:

1. The multi-section \(s_n\) is \(C^0\) and is \(C^1\) in a neighborhood of \(s_n^{-1}(0)\). The multi-sections \(\{s_n\}\) are transversal to 0. The sequence of multi-sections \(\{s_n\}\) converges to the Kuranishi map in \(C^0\). Moreover, this convergence is in \(C^1\) in a neighborhood of the zero locus of the Kuranishi map.
2. The multivalued perturbations \(\{s_n\}\) are compatible with the description of the boundary given by [DF2, Lemmata 3.67, 3.70].
3. The multivalued perturbations \(\{s_n\}\) are compatible with the forgetful map of the marked points given by [DF2, Lemma 3.75].
4. Suppose that the (virtual) dimension of \(M_{RGW}^{k_1, k_0}(L_1, L_0; p, q; \beta)\) is not greater than 1. Then the multi-section \(s_n\) does not vanish on the

\[^{43}\text{We need (relative) spin structure to define orientations.}\]
codimension 2 stratum $\mathcal{M}^\text{RGW}_{k_1,k_0}(L_1,L_0;p,q;\beta)^{(1)}$ described by [DF2, Proposition 3.63].

Proposition 12.5 is a slightly simpler version of [DF2, Proposition 4.7] where we claimed similar results for the more general case of moduli spaces $\mathcal{M}^\text{RGW}_{k_1,k_0}(L_1,L_0;p,q;\beta)$ with dimensions possibly greater than 1. As it is pointed out there, Proposition 12.5 suffices for our purposes in [DF2, Section 4] (including the proof of [DF2, Lemma 4.17]) and we content ourselves to the proof of this simpler result.

**Proof.** For $j = 1, 2$ and $\alpha \in \Pi_2(X;L_j)$ with Maslov index 2, we fix a multivalued perturbation on $\mathcal{M}^\text{RGW}_1(L_j;\alpha)$ such that it induces a transversal multi-valued perturbation on $\mathcal{M}^\text{RGW}_1(L_j;\alpha;p)$ for any $p \in L_0 \cap L_1$. We extend these multi-valued perturbations to all moduli spaces $\mathcal{M}_{k+1}(L_j;\alpha)$ in the $C^0$ sense such that they are compatible over the boundary in a similar sense as in Proposition 12.1. We use these multi-valued perturbations and induction to define the required multi-valued perturbations on the moduli space $\mathcal{M}^\text{RGW}_{k_1,k_0}(L_1,L_0;p,q;\beta)$. To be more detailed, we construct multi-valued perturbations on $\mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\beta)$ by induction on $\omega \cap \beta$. The perturbation on the general moduli space the multi-valued perturbation on $\mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\beta)$ is given by pulling back from $\mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\beta)$. Here we use consistency of Kuranishi structures with the forgetful map.

Suppose we have constructed required multi-valued perturbations for $\beta$ with $\beta \cap \omega < \alpha \cap \omega$. We use the induction hypothesis and the constructed multi-valued perturbations for the moduli spaces of discs to define a perturbation on $\partial \mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\beta)$ in the same way as in Proposition 12.1. We wish to analyze zeros of our induced multi-section on the boundary of $\mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\beta)$. In compare to Proposition 12.1, the new types of zeros are given by disc bubbles with Maslov index 2. Such boundary components have the form:

\[(12.4) \quad \mathcal{M}^\text{RGW}_{1,0}(L_1,L_0;p,q;\beta_0) \times_{L_1} \mathcal{M}^\text{RGW}_1(L_1;\beta_1)\]

or

\[(12.5) \quad \mathcal{M}^\text{RGW}_{0,1}(L_1,L_0;p,q;\beta_0) \times_{L_0} \mathcal{M}^\text{RGW}_1(L_0;\beta_2).\]

where $\beta_0 + \beta_1 = \alpha$ and the Maslov index of $\beta_1$ is 2. We focus on the boundary components of the form in (12.4). The other case is similar. There are two cases to consider:

**(Case 1) ($\beta_0 \neq 0$):** The virtual dimension of $\mathcal{M}^\text{RGW}_{1,0}(L_1,L_0;p,q;\beta_0)$ is:

$$\dim(\mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\alpha)) - 1$$

If the virtual dimension of $\mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\alpha)$ is not greater than 0, then the multi-section does not vanish on this component. To treat the case that the virtual dimension of $\mathcal{M}^\text{RGW}_{0,0}(L_1,L_0;p,q;\alpha)$ is 1, note that
the multi-section of $\mathcal{M}_{1,0}^{\text{RGW}}(L_1, L_0; p, q; \beta_0)$ is the pull-back of the multi-valued perturbation on $\mathcal{M}_{0,0}^{\text{RGW}}(L_1, L_0; p, q; \beta_0)$. This latter moduli space has virtual dimension $-1$ and hence the multi-section does not vanish on it. Therefore, the multi-section does not have any zero on the moduli spaces $\mathcal{M}_{1,0}^{\text{RGW}}(L_1, L_0; p, q; \beta_0)$ and (12.4).

(Case 2) ($\beta_0 = 0$): In this case, $p = q$ and $\alpha = 0#\beta_1$ where $\beta_1$ is a homology class in $\Pi_2(X; L_1)$ with Maslov index 2. Therefore, the corresponding boundary component is identified with $\mathcal{M}_{1}^{\text{RGW}}(L_1; \alpha; p)$ where $p \in L_0 \cap L_1$. We defined a multi-valued perturbation on this moduli space such that its zero set is cut down transversely and consists of isolated points. Now we can proceed as in Proposition 12.1 to complete the construction of multi-valued perturbations.

□

References


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