MONOTONE LAGRANGIAN FLOER THEORY IN SMOOTH DIVISOR COMPLEMENTS: II

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Abstract. In [DF18a], the first part of the present series of papers, we studied the moduli spaces of holomorphic discs and strips into an open symplectic manifold, isomorphic to the complement of a smooth divisor in a closed symplectic manifold. In particular, we introduced a compactification of this moduli space, which is called the *RGW compactification*. The goal of this paper is to show that the RGW compactifications admit Kuranishi structures. This result provides the crucial ingredient for the main construction of [DF18a, DF18b]: Floer homology for monotone Lagrangians in a smooth divisor complement.

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1. Introduction

In this series of papers, the authors study Lagrangian intersection Floer homology of a pair of monotone Lagrangians in an open symplectic manifold, which is isomorphic to a divisor complement. At the heart of the construction, there is a compactification of the moduli spaces of holomorphic discs and strips satisfying Lagrangian boundary condition. The main purpose of this sequel to [DF18a] is to show that this compactification, called the *RGW compactification*, admits a *Kuranishi structure*. The virtual count of the elements of these Kuranishi spaces is used in [DF18b] to define the desired Lagrangian Floer homology.

To be more detailed, let (X, ω) be a symplectic manifold and $\mathcal{D}$ be a symplectic submanifold of X with codimension 2. Using a compatible almost complex structure of $\mathcal{D}$ and a unitary connection of the normal bundle, we defined in [DF18a, Subsections 3.1 and 3.2] a compatible almost complex structure in a neighborhood of $\mathcal{D}$ in X, which is invariant under a *partial* $\mathbb{C}_*$*-action*. Then we extend this into an ω-compatible almost complex structure J to X.

Let L_0 and L_1 be compact orientable and transversal Lagrangians in $X \backslash \mathcal{D}$. For any $\beta \in \Pi_2(X; L_i)$ with $\beta \cap \mathcal{D} = 0$, let $\mathcal{M}^{\rm reg}_{k+1}(L_i; \beta)$ be the moduli space of J-holomorphic disks of homology class β with $k+1$ boundary marked points and Lagrangian boundary condition associated to L_i. In [DF18a], we introduced the RGW compactification $\mathcal{M}^{\rm RGW}_{k+1}(L_i; \beta)$ of $\mathcal{M}^{\rm reg}_{k+1}(L_i; \beta)$. (See [DF18a, Section 3] for the definition of this moduli space as a set and [DF18a, Section 4] for the definition of topology on this moduli space. Note that this compactification is different from the stable map compactification.) We also defined the RGW compactification $\mathcal{M}^{\rm RGW}_{k_1,k_0}(L_1, L_0; p, q; \beta)$ of the moduli space $\mathcal{M}^{\rm reg}_{k_1,k_0}(L_1, L_0; p, q; \beta)$ of J-holomorphic strips of homology class β with boundary marked points and Lagrangian boundary condition associated to L_0, L_1 in a similar way as in [DF18a, Section 3].

Theorem 1. *The moduli spaces* $\mathcal{M}^{\rm RGW}_{k+1}(L_i; \beta)$ *and* $\mathcal{M}^{\rm RGW}_{k_1,k_0}(L_1, L_0; p, q; \beta)$ *admit Kuranishi structures.*

A topological space with a Kuranishi structure is locally modeled by the vanishing locus of an equation defined on a manifold, or more generally an orbifold. These orbifolds and equations for different points need to satisfy some compatibility conditions. (See [FOOO09, Definition A.1.1] for a more precise definition of Kuranishi structures.) Given a point of a space with Kuranishi structure, the zeros of the corresponding equation might be cut down transversally. In that case, our space looks like an orbifold in a neighborhood of such point. The main point is that such equations might not be transversal to zero and we might end up with a space which is not as regular as an orbifold. Nevertheless, a Kuranishi structure would be sufficient to have some of the interesting properties of smooth orbifolds. For example, it makes sense to talk about a space with Kuranishi structure which has boundary and corners. In fact, the Kuranishi structures of Theorem 1 have boundary and corners, which can be described in terms of similar moduli spaces. We will construct a system of Kuranishi structures which are compatible at the boundary and the corners in [DF18b].

One of the novel features of the RGW compactification is that it has some strata which consist of obstructed objects by default. To make this point more clear, we make a comparison with the stable map compactification. In the stable map compactification of the moduli space of holomorphic discs, each stratum is described by a fiber product of the moduli spaces of holomorphic discs and strips. If each of the moduli spaces appearing in such a fiber product consists of Fredholm regular elements and the fiber product is transversal, then the moduli space in a neighborhood of this stratum consists of regular objects and hence it is a smooth orbifold in this neighborhood. However, the situation in the case of the RGW compactification looks significantly different. There are strata of the compactification which belong to the singular locus of the moduli space, even if each element of the associated fiber product is Fredholm regular and the fiber product is cut down transversely. This subtlety is the main point that our treatment diverges from the proof of the analogues of Theorem 1 for the stable map compactification of the moduli spaces of holomorphic discs and strips. (See [FOOO09, FOOO12, FOOO15, FOOO17] for such results in the context of the stable map compactification.)

We resolve the above issue by introducing the notion of *inconsistent solutions* to the Cauchy-Riemann equation. Under the assumption of the previous paragraph, the space of inconsistent solutions forms a smooth orbifold. Moreover, the elements of the moduli spaces $\mathcal{M}^{\rm RGW}_{k+1}(L_i;\beta)$ and $\mathcal{M}^{\rm RGW}_{k_1,k_0}(L_1,L_0;p,q;\beta)$ can be regarded as the zero sets of appropriate equations on the moduli space of inconsistent solutions. We treat these equations as extra terms for Kuranishi maps. We believe this approach could be also useful for the analysis of the relative Gromov-Witten invariants in symplectic category. We also believe that this idea as well as some of the arguments provided in this paper can be generalized to study various conjectures proposed in [DF18b, Section 6].

The main steps of the construction of the Kuranishi structures required for the proof of Theorem 1 are parallel to the ones for the stable map compactification. Throughout the paper, we point out relevant references for the corresponding results in the context of the stable map compactification. At the same time, we try to make our exposition as self-contained as possible. One of the exceptions is the exponential decay result of [FOOO16a] where the same arguments can be used to deal with the exponential decay result which we need for this paper. (However, in our application to prove [DF18a, Theorem 1], we do not use the smoothness of our Kuranishi structure and so one can avoid using [FOOO16a].)

Remark 1.1. The fact that certain strata of the RGW compactification consists of obstructed objects by default (that is to say, such a strata are singular) was observed inndependetly by M. Tehrani [Teh22]. A similar gluing analysis is studied in [LS19] which considers only the case that the neck region is connected. In such a case, the main point of the concern of this paper does not appear. B. Parker [Par12] studies a related problem (in the case of pseudo-holomorphic map from curves without boundary) in a different way. For example, the 'Kuranishi structure' obtained is different from those in the sense of [FOOO20]. In fact the Kuranishi neighborhood obtained in [Par12] is an exploded manifold (or rather its orbifold analogue), which may not be a manifold or an orbifold. By a similar reason the method of this paper does not give a Kuranishi structure for log compactification in [Teh22].

Outline of Contents. In order to make the main ideas of the construction more clear, we devote the first part of the paper to the construction of a Kuranishi chart around a special point in the RGW compactification of moduli spaces of discs. This special point, described in Section 2, belongs to a stratum of the moduli space which is always obstructed. Motivated by this example, we introduce the notion of inconsistent solutions in Section 5. The stratum of this special example is given by the fiber product of a moduli space of discs and two moduli spaces of spheres. In Sections 3 and 4, we study the deformation theory of the elements of the moduli space within this stratum. A Kuranishi chart for each element of this stratum is constructed in Section 6. The main analytical results required for the construction of the Kuranishi chart is verified in Section 7.

In Section 8, we explain how the method of the first part of the paper can be used to construct a Kuranishi chart around any point of the RGW moduli space. Section 9 is devoted to showing that these Kuranishi charts are compatible with each other using appropriate coordinate changes. This completes the proof of Theorem 1.

2. A Special Point of the Moduli Spaces of Discs

In the first half of the paper, we focus on the analysis of a special case. We hope that this allows the main features of our construction stand out. The special case can be described as follows. Let Σ be a surface with nodal singularities, which has three irreducible components $\Sigma_{\rm d}$, $\Sigma_{\rm s}$ and $\Sigma_{\rm D}$. The irreducible component $\Sigma_{\rm d}$ is a disc and the remaining ones are spheres. The components $\Sigma_{\rm d}$, $\Sigma_{\rm s}$ and $\Sigma_{\rm D}$ are respectively called the *disc component*, the *sphere component* and the *divisor component*. The divisor component $\Sigma_{\rm D}$ intersects $\Sigma_{\rm d}$ and $\Sigma_{\rm s}$ at the points, $z_{\rm d}$ and $z_{\rm s}$, respectively. When we want to emphasize that we consider these points as elements of $\Sigma_{\rm D}$, we denote them by $z_{\rm D,d}$ and $z_{\rm D,s}$. There is no intersection between $\Sigma_{\rm d}$ and $\Sigma_{\rm s}$.

We are given a J-holomorphic map $u : (\Sigma, \partial\Sigma) \to (X, L)$, where J is given as in [DF18a, Subsection 3.2]. The restriction of this map to $\Sigma_{\rm d}$, $\Sigma_{\rm s}$, $\Sigma_{\rm D}$ are denoted by $u_{\rm d}$, $u_{\rm s}$, $u_{\rm D}$. We assume that the image of $u_{\rm D}$ is contained in the divisor $\mathcal{D}$. The images of $\Sigma_{\rm d}$, $\Sigma_{\rm s}$ intersect $\mathcal{D}$ only at the points $z_{\rm d}$ and $z_{\rm s}$ with multiplicities p_1 and p_2, respectively. Here p_1, p_2 are positive integers. (See [DF18a, Lemma 3.8].) Following [DF18a, Section 3], we also associate a *level function* λ that evaluates to 0 at the components $\Sigma_{\rm d}$ and $\Sigma_{\rm s}$ and to 1 at $\Sigma_{\rm D}$. We assume that there is one boundary marked point z_0 on $\Sigma_{\rm d}$.

We also assume that the homology class $(u_{\rm D})_*([\Sigma_{\rm D}])$ satisfies the following identity (compare with [DF18a, Condition (3.28)]):

$$p_1 + p_2 + c_1(\mathcal{N}_{\mathcal{D}}(X)) \cap (u_{\rm D})_*([\Sigma_{\rm D}]) = 0.$$

This condition implies that there exists a meromorphic section[1] $\mathfrak{s}$ of $(u_{\rm D})^*\mathcal{N}_D X$ such that $\mathfrak{s}$ has a pole of order p_1 (resp. p_2) at $z_{\rm d}$, (resp. $z_{\rm s}$), and $\mathfrak{s}$ has no other pole or zero. The choice of this section $\mathfrak{s}$ is unique up to a multiplicative constant in $\mathbb{C}_*$. We fix one such section $\mathfrak{s}$ and define:

$$U_{\rm D} : \Sigma_{\rm D} \backslash \{z_{\rm d}, z_{\rm s}\} \to \mathcal{N}_{\mathcal{D}}(X) \backslash \mathcal{D} = \mathbb{R} \times S\mathcal{N}_{\mathcal{D}}(X) \tag{2.1}$$

where $U_{\rm D}(z)$, for $z \in \Sigma_{\rm D} \backslash \{z_{\rm d}, z_{\rm s}\}$, is defined to be $(u_{\rm D}(z), \mathfrak{s}(z))$. This map is J-holomorphic by [DF18a, Lemma 3.7].

The nodal curve Σ and the *detailed ribbon tree* corresponding to u are sketched in Figures 1, 2. These data define an element of $\mathcal{M}_1^{\rm RGW}(L;\beta)$ for an appropriate choice of $\beta \in H_2(X, L; \mathbb{Z})$. See [DF18a, Section 3] for the definitions of detailed ribbon trees and moduli spaces $\mathcal{M}_{k+1}^{\rm RGW}(L;\beta)$. Constructing a Kuranishi neighborhood for this element of $\mathcal{M}_1^{\rm RGW}(L;\beta)$ is the main goal of the first half of the paper.

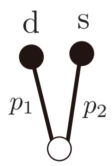

Figure 1. Detailed tree of the element we study.

[1]Using the $U(1)$ connection, the pulled back bundle $(u_{\rm D})_*([\Sigma_{\rm D}])$ has a canonical holomorphic structure. We use this to define meromorphicity.

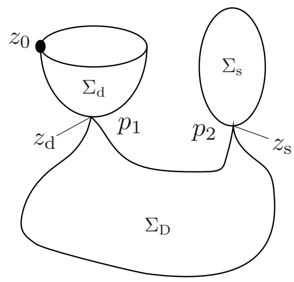

FIGURE 2. The source curves of $(u_{\rm d}, u_{\rm D}, u_{\rm s})$

3. Fredholm theory of the Irreducible Components

In this section, we shall be concerned with the deformations of the restrictions of the map u to the irreducible components $\Sigma_{\rm d}$, $\Sigma_{\rm s}$ and $\Sigma_{\rm D}$. We will see that the deformation theory of each irreducible component is governed by a Fredholm operator.

Throughout this section, we use cylindrical coordinates both for the target and the source. There is a neighborhood $\mathfrak{U}$ of the divisor $\mathcal{D}$ with a *partial* $\mathbb{C}_*$*-action* such that the almost complex structure J on $\mathfrak{U}$ is invariant under the partial $\mathbb{C}_*$-action. (See [DF18a, Subsections 3.1] for the definition of partial $\mathbb{C}_*$-actions and [DF18a, Subsections 3.1 and 3.2] for the existence of $\mathfrak{U}$.) We may assume that the open set $\mathfrak{U}$ is chosen such that its closure minus $\mathcal{D}$ is diffeomorphic to:

$$[0,\infty)_\tau \times S\mathcal{N}_{\mathcal{D}}(X) \tag{3.1}$$

where $S\mathcal{N}_{\mathcal{D}}(X)$ is the unit S^1-bundle associated to the normal bundle $\mathcal{N}_{\mathcal{D}}(X)$ of $\mathcal{D}$ in X. We use τ to denote the standard coordinate on $[0,\infty)$. The 1-form $\theta = -d\tau \circ J$ determines a connection 1-form for the S^1-bundle $S\mathcal{N}_{\mathcal{D}}(X)$. Let g' be a metric on $\mathcal{D}$ which is fixed for the rest of the paper. We also fix a metric g on $X\backslash\mathcal{D}$ such that its restriction to (3.1) is given by:

$$g|_{[0,\infty)_\tau \times S\mathcal{N}_{\mathcal{D}}(X)} = d\tau^2 + \theta^2 + g'. \tag{3.2}$$

In particular, g is invariant with respect to the partial $\mathbb{C}_* \cong (-\infty,\infty) \times S^1$-action on (3.1), where $(-\infty,\infty)$ acts (partially) by translation along the factor $[0,\infty)_\tau$ and the action of S^1 is induced by the obvious circle action on $S\mathcal{N}_{\mathcal{D}}(X)$. We also fix another metric g_{NC} on $X\backslash\mathcal{D}$ whose restriction to (3.1) has the following form:

$$g_{NC}|_{[0,\infty)_\tau \times S\mathcal{N}_{\mathcal{D}}(X)} = e^{-2\tau}(d\tau^2 + \theta^2) + g'. \tag{3.3}$$

This non-cylindrical metric extends to $\mathcal{D}$ to give a smooth metric on X which is also denoted by g_{NC}.

Remark 3.4. We do not make any assumption on compatibility of the metric g' with the almost complex structure or the symplectic structure.

The unitary connection on the normal bundle $\mathcal{N}_{\mathcal{D}}X$ determines the decomposition

$$TX|_{\mathfrak{U}} = \underbrace{\mathbb{R} \oplus \mathbb{R}}_{\mathbb{C}} \oplus \pi^*(T\mathcal{D}), \tag{3.5}$$

where the first factor is given by the action of $(-\infty,\infty) \times \{1\} \subset (-\infty,\infty) \times S^1$, the second factor is given by the action of $\{1\} \times S^1 \subset (-\infty,\infty) \times S^1$, and the third factor is given by the vectors orthogonal to the first two factors. Note that the last factor and the direct sum of the first two factors determine complex subspaces of $TX|_{\mathfrak{U}}$.

3.1. **The Disk Component.** The surface $\Sigma_{\rm d}$ can be identified uniquely with the standard unit disc $D^2 \subset \mathbb{C}$ such that z_0 and $z_{\rm d}$ are mapped to 1 and 0. The map $u_{\rm d} : D^2 \to X$ induces a map from $D^2 \backslash \{0\}$ to $X \backslash \mathcal{D}$, which we also denote by $u_{\rm d}$. We identify $D^2 \backslash \{0\}$ with $[0,\infty)_{r_1} \times S^1_{s_1}$ and denote the standard coordinates on the $[0,\infty)$ and S^1 factors with r_1 and s_1. Namely, the point $(r_1, s_1) \in [0,\infty) \times S^1$ is mapped to $\exp(-r_1 - \sqrt{-1}s_1)$. (Here and in what follows, S^1 is identified with $\mathbb{R}/2\pi\mathbb{Z}$.)

Lemma 3.6. *There exist $R_{\rm d} \in \mathbb{R}$ and $x_{\rm d} \in S\mathcal{N}_{\mathcal{D}}(X)$ such that:*

$$d_{C^m}(u_{\rm d}(r_1, s_1), (p_1 r_1 + R_{\rm d}, p_1 s_1 + x_{\rm d})) \leqslant C_m e^{-\delta_1 r_1} \tag{3.7}$$

for some $C_m, \delta_1 > 0$. The constant δ_1 is independent of m.

Here we regard $(p_1 r_1 + R_{\rm d}, p_1 s_1 + x_{\rm d})$ as an element of $[0,\infty)_\tau \times S\mathcal{N}_{\mathcal{D}}(X)$ using the partial action of $(-\infty,\infty) \times S^1$ on $[0,\infty)_\tau \times S\mathcal{N}_{\mathcal{D}}(X)$. The expression on the left hand side of (3.7) is the C^m distance between the following two maps from $[0,\infty)_{r_1} \times S^1_{s_1}$ to $X \backslash \mathcal{D}$

$$(r_1, s_1) \mapsto u_{\rm d}(r_1, s_1) \qquad (r_1, s_1) \mapsto (p_1 r_1 + R_{\rm d}, p_1 s_1 + x_{\rm d})$$

Note that there exists $R'_{\rm d} > 0$ such that $u_{\rm d}(s_1, r_1)$ is an element of (3.1) for $r_1 > R'_{\rm d}$. The C^m norm is defined with respect to the cylindrical metric on $D^2 \backslash \{0\}$ and the metric g on $X \backslash \mathcal{D}$.

Proof. The claim is a consequence of [DF18a, Lemma 3.7] and the fact that the multiplicity of the intersection of u_d and $\mathcal{D}$ at $z_{\rm d}$ is p_1. □

Definition 3.8. We define $C^\infty(([0,\infty) \times S^1, \{0\} \times S^1); (u_{\rm d}^*TX, u_{\rm d}^*TL))$ to be the space of all smooth sections V of $u_{\rm d}^*TX$ on the space $[0,\infty) \times S^1$ with the boundary condition

$$V(0, s_1) \in T_{u_{\rm d}(0,s_1)}L.$$

We extend each vector $v \in T_{u_{\rm d}(z_{\rm d})}\mathcal{D}$ to a vector field defined on a neighborhood of $u_{\rm d}(z_{\rm d})$ in $\mathcal{D}$. Let $\hat{v}$ be the horizontal lift of this vector field using the decomposition in (3.5) to a neighborhood of $u_{\rm d}(z_{\rm d})$ in X. We may assume that the map $v \mapsto \hat{v}$ is linear. Using (3.5), we can also obtain a vector field $[\mathfrak{r}_\infty, \mathfrak{s}_\infty]$ on $X \backslash \mathcal{D}$ for each $(\mathfrak{r}_\infty, \mathfrak{s}_\infty) \in \mathbb{R} \times \mathbb{R}$. These vector fields are also $(-\infty,\infty) \times S^1$-invariant.

Definition 3.9. Let $C_0^\infty(([0,\infty) \times S^1, \{0\} \times S^1); (u_{\rm d}^*TX, u_{\rm d}^*TL))^+$ be the space of all triples $(V, (\mathfrak{r}_\infty, \mathfrak{s}_\infty), v)$ such that $V \in C^\infty(([0,\infty) \times S^1, \{0\} \times S^1); (u_{\rm d}^*TX, u_{\rm d}^*TL))$, $(\mathfrak{r}_\infty, \mathfrak{s}_\infty) \in \mathbb{R} \times \mathbb{R}$, $v \in T_{u_{\rm d}(z_{\rm d})}\mathcal{D}$, and

$$V - [\mathfrak{r}_\infty, \mathfrak{s}_\infty] - \hat{v}$$

has compact support. We define a weighted Sobolev norm on this vector space as follows:

$$\begin{aligned} &\|(V, (\mathfrak{r}_\infty, \mathfrak{s}_\infty), v)\|^2_{W^2_{m,\delta}} \\ =&\|V\|^2_{L^2_m([0,R'_{\rm d}] \times S^1)} \\ &+ \sum_{j=0}^m \int_{[R'_{\rm d},\infty) \times S^1} e^{\delta r_1} |\nabla^j (V - [\mathfrak{r}_\infty, \mathfrak{s}_\infty] - \hat{v})|^2 dr_1 ds_1 \\ &+ |(\mathfrak{r}_\infty, \mathfrak{s}_\infty)|^2 + |v|^2. \end{aligned} \tag{3.10}$$

Later we shall be concerned with the case that $\delta > 0$ is a sufficiently small positive number and m is a sufficiently large positive integer. We denote by

$$W^2_{m,\delta}(\Sigma_{\rm d} \backslash \{z_{\rm d}\}; (u_{\rm d}^*TX, u_{\rm d}^*TL))$$

the completion of $C_0^\infty(([0,\infty)_{r_1} \times S^1_{s_1}, \{0\} \times S^1_{s_1}); (u_{\rm d}^*TX, u_{\rm d}^*TL))^+$ with respect to the norm $\|\cdot\|_{W^2_{m,\delta}}$. This completion is a Hilbert space and is independent of how we extend the vectors v to $\hat{v}$.

Definition 3.11. Let $C_0^\infty([0,\infty)\times S^1;u_{\rm d}^*TX\otimes\Lambda^{0,1})$ be the space of all smooth sections with compact supports, and define a weighted Sobolev norm on it by:

$$\|V\|^2_{L^2_{m,\delta}} = \|V\|^2_{L^2_m([0,R'_{\rm d}]\times S^1)} + \sum_{j=0}^m \int_{[R'_{\rm d},\infty)\times S^1} e^{\delta r_1}|\nabla^j(V)|^2 dr_1 ds_1.$$

The completion of $C_0^\infty([0,\infty)\times S^1;u_{\rm d}^*TX\otimes\Lambda^{0,1})$ with respect to the norm $\|\cdot\|_{L^2_{m,\delta}}$ is denoted by

$$L^2_{m,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\};u_{\rm d}^*TX\otimes\Lambda^{0,1}). \tag{3.12}$$

Lemma 3.13. *Linearization of the Cauchy-Riemann equation at $u_{\rm d}$ gives a first order differential operator*

$$\begin{aligned} D_{u_{\rm d}}\overline\partial :& C_0^\infty(([0,\infty)\times S^1,\{0\}\times S^1);(u_{\rm d}^*TX,u_{\rm d}^*TL)) \\ &\to C_0^\infty([0,\infty)\times S^1;u_{\rm d}^*TX\otimes\Lambda^{0,1}), \end{aligned}$$

which has the following properties.

(1) *The operator $D_{u_{\rm d}}\overline\partial$ induces a continuous linear map*

$$\begin{aligned} D_{u_{\rm d}}\overline\partial :& W^2_{m+1,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\};(u_{\rm d}^*TX,u_{\rm d}^*TL)) \\ &\to L^2_{m,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\};u_{\rm d}^*TX\otimes\Lambda^{0,1}). \end{aligned} \tag{3.14}$$

In particular, for an element

$$(V,(\mathfrak r_\infty,\mathfrak s_\infty),v)\in C_0^\infty(([0,\infty)\times S^1,\{0\}\times S^1);(u_{\rm d}^*TX,u_{\rm d}^*TL))^+$$

we have $D_{u_{\rm d}}\overline\partial(V,(\mathfrak r_\infty,\mathfrak s_\infty),v) = D_{u_{\rm d}}\overline\partial(V)$.

(2) *(3.14) is a Fredholm operator.*

(3) *The index of the operator (3.14) is equal to the virtual dimension of the moduli space $\mathcal M_1^{\rm reg,d}(\beta_{\rm d};(p_1))$[2] which contains $u_{\rm d}$.*

Proof. (1) is a consequence of (3.7). (We choose δ to be smaller than the constant δ_1 in (3.7).) The differential operator $D_{u_{\rm d}}\overline\partial$ is asymptotic to an operator of the form

$$\frac{\partial}{\partial r_1} + P$$

as r_1 goes to infinity. Furthermore, $P = J\partial/\partial s_1$ and the kernel of this operator can be identified with $\mathbb R\oplus\mathbb R\oplus T_{u_{\rm d}(z_{\rm d})}\mathcal D$. Part (2) is a consequence of this observation and general results about elliptic operators on manifolds with cylindrical ends [APS75, Theorem (3.10)].

To prove Part (3), we need to show that the Fredholm index of two elliptic operators defined agree with each other: one is the linearized Cauchy-Riemann operator on D^2 and the other one is the linearized Cauchy-Riemann operator on $[0,\infty)\times S^1$. We relate the indices of these operators to the indices of two other operators with smaller indices. First consider the subspace

$$L^2_{m+1,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\};(u_{\rm d}^*TX,u_{\rm d}^*TL)) \tag{3.15}$$

of $W^2_{m+1,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\};(u_{\rm d}^*TX,u_{\rm d}^*TL))$. This subspace consists of the closure of elements $(V,(\mathfrak r_\infty,\mathfrak s_\infty),v)$ with extra condision that $v=0$. In particular, its codimension is equal to $\dim(\mathcal D)$. Let $D'_{u_{\rm d}}\overline\partial$ be the elliptic complex given by the variation of (3.14) where we replace the domain with (3.15). In particular, we have

$$\operatorname{ind}(D'_{u_{\rm d}}\overline\partial) = \operatorname{ind}(D_{u_{\rm d}}\overline\partial) - \dim(\mathcal D).$$

[2]See [DF18a, Definition 3.57] for the definition of this moduli space. Here (p_1) stands for the multiplicity number p_1.

Let $L^2_m(D^2; u_{\rm d}; X, L)$ be the space of L^2_m sections of $u_{\rm d}^*TX$ whose boundary values are in $u_{\rm d}|^*_{\partial D^2}TL$. Consider the subspace of this space consisting of elements whose derivatives up to order $p_1 - 1$ vanish at the origin, and denote this space by $L^2_m(D^2; u_{\rm d}; X, L; (p_1))$. Then the linearization of the Cauchy-Riemann operator defines an operator

$$D''_{u_{\rm d}}\overline{\partial} : L^2_m(D^2; u_{\rm d}; X, L; (p_1)) \to L^2_{m-1}(D^2; u_{\rm d}^*TX \otimes \Lambda^{0,1}), \tag{3.16}$$

This elliptic complex controls the deformation theory of a subspace of $\mathcal{M}_1^{\rm reg,d}(\beta_{\rm d}; (p_1))$ given by J-holomorphic curves that the intersection point with the divisor is constrained to be the fixed point $u_{\rm d}(D^2) \cap \mathcal{D}$. In particular, we have

$$\operatorname{ind}(D''_{u_{\rm d}}\overline{\partial}) = \operatorname{virdim}(\mathcal{M}_1^{\rm reg,d}(\beta_{\rm d}; (p_1))) - \dim(\mathcal{D}).$$

To prove the claim in Part (3), it suffices to show that the indices of the operators $D'_{u_{\rm d}}\overline{\partial}$ and $D''_{u_{\rm d}}\overline{\partial}$ agree with each other. Any element in the kernel (resp. the cokernel) of $D''_{u_{\rm d}}\overline{\partial}$ determines an element of the the kernel (resp. the cokernel) of $D'_{u_{\rm d}}\overline{\partial}$ by a reparametrization of the domain. In the other direction, we may use removability of singularity. Consequently, the kernels and the cokernels of these two operators are in correspondence with each other. □

3.2. **The Sphere Component.** In this part, we study the linearization of the problem governing the map $u_{\rm s}$. This can be done similar to the case of $u_{\rm d}$. We take a compact subset $K_{\rm s}$ of $\Sigma_{\rm s}\backslash\{z_{\rm s}\}$ such that $u_{\rm s}(\Sigma_{\rm s}\backslash K_{\rm s})$ is contained in (3.1). We may assume that $\Sigma_{\rm s}\backslash K_{\rm s}$ is a disk. We take a coordinate $(r_2, s_2) \in \mathbb{R} \times S^1$ of $\Sigma_{\rm s}\backslash(K_{\rm s} \cup \{z_{\rm s}\})$ such that (r_2, s_2) is identified with $\exp(-r_2 - \sqrt{-1}s_2) \in D^2\backslash\{0\} \cong \Sigma_{\rm s}\backslash(K_{\rm s} \cup \{z_{\rm s}\})$. In the same way as in (3.7), we have the inequality

$$d_{C^m}(u_{\rm s}(r_2, s_2), (p_2 r_2 + R_{\rm s}, p_2 s_2 + x_{\rm s})) \leqslant C_m e^{-\delta_1 r_2}, \tag{3.17}$$

for a constant $R_{\rm s}$ and $x_{\rm s} \in S\mathcal{N}_{\mathcal{D}}(X)$.

Definition 3.18. (Compare to [FOOO09, Lemma 7.1.5].) Let

$$C^\infty_0(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX)^+$$

be the space of all triples $(V, (\mathfrak{r}_\infty, \mathfrak{s}_\infty), v)$ such that $V \in C^\infty(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX)$, $(\mathfrak{r}_\infty, \mathfrak{s}_\infty) \in \mathbb{R} \times \mathbb{R}$, $v \in T_{u_{\rm s}(z_{\rm s})}\mathcal{D}$ and

$$V - [\mathfrak{r}_\infty, \mathfrak{s}_\infty] - \hat{v}$$

is compactly supported.[3] Analogous to (3.10), we define a Sobolev norm on this space as follows:

$$\begin{aligned} &\|(V, (\mathfrak{r}_\infty, \mathfrak{s}_\infty), v)\|^2_{W^2_{m,\delta}} \\ =&\|V\|^2_{L^2_m(K_{\rm s})} \\ &+ \sum_{j=0}^m \int_{[0,\infty)\times S^1} e^{\delta r_2} |\nabla^j (V - [\mathfrak{r}_\infty, \mathfrak{s}_\infty] - \hat{v})|^2 dr_2 ds_2 \\ &+ |(\mathfrak{r}_\infty, \mathfrak{s}_\infty)|^2 + |v|^2. \end{aligned} \tag{3.19}$$

We shall be concerned with the case that δ is a sufficiently small positive number and m is a sufficiently large positive integer. We denote by

$$W^2_{m,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX)$$

the completion of $C^\infty_0(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX)^+$ with respect to the norm $\|\cdot\|_{W^2_{m,\delta}}$. This completion is a Hilbert space.

[3] Note that $(\mathfrak{r}_\infty, \mathfrak{s}_\infty)$ and v determine a vector fields $[\mathfrak{r}_\infty, \mathfrak{s}_\infty]$ and $\hat{v}$ on (3.1) in the same way as in the last subsection.

We can also define the Hilbert space:

$$L^2_{m,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX \otimes \Lambda^{0,1}),$$

in the same way as in (3.12).

Lemma 3.20. (1) *The linearization of the Cauchy-Riemann equation at $u_{\rm s}$ defines a continuous linear map*

$$D_{u_{\rm s}}\overline{\partial} : W^2_{m+1,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX) \to L^2_{m,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX \otimes \Lambda^{0,1}). \tag{3.21}$$

(2) *(3.21) is a Fredholm operator.*
(3) *The index of the operator (3.21) is equal to 4 plus the virtual dimension of the moduli space $\mathcal{M}^{\rm reg,s}(\beta_{\rm s}; (p_2))$[4] which contains $u_{\rm s}$.*

The proof is similar to the proof of Lemma 3.13. The number 4, that appears in Item (3), is the dimension of the group of automorphisms of $(S^2, z_{\rm s})$.

3.3. **The Divisor Component.** Finally, we analyze the deformation theory of $u_{\rm D}$. Note that $u_{\rm D}$ is a map to the symplectic manifold $\mathcal{D}$. So we firstly describe a Fredholm theory for the deformation of $u_{\rm D}$ as a map to $\mathcal{D}$. This is a standard task in Gromov-Witten theory. We have a Fredholm operator

$$D_{u_{\rm D}}\overline{\partial} : L^2_{m+1}(\Sigma_{\rm D}; u_{\rm D}^*T\mathcal{D}) \to L^2_m(\Sigma_{\rm D}; u_{\rm D}^*T\mathcal{D} \otimes \Lambda^{0,1}). \tag{3.22}$$

To perform gluing analysis, we compare this Fredholm operator with another Fredholm operator associated to the map $U_{\rm D}$ in (2.1).

Definition 3.23. As in previous two subsections, we extend any $v_{\rm d} \in T_{u_{\rm d}(z_{\rm d})}\mathcal{D}$, $v_{\rm s} \in T_{u_{\rm s}(z_{\rm s})}\mathcal{D}$, to vector fields $\hat{v}_{\rm d}$, $\hat{v}_{\rm s}$ on open neighborhoods of the fibers of $\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)$ over $u_{\rm d}(z_{\rm d})$ and $u_{\rm s}(z_{\rm s})$. For any $(\mathfrak{r}_\infty, \mathfrak{s}_\infty) \in \mathbb{R} \times \mathbb{R}$, we can also define a vector field $[\mathfrak{r}_\infty, \mathfrak{s}_\infty]$ in a neighborhood of any of the points $u_{\rm d}(z_{\rm d})$ and $u_{\rm s}(z_{\rm s})$, as in the last two subsections. Let

$$C^\infty_0(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; U_{\rm D}^*T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)))^+$$

be the space of all 5-tuples $(V, (\mathfrak{r}_{\infty,\rm d}, \mathfrak{s}_{\infty,\rm d}), (\mathfrak{r}_{\infty,\rm s}, \mathfrak{s}_{\infty,\rm s}), v_{\rm d}, v_{\rm s})$ such that

$$V \in C^\infty(\Sigma_{\rm D}\backslash\{z_{\rm s}\}\backslash\{z_{\rm d}\}; U_{\rm D}^*T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)))$$

and that:

(i) The restriction of $V - [\mathfrak{r}_{\infty,\rm d}, \mathfrak{s}_{\infty,\rm d}] - \hat{v}_{\rm d}$ to a punctured neighborhood of $z_{\rm d}$ in $\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}$ vanishes;
(ii) The restriction of $V - [\mathfrak{r}_{\infty,\rm s}, \mathfrak{s}_{\infty,\rm s}] - \hat{v}_{\rm s}$ to a punctured neighborhood of $z_{\rm s}$ in $\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}$ vanishes.

We define a weighted Sobolev norm on this space as follows:

$$\begin{aligned}
&\|(V, (\mathfrak{r}_{\infty,\rm d}, \mathfrak{s}_{\infty,\rm d}), (\mathfrak{r}_{\infty,\rm s}, \mathfrak{s}_{\infty,\rm s}), v_{\rm d}, v_{\rm s})\|^2_{W^2_{m,\delta}} \\
=&\|V\|^2_{L^2_m(K_{\rm D})} \\
&+ \sum_{j=0}^m \int_{[0,\infty)\times S^1} e^{\delta r_1} |\nabla^j (V - [\mathfrak{r}_{\infty,\rm d}, \mathfrak{s}_{\infty,\rm d}] - \hat{v}_{\rm d})|^2 dr_1 ds_1 \\
&+ \sum_{j=0}^m \int_{[0,\infty)\times S^1} e^{\delta r_2} |\nabla^j (V - [\mathfrak{r}_{\infty,\rm s}, \mathfrak{s}_{\infty,\rm s}] - \hat{v}_{\rm s})|^2 dr_2 ds_2 \\
&+ |(\mathfrak{r}_{\infty,\rm d}, \mathfrak{s}_{\infty,\rm d})|^2 + |(\mathfrak{r}_{\infty,\rm s}, \mathfrak{s}_{\infty,\rm s})|^2 + |v_{\rm d}|^2 + |v_{\rm s}|^2.
\end{aligned} \tag{3.24}$$

In order to clarify the notation in (3.24), the following comments are in order. We take a compact subset $K_{\rm D} \subset \Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}$ such that $\Sigma_{\rm D}\backslash K_{\rm D}$ is the union of two discs. We fix

[4] See [DF18a, Definition 3.60]. (p_2) stands for the multiplicity number p_2.

coordinates $(r_1, s_1) \in [0,\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ and $(r_2, s_2) \in [0,\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ on the complement of the origins of these two discs. That is, we identify $[0,\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ with $D^2\backslash\{0\}$ using $(r_i, s_i) \mapsto \exp(-(r_i + \sqrt{-1}s_i))$.

We denote the completion of $C_0^\infty(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; U_{\rm D}^* T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)))^+$ with respect to the norm $\|\cdot\|_{W^2_{m,\delta}}$ by:

$$H_1 := W^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; U_{\rm D}^* T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X))).$$

This completion is a Hilbert space.

We also define a Hilbert space

$$H_2 := L^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; U_{\rm D}^* T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)) \otimes \Lambda^{0,1}),$$

in the same way as in (3.12).

We have a short exact sequence of holomorphic bundles on $\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}$ as follows:

$$0 \to \underline{\mathbb{C}} \to U_{\rm D}^* T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)) \to u_{\rm D}^* T\mathcal{D} \to 0.$$

Here the first map is defined by the $\mathbb{C}_*$-action. This short exact sequence induces a diagram of the following form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & H_1 & \longrightarrow & B_1 & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & H_2 & \longrightarrow & B_2 & \longrightarrow & 0 \end{array} \tag{3.25}$$

where we have:

$$A_1 = W^2_{m+1,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; \underline{\mathbb{C}}), \qquad A_2 = L^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; \Lambda^{0,1}),$$

and

$$B_1 = W^2_{m+1,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; u_{\rm D}^* T\mathcal{D}), \quad B_2 = L^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; u_{\rm D}^* T\mathcal{D} \otimes \Lambda^{0,1}).$$

The spaces A_1 and B_1 are defined similar to H_1 in an obvious way. In the same way as in the proof of Lemma 3.13, we can show that the linearization of the Cauchy-Riemann equation at $U_{\rm D}$ defines a continuous linear map:

$$\begin{aligned} D_{U_{\rm D}}\overline{\partial} :& W^2_{m+1,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; U_{\rm D}^* T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X))) \\ &\to L^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; U_{\rm D}^* T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)) \otimes \Lambda^{0,1}). \end{aligned} \tag{3.26}$$

which is the map g in (3.25). The map f is the standard Cauchy-Riemann operator and h is the linearized Cauchy-Riemann operator associated to the map $u_{\rm D}$. The diagram in (3.25) commutes and each row of the diagram forms an exact sequence.

Lemma 3.27. (1) *The operator in (3.26) is Fredholm.*

(2) *The kernel and the cokernel of the operator h in (3.25) can be identified with the kernel and the cokernel of $D_{u_{\rm D}}\overline{\partial}$ in (3.22). Moreover, (3.25) induces a short exact sequence of the following form:*

$$0 \to \mathbb{C} \to {\rm Ker} D_{U_{\rm D}}\overline{\partial} \to {\rm Ker} D_{u_{\rm D}}\overline{\partial} \to 0.$$

and an isomorphism

$$\text{CoKer} D_{u_{\rm D}}\overline{\partial} \cong \text{CoKer} D_{U_{\rm D}}\overline{\partial}.$$

Proof. The proof of the claim in (1) is similar to the proof of Lemma 3.13. Identification of the kernels and cokernels of the operators h and $D_{u_{\rm D}}\overline{\partial}$ in (3.22) is straightforward. Similarly, the kernels and cokernels of the operators f and the Cauchy-Riemann operator associated to the trivial bundle on the sphere $\Sigma_{\rm D}$ can be identified with each other. The latter operator is surjective and its kernel is a copy of $\mathbb{C}$, consists of constant sections of the trivial bundle. Using this observation, the remaining claims in part (2) follow from the diagram chase of the diagram (3.25). □

4. Stabilization of the Source Curves and the Obstruction Bundles

The operators $D_{u_{\rm d}}\overline{\partial}$, $D_{u_{\rm s}}\overline{\partial}$, $D_{u_{\rm D}}\overline{\partial}$ are not necessarily surjective. If these operators are not surjective, then the deformation theories of $u_{\rm d}, u_{\rm s}, u_{\rm D}$ are obstructed. Following a general idea due to Kuranishi, we introduce *obstruction spaces*.

Definition 4.1. A triple $E = (E_{\rm d}, E_{\rm s}, E_{\rm D})$ of finite dimensional vector spaces that

$$\begin{aligned} E_{\rm d} &\subset C^\infty(\Sigma_{\rm d}\backslash\{z_{\rm d}\}; u_{\rm d}^*TX \otimes \Lambda^{0,1}),\\ E_{\rm s} &\subset C^\infty(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX \otimes \Lambda^{0,1}),\\ E_{\rm D} &\subset C^\infty(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; u_{\rm D}^*T\mathcal{D} \otimes \Lambda^{0,1}) \end{aligned}$$

is called an *obstruction space* for u if it satisfies the following properties.

1. Elements of $E_{\rm d}$, $E_{\rm s}$, $E_{\rm D}$ have compact supports away from $z_{\rm d}$, $z_{\rm s}$, $\{z_{\rm d}, z_{\rm s}\}$, respectively;
2. $\operatorname{Im}(D_{u_{\rm d}}\overline{\partial}) + E_{\rm d} = L^2_{m,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\}; u_{\rm d}^*TX \otimes \Lambda^{0,1})$;
3. $\operatorname{Im}(D_{u_{\rm s}}\overline{\partial}) + E_{\rm s} = L^2_{m,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX \otimes \Lambda^{0,1})$;
4. $\operatorname{Im}(D_{u_{\rm D}}\overline{\partial}) + E_{\rm D} = L^2_{m}(\Sigma_{\rm D}; u_{\rm D}^*T\mathcal{D} \otimes \Lambda^{0,1})$.

For the purpose of the gluing analysis, we need our obstruction spaces satisfy the *mapping transversality condition* defined as follows.

Definition 4.2. Let

$$\mathcal{EV}_{\rm d} : W^2_{m+1,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\}; (u_{\rm d}^*TX, u_{\rm d}^*TL)) \to T_{u_{\rm d}(z_{\rm d})}\mathcal{D}$$

be the continuous linear map that associates to a triple $(V, (\mathfrak{r}_\infty, \mathfrak{s}_\infty), v)$ the vector v. The map

$$\mathcal{EV}_{\rm s} : W^2_{m+1,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; u_{\rm s}^*TX) \to T_{u_{\rm s}(z_{\rm s})}\mathcal{D},$$

is defined similarly. Finally, let:

$$\mathcal{EV}_{\rm D} := (\mathrm{EV}_{\rm D,d}, \mathrm{EV}_{\rm D,s}) : L^2_{m+1}(\Sigma_{\rm D}; u_{\rm D}^*T\mathcal{D}) \to T_{u_{\rm d}(z_{\rm d})}\mathcal{D} \oplus T_{u_{\rm s}(z_{\rm s})}\mathcal{D}.$$

be the map that associates to $V \in L^2_{m+1}(\Sigma_{\rm D}; u_{\rm D}^*T\mathcal{D})$ the pair of vectors $(V(z_{\rm d}), V(z_{\rm s}))$. We say that an obstruction space $E = (E_{\rm s}, E_{\rm d}, E_{\rm D})$ satisfies the *mapping transversality condition*, if the following map is surjective:

$$\begin{aligned} &(\mathcal{EV}_{\rm d} + \mathcal{EV}_{\rm D,d}, \mathcal{EV}_{\rm s} + \mathcal{EV}_{\rm D,s}) :\\ &\qquad (D_{u_{\rm d}}\overline{\partial})^{-1}E_{\rm d} \oplus (D_{u_{\rm s}}\overline{\partial})^{-1}E_{\rm s} \oplus (D_{u_{\rm D}}\overline{\partial})^{-1}E_{\rm D}\\ &\qquad \to T_{u_{\rm d}(z_{\rm d})}\mathcal{D} \oplus T_{u_{\rm s}(z_{\rm s})}\mathcal{D}. \end{aligned} \tag{4.3}$$

We shall use obstruction spaces to define Kuranishi neighborhoods of the elements represented by $u_{\rm d}$, $u_{\rm s}$, $u_{\rm D}$, respectively. Note that at this stage we are studying three irreducible components separately. The process of gluing them will be discussed in the next stage.

We next introduce the notion of *added marked points* and *transversal submanifolds*. The primary purpose of these auxiliary data is to fix coordinate on the domain of maps close to u. Note that the domain of u is unstable, i.e., there are non-trivial automorphisms of the domain. This automorphism group gives rise to an ambiguity when we want to fix coordinate for the domain of u and nearby maps. We remove this ambiguity using added marked points and transversal submanifolds. This is a standard technique which is used, for example, in [FO99, appendix]. (See also [FOOO12, Section 20], [FOOO18, Subsection 9.3].)

The source curve $\Sigma_{\rm d}$ of $u_{\rm d}$ comes with one interior nodal point $z_{\rm d}$ and one boundary marked point z_0. The group of isometries of $\Sigma_{\rm d}$ preserving $z_{\rm d}$ and z_0 is trivial and hence $\Sigma_{\rm d}$ together with these marked points is stable. Moreover, this source curve does not have any deformation parameter. However, the source curve $\Sigma_{\rm s}$ of $u_{\rm s}$ comes with only one interior nodal point $z_{\rm s}$. Therefore, it is unstable and we add two extra marked points $w_{{\rm s},1}$, $w_{{\rm s},2}$ such that it becomes stable with no deformation parameter. Similarly, the source curve $\Sigma_{\rm D}$ of $u_{\rm D}$ comes with two interior nodal points $z_{\rm s}$, $z_{\rm d}$ and is unstable. We add one marked point $w_{\rm D}$ so that it becomes stable without any deformation parameter. The transversal submanifolds in the following definition are used for the purpose of killing the extra freedom of moving the auxiliary marked points.

Definition 4.4. Suppose $\mathcal{N}_{{\rm s},1}$, $\mathcal{N}_{{\rm s},2}$ are codimension 2 smooth submanifolds of X and $\mathcal{N}_{\rm D}$ is a codimension 2 smooth submanifold of $\mathcal{D}$. We say the data Ξ of the extra marked points $w_{{\rm s},1}$, $w_{{\rm s},2}$, $w_{\rm D}$ as above and the submanifolds $\mathcal{N}_{{\rm s},1}$, $\mathcal{N}_{{\rm s},2}$ and $\mathcal{N}_{\rm D}$ form *stabilization data* for u, if they satisfy the following properties.

(1) For $i = 1, 2$, there exists an open neighborhood $\mathcal{V}_{\rm s}(w_{{\rm s},i})$ of $w_{{\rm s},i}$ such that $\mathcal{V}_{\rm s}(w_{{\rm s},i}) \cap u_{\rm s}^{-1}(\mathcal{N}_{{\rm s},i}) = \{w_{{\rm s},i}\}$ and $u_{\rm s}|_{\mathcal{V}_{\rm s}(w_{{\rm s},i})}$ is transversal to $\mathcal{N}_{{\rm s},i}$ at $w_{{\rm s},i}$.
(2) There exists an open neighborhood $\mathcal{V}_{\rm D}(w_{\rm D})$ of $w_{\rm D}$ such that $\mathcal{V}_{\rm D}(w_{\rm D}) \cap u_{\rm D}^{-1}(\mathcal{N}_{\rm D}) = \{w_{\rm D}\}$ and $u_{\rm D}|_{\mathcal{V}_{\rm D}(w_{\rm D})}$ is transversal to $\mathcal{N}_{\rm D}$ at $w_{\rm D}$.

We say a pair $\Upsilon := (\Xi, E)$ of stabilization data Ξ together with an obstruction space E as in Definition 4.1 provide *stabilization and obstruction data* for u.

To define Kuranishi neighborhoods for the components of u, we need to transfer the obstruction space E of u to nearby maps. This is done using *target parallel transportation*. Let $u'_{\rm d} : (\Sigma_{\rm d} \backslash \{z_{\rm d}\}, \partial \Sigma_{\rm d}) \to (X \backslash \mathcal{D}, L)$ (resp. $u'_{\rm s} : \Sigma_{\rm s} \backslash \{z_{\rm s}\} \to X \backslash \mathcal{D}$, $u'_{\rm D} : \Sigma_{\rm D} \to \mathcal{D}$) be an $L^2_{m+1,loc}$ map such that:

$$d(u_{\rm d}(x), u'_{\rm d}(x)) \leqslant \varepsilon \quad (\text{resp. } d(u_{\rm s}(x), u'_{\rm s}(x)) \leqslant \varepsilon, \; d(u_{\rm D}(x), u'_{\rm D}(x)) \leqslant \varepsilon.) \tag{4.5}$$

for any $x \in {\rm Int}(\Sigma_{\rm d} \backslash \{z_{\rm d}\})$ (resp. $x \in \Sigma_{\rm s} \backslash \{z_{\rm s}\}$, $x \in \Sigma_{\rm D}$). Here d is defined with respect to the metric g on $X \backslash \mathcal{D}$ or the metric g' on $\mathcal{D}$, introduced at the beginning of Section 3. We wish to define:

$$\begin{aligned} E_{\rm d}(u'_{\rm d}) &\subset L^2_m(\Sigma_{\rm d} \backslash \{z_{\rm d}\}; (u'_{\rm d})^* TX \otimes \Lambda^{0,1}) \\ E_{\rm s}(u'_{\rm s}) &\subset L^2_m(\Sigma_{\rm s} \backslash \{z_{\rm s}\}; (u'_{\rm s})^* TX \otimes \Lambda^{0,1}) \\ E_{\rm D}(u'_{\rm D}) &\subset L^2_m(\Sigma_{\rm D}; (u'_{\rm D})^* T\mathcal{D} \otimes \Lambda^{0,1}) \end{aligned} \tag{4.6}$$

which are finite dimensional subspaces consisting of elements with compact supports. To define target parallel transportation, we need to impose an additional constraint on $E_{\rm d}$, $E_{\rm s}$, $E_{\rm D}$.

Definition 4.7. Given an obstruction space $E = (E_{\rm d}, E_{\rm s}, E_{\rm D})$, if for any $x \in \Sigma_{\rm d}$ (resp. $x \in \Sigma_{\rm s}$, $x \in \Sigma_{\rm D}$) in the support of an element of $E_{\rm d}$ (resp. $E_{\rm s}$, $E_{\rm D}$), the map $u_{\rm d}$ (resp. $u_{\rm s}$, $u_{\rm D}$) is an immersion at x, then E is called *support-immersive*.

This condition in particular implies that $E_{\rm d}$ (resp. $E_{\rm s}$, $E_{\rm D}$) is zero, if u is constant on $\Sigma_{\rm d}$ (resp. $\Sigma_{\rm s}$, $\Sigma_{\rm D}$). Using the fact that Σ has genus 0, we can always take $E_{\rm d}$, $E_{\rm s}$, $E_{\rm D}$ satisfying this additional condition. In the following, we assume that our obstruction spaces satisfy the mapping transversality condition of Definition 4.2 and are support-immersive in the sense of Definition 4.7.[5]

In the following definition, ${\rm Supp}(E_{\rm d})$ (resp. ${\rm Supp}(E_{\rm s})$, ${\rm Supp}(E_{\rm D})$) denotes the union of the supports of the elements of $E_{\rm d}$ (resp. $E_{\rm s}$, $E_{\rm D}$).

[5] There are several other methods to define (4.6), where we do not need to assume Condition 4.7.

Definition 4.8. For any triple of maps $u'_{\rm d}$, $u'_{\rm s}$, $u'_{\rm D}$ as above, let $I^{\rm t}_{\rm d} : {\rm Supp}(E_{\rm d}) \to \Sigma_{\rm d}$ (resp. $I^{\rm t}_{\rm s} : {\rm Supp}(E_{\rm s}) \to \Sigma_{\rm s}$, $I^{\rm t}_{\rm D} : {\rm Supp}(E_{\rm D}) \to \Sigma_{\rm D}$) be the maps that is defined as follows. For $x \in {\rm Supp}(E_{\rm d})$ (resp. $x \in {\rm Supp}(E_{\rm s})$, $x \in {\rm Supp}(E_{\rm D})$), the point $I^{\rm t}_{\rm d}(x)$ (resp. $I^{\rm t}_{\rm s}(x)$, $I^{\rm t}_{\rm D}(x)$) is the unique point which satisfies the following two conditions.

(1) The distance between x and $I^{\rm t}_{\rm d}(x)$ (resp. $I^{\rm t}_{\rm s}(x)$, $I^{\rm t}_{\rm D}(x)$) is smaller than the constant ε. We choose ε small enough such that (4.5) and this condition imply that
$$d(u_{\rm d}(x), u'_{\rm d}(I^{\rm t}_{\rm d}(x))) \leqslant o, \quad (\text{resp. } d(u_{\rm s}(x), u'_{\rm s}(I^{\rm t}_{\rm s}(x))) \leqslant o, \quad d(u_{\rm D}(x), u'_{\rm D}(I^{\rm t}_{\rm D}(x))) \leqslant o.)$$
where o is a constant smaller than the injectivity radii of $X\backslash\mathcal{D}$ and $\mathcal{D}$.

(2) Condition (1) implies that there exists a unique minimal geodesic $\gamma_{\rm d} : [0,1] \to X\backslash\mathcal{D}$ (resp. $\gamma_{\rm s} : [0,1] \to X\backslash\mathcal{D}$, $\gamma_{\rm D} : [0,1] \to \mathcal{D}$) joining $u_{\rm d}(x)$ to $u'_{\rm d}(I^{\rm t}_{\rm d}(x))$ (resp. $u_{\rm s}(x)$ to $u'_{\rm s}(I^{\rm t}_{\rm s}(x))$, $u_{\rm D}(x)$ to $u'_{\rm D}(I^{\rm t}_{\rm D}(x))$.)[6] We require that the vector $(d\gamma_{\rm d}/dt)(0)$ (resp. $(d\gamma_{\rm s}/dt)(0)$, $(d\gamma_{\rm D}/dt)(0)$) is perpendicular to the image of $u_{\rm d}$ (resp. $u_{\rm s}$, $u_{\rm D}$) at $t = 0$.

Here t in $I^{\rm t}_{\rm d}$, $I^{\rm t}_{\rm s}(x)$ and $I^{\rm t}_{\rm D}(x)$ stands for target.

We fix a unitary connection on $T(X\backslash\mathcal{D})$, whose restriction to $\mathfrak{U}$ is given by the direct sum of the trivial connection on $\underline{\mathbb{C}}$ and a unitary connection on $T\mathcal{D}$. In particular, this connection is invariant with respect to the partial $\mathbb{C}_*$-action. The parallel transport along the geodesics $\gamma_{\rm d}$, $\gamma_{\rm s}$ with respect to this unitary connection induces complex linear maps:
$$T_{u_{\rm d}(x)}X \to T_{u'_{\rm d}(I^{\rm t}_{\rm d}(x))}X, \quad T_{u_{\rm s}(x)}X \to T_{u'_{\rm s}(I^{\rm t}_{\rm s}(x))}X.$$
We thus obtain bundle maps:
$$u_{\rm d}^*TX \to (u'_{\rm d} \circ I^{\rm t}_{\rm d})^*TX, \quad u_{\rm s}^*TX \to (u'_{\rm s} \circ I^{\rm t}_{\rm s})^*TX.$$
By differentiating and projecting to the $(0,1)$ part, we also obtain bundle maps:
$$d^{0,1}I^{\rm t}_{\rm d} : \Lambda^{0,1} \to (I^{\rm t}_{\rm d})^*\Lambda^{0,1}, \quad d^{0,1}I^{\rm t}_{\rm s} : \Lambda^{0,1} \to (I^{\rm t}_{\rm s})^*\Lambda^{0,1}.$$
We may assume that these maps are isomorphisms by choosing ε to be small enough. Taking tensor product gives rise to the maps:
$$u_{\rm d}^*TX \otimes \Lambda^{0,1} \to (u'_{\rm d} \circ I^{\rm t}_{\rm d})^*TX \otimes (I^{\rm t}_{\rm d})^*\Lambda^{0,1},$$
$$u_{\rm s}^*TX \otimes \Lambda^{0,1} \to (u'_{\rm s} \circ I^{\rm t}_{\rm s})^*TX \otimes (I^{\rm t}_{\rm s})^*\Lambda^{0,1}.$$
which induce linear maps:
$$\mathcal{PAL} : L^2_m({\rm Supp}E_{\rm d}; u_{\rm d}^*TX \otimes \Lambda^{0,1}) \to L^2_{m,loc}(\Sigma_{\rm d}\backslash\{z_{\rm d}\}; (u'_{\rm d})^*TX \otimes \Lambda^{0,1}), \tag{4.9}$$
$$\mathcal{PAL} : L^2_m({\rm Supp}E_{\rm s}; u_{\rm s}^*TX \otimes \Lambda^{0,1}) \to L^2_{m,loc}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; (u'_{\rm s})^*TX \otimes \Lambda^{0,1}). \tag{4.10}$$
We now define
$$E_{\rm d}(u'_{\rm d}) = \mathcal{PAL}(E_{\rm d}), \quad E_{\rm s}(u'_{\rm s}) = \mathcal{PAL}(E_{\rm s}). \tag{4.11}$$

Remark 4.12. Since $\Sigma_{\rm d}$, $\Sigma_{\rm s}$ and $\Sigma_{\rm D}$ with the added marked points are stable, we can use the identity map instead of $I^{\rm t}_{\rm d}$, $I^{\rm t}_{\rm s}$ and $I^{\rm t}_{\rm D}$. (Note that adding marked points is essential, otherwise we need to fix representatives for the components $\Sigma_{\rm s}$, $\Sigma_{\rm D}$, and then the identity maps depend on these representatives.) This approach of using the identity map after adding marked points is employed in [FO99, FOOO09] and many other places in the literature. We call our choice here the *target parallel transportation.* (A similar method is used in [FOOO16b, page 250, Condition 4.3.27].) This method works better for the construction of [DF18b]. The advantage of target parallel transport lies in the

[6] Here the geodesics are defined with respect to the metric g on $X\backslash\mathcal{D}$ and the metric g' on $\mathcal{D}$.

fact that it is more canonical and independent of the choice of domain coordinate. This fact is useful to obtain a system of Kuranishi structures which are compatible at the boundary and corners. For example, the maps $I^{\rm t}_{\rm d}$, $I^{\rm t}_{\rm S}$, $I^{\rm t}_{\rm D}$ do *not* change when we slightly perturb added marked points or transversals.

We now define (Kuranishi) neighborhoods of $u_{\rm d}$, $u_{\rm s}$, $u_{\rm D}$ as follows.

Definition 4.13. We denote by $\mathcal{U}_{\rm d}$ (resp. $\mathcal{U}_{\rm s}$) the set of all $L^2_{m+1,loc}$ maps $u'_{\rm d} : (\Sigma_{\rm d}\backslash\{z_{\rm d}\}, \partial\Sigma_{\rm d}) \to (X\backslash\mathcal{D}, L)$ (resp. $u'_{\rm s} : \Sigma_{\rm s}\backslash\{z_{\rm s}\} \to X\backslash\mathcal{D}$) with the following properties:

(1) The C^2-distance between $u_{\rm d}$ and $u'_{\rm d}$ (resp. $u_{\rm s}$ and $u'_{\rm s}$) is less than ε.
(2) The equation
$$\overline{\partial}u'_{\rm d} \in E_{\rm d}(u'_{\rm d}), \quad (\text{resp. } \overline{\partial}u'_{\rm s} \in E_{\rm s}(u'_{\rm s}))$$
is satisfied.
(3) There exists $p \in \mathcal{D}$ such that
$$\lim_{x\to z_{\rm d}} u'_{\rm d}(x) = p, \quad (\text{resp. } \lim_{x\to z_{\rm s}} u'_{\rm s}(x) = p).$$
(4) In the latter case, $u'_{\rm s}(w_{{\rm s},1}) \in \mathcal{N}_{{\rm s},1}$ and $u'_{\rm s}(w_{{\rm s},2}) \in \mathcal{N}_{{\rm s},2}$.

We define $\mathcal{U}^+_{\rm s}$ to be the set of maps $u'_{\rm s}$ satisfying (1), (2) and (3), but not necessarily (4).

Note that standard regularity results imply that elements of $\mathcal{U}_{\rm d}$ and $\mathcal{U}_{\rm s}$ are smooth.

In the same way as in the case of $u'_{\rm d}$, $u'_{\rm s}$, for $u'_{\rm D} : \Sigma_{\rm D} \to \mathcal{D}$ with $d(u_{\rm D}(x), u'_{\rm D}(x)) \leqslant \varepsilon$, we define:

$$\mathcal{PAL} : L^2_m(\text{Supp}E_{\rm D}; u^*_{\rm D}T\mathcal{D} \otimes \Lambda^{0,1}) \to L^2_m(\Sigma_{\rm D}; (u'_{\rm D})^*T\mathcal{D} \otimes \Lambda^{0,1}) \tag{4.14}$$

using the map $I^{\rm t}_{\rm D}$ and parallel transport with respect to the chosen unitary connection on $T\mathcal{D}$. We also define:

$$E_{\rm D}(u'_{\rm D}) = \mathcal{PAL}(E_{\rm D}). \tag{4.15}$$

Definition 4.16. We denote by $\mathcal{U}_{\rm D}$ the set of L^2_m maps $u'_{\rm D} : \Sigma_{\rm D} \to \mathcal{D}$ with the following properties:

(1) The C^2-distance between $u_{\rm D}$ and $u'_{\rm D}$ is less than ε.
(2) The equation
$$\overline{\partial}u'_{\rm D} \in E_{\rm D}(u'_{\rm D}) \tag{4.17}$$
is satisfied.
(3) $u'_{\rm D}(w_{\rm D}) \in \mathcal{N}_{\rm D}$.

We define $\mathcal{U}^+_{\rm D}$ to be the set of maps $u'_{\rm D}$ satisfying (1) and (2), but not necessarily (3).

We define maps:
$$\text{ev}_{\rm d} : \mathcal{U}_{\rm d} \to \mathcal{D}, \quad \text{ev}_{\rm s} : \mathcal{U}_{\rm s} \to \mathcal{D}, \quad (\text{ev}_{{\rm D},{\rm d}}, \text{ev}_{{\rm D},{\rm s}}) : \mathcal{U}_{\rm D} \to \mathcal{D} \times \mathcal{D},$$
by
$$\text{ev}_{\rm d}(u'_{\rm d}) := u'_{\rm d}(z_{\rm d}), \quad \text{ev}_{\rm s}(u'_{\rm s}) := u'_{\rm s}(z_{\rm s}),$$
$$\text{ev}_{\rm d}(u'_{\rm D}) := u'_{\rm D}(z_{\rm d}), \quad \text{ev}_{\rm s}(u'_{\rm D}) := u'_{\rm D}(z_{\rm s}).$$
We summarize their properties as follows.

Lemma 4.18. *If ε is small enough, then we have:*

(1) $\mathcal{U}_{\rm d}$*,* $\mathcal{U}_{\rm D}$*,* $\mathcal{U}_{\rm s}$ *are smooth manifolds.*
(2) *The maps* $\text{ev}_{\rm d}$*,* $\text{ev}_{{\rm D},{\rm d}}$*,* $\text{ev}_{{\rm D},{\rm s}}$*,* $\text{ev}_{\rm s}$ *are smooth.*

(3) *The fiber product*

$$\mathcal{U}_{\rm d} \;{}_{\rm ev_d}\times_{\rm ev_{D,d}} \mathcal{U}_{\rm D} \;{}_{\rm ev_{D,s}}\times_{\rm ev_s} \mathcal{U}_{\rm s} \tag{4.19}$$

is transversal.

Proof. Part (1) is a consequence of the implicit function theorem using the assumptions in Definition 4.1. Part (2) follows from the way we set up Fredholm theory. Part (3) follows from the surjectivity of the map (4.3). ☐

The fiber product (4.19) describes a Kuranishi neighborhood of any element $[\Sigma, z_0, u]$ of the stratum of $\mathcal{M}_1^{\rm RGW}(L;\beta)$, consisting of objects with the combinatorial data given in Section 2. Next, we include the gluing construction and construct a Kuranishi neighborhood of $[\Sigma, z_0, u]$ in the moduli space $\mathcal{M}_1^{\rm RGW}(L;\beta)$. Let D^2 be the unit disk in the complex plane and $D^2(r)$ denote $r \cdot D^2$. We fix coordinate charts:

$$\begin{aligned} &\varphi_{\rm d} : {\rm Int}(D^2) \to \Sigma_{\rm d}, && \varphi_{\rm D,d} : {\rm Int}(D^2) \to \Sigma_{\rm D}, \\ &\varphi_{\rm D,s} : {\rm Int}(D^2) \to \Sigma_{\rm D}, && \varphi_{\rm s} : {\rm Int}(D^2) \to \Sigma_{\rm s}, \end{aligned} \tag{4.20}$$

which are bi-holomorphic maps onto the image and $\varphi_{\rm d}(0) = z_{\rm d}$, $\varphi_{\rm D,d}(0) = z_{\rm D,d}$, $\varphi_{\rm D,s}(0) = z_{\rm D,s}$, $\varphi_{\rm s}(0) = z_{\rm s}$. We assume that the marked points $w_{\rm D}$, $w_{{\rm s},i}$ do not belong to the image of the above coordinate charts. Moreover we assume that the image of $\varphi_{\rm D,d}$ is disjoint from the image of $\varphi_{\rm D,s}$.

For $\sigma_1, \sigma_2 \in D^2\backslash\{0\}$, we form the disk $\Sigma(\sigma_1, \sigma_2)$ as follows. Consider the disjoint union:

$$\begin{aligned} &(\Sigma_{\rm d}\backslash\varphi_{\rm d}(D^2(|\sigma_1|))) \sqcup (\Sigma_{\rm D}\backslash(\varphi_{\rm D,d}(D^2(|\sigma_1|)) \cup \varphi_{\rm D,s}(D^2(|\sigma_2|))) \\ &\qquad \sqcup (\Sigma_{\rm s}\backslash\varphi_{\rm s}(D^2(|\sigma_2|))). \end{aligned} \tag{4.21}$$

and define the equivalence relation $\sim$ on (4.21) as follows:

(gl-i) If $z_1 z_2 = \sigma_1$, $z_1, z_2 \in D^2$, then $\varphi_{\rm d}(z_1) \sim \varphi_{\rm D,d}(z_2)$.
(gl-ii) If $z_1 z_2 = \sigma_2$, $z_1, z_2 \in D^2$, then $\varphi_{\rm s}(z_1) \sim \varphi_{\rm D,s}(z_2)$.

Then $\Sigma(\sigma_1, \sigma_2)$ is the quotient space of (4.21) by this equivalence relation. See Figure 3 below. The above definition can be extended to the case that σ_1 or σ_2 vanishes. For example, if $\sigma_2 = 0$, then (4.21) is replaced with:

$$(\Sigma_{\rm d}\backslash\varphi_{\rm d}(D^2(|\sigma_1|))) \sqcup (\Sigma_{\rm D}\backslash\varphi_{\rm D,d}(D^2(|\sigma_1|))) \sqcup \Sigma_{\rm s}. \tag{4.22}$$

where we use the identification in (gl-i), and the identification in (gl-ii) is replaced with $\varphi_{\rm s}(0) \sim \varphi_{\rm D,s}(0)$.

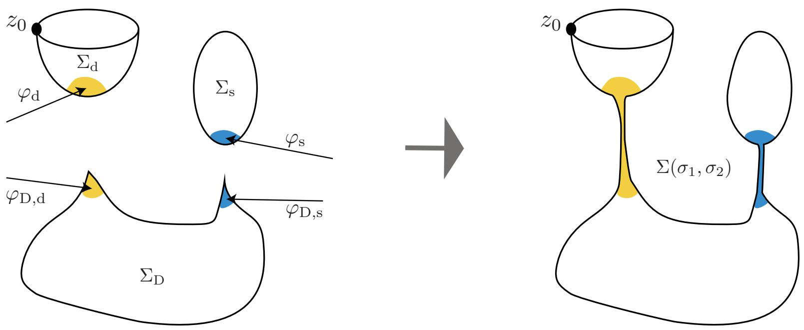

FIGURE 3. $\Sigma(\sigma_1, \sigma_2)$

We also define:

$$\begin{aligned}\Sigma_{\rm d}(\sigma_1) &= \Sigma_{\rm d}\backslash\varphi_{\rm d}(D^2(|\sigma_1|)),\\ \Sigma_{\rm s}(\sigma_2) &= \Sigma_{\rm s}\backslash\varphi_{\rm s}(D^2(|\sigma_2|)),\\ \Sigma_{\rm D}(\sigma_1,\sigma_2) &= \Sigma_{\rm D}\backslash(\varphi_{\rm D,d}(D^2(|\sigma_1|))\cup\varphi_{\rm D,s}(D^2(|\sigma_2|))).\end{aligned} \tag{4.23}$$

By construction, there exist bi-holomorphic embeddings:

$$\begin{aligned}&I_{\rm d}:\Sigma_{\rm d}(\sigma_1)\to\Sigma(\sigma_1,\sigma_2),\quad I_{\rm s}:\Sigma_{\rm s}(\sigma_2)\to\Sigma(\sigma_1,\sigma_2),\\ &I_{\rm D}:\Sigma_{\rm D}(\sigma_1,\sigma_2)\to\Sigma(\sigma_1,\sigma_2).\end{aligned} \tag{4.24}$$

Let $u'_{\rm d}:\Sigma_{\rm d}(\sigma_1)\to X\backslash\mathcal{D}$, $u'_{\rm s}:\Sigma_{\rm s}(\sigma_2)\to X\backslash\mathcal{D}$, $U'_{\rm D}:\Sigma_{\rm D}(\sigma_1,\sigma_2)\to\mathfrak{U}\subset X\backslash\mathcal{D}$ be L^2_{m+1} maps such that $u'_{\rm d}$, $u'_{\rm s}$, $u'_{\rm D}:=\pi\circ U'_{\rm D}$ are close to the restrictions of $u_{\rm d}$, $u_{\rm s}$, $u_{\rm D}$ in the same sense as in (4.5). We define:

$$\begin{aligned}E_{\rm d}(u'_{\rm d})&\subset L^2_m(\Sigma_{\rm d}(\sigma_1);(u'_{\rm d})^*TX\otimes\Lambda^{0,1})\\ E_{\rm s}(u'_{\rm s})&\subset L^2_m(\Sigma_{\rm s}(\sigma_2);(u'_{\rm s})^*TX\otimes\Lambda^{0,1})\end{aligned} \tag{4.25}$$

similar to (4.11), using target parallel transportations. Next, we define:

$$E_{\rm D}(U'_{\rm D})\subset L^2_m(\Sigma_{\rm D}(\sigma_1,\sigma_2);(U'_{\rm D})^*TX\otimes\Lambda^{0,1}) \tag{4.26}$$

as follows. Since $u'_{\rm D}$ is close to $u_{\rm D}$, we can use the same construction as in (4.15) to define:

$$E'_{\rm D}(u'_{\rm D})\subset L^2_m(\Sigma_{\rm D};(u'_{\rm D})^*T\mathcal{D}\otimes\Lambda^{0,1})$$

Then the decomposition in (3.5) allows us to define:

$$E_{\rm D}(U'_{\rm D})\subset L^2_m(\Sigma_{\rm D};(U'_{\rm D})^*TX\otimes\Lambda^{0,1}). \tag{4.27}$$

By construction, we have isomorphisms

$$\mathcal{P}_{\rm d}:E_{\rm d}(u'_{\rm d})\to E_{\rm d},\quad \mathcal{P}_{\rm s}:E_{\rm s}(u'_{\rm s})\to E_{\rm s},\quad \mathcal{P}_{\rm D}:E_{\rm D}(u'_{\rm D})\to E_{\rm D}. \tag{4.28}$$

Recall that we fixed a codimension 2 submanifold $\mathcal{N}_{\rm D}\subset\mathcal{D}$. We define $\widehat{\mathcal{N}}_{\rm D}\subset X$ to be its inverse image in the tubular neighborhood $\mathfrak{U}$ of $\mathcal{D}$ in X by the projection map π. In the following definition ε is the same constant as in Lemma 4.18. We may make this constant smaller as we move through the paper whenever it is necessary.

Definition 4.29. We denote by $\mathcal{U}_0$ the set of all triples (u',σ_1,σ_2) where $\sigma_1,\sigma_2\in D^2(\varepsilon)$. In the case that σ_1 and σ_2 are non-zero, (u',σ_1,σ_2) needs to satisfy the following properties:

(1) $u':\Sigma(\sigma_1,\sigma_2)\to X\backslash\mathcal{D}$ is a smooth map.
(2) Let:

$$u'_{\rm d}:=u'\circ I_{\rm d},\quad u'_{\rm s}:=u'\circ I_{\rm s},\quad U'_{\rm D}=u'\circ I_{\rm D}.$$

Then the C^2 distance of $u'_{\rm d}$ (resp. $u'_{\rm s}$) with the restriction of $u_{\rm d}$ (resp. $u_{\rm s}$) to $\Sigma_{\rm d}(\sigma_1)$ (resp. $\Sigma_{\rm s}(\sigma_2)$) is less than ε. The maps $U'_{\rm D}$ and $U_{\rm D}$ are also C^2-close to each other in the sense that the image of $U'_{\rm D}$ is contained in the open set $\mathfrak{U}$ and there is a constant r such that the C^2 distance of $U'_{\rm D}$ and ${\rm Dil}_r\circ U_{\rm D}$, restricted to $\Sigma_{\rm D}(\sigma_1,\sigma_2)$, is less than ε.[7]
(3) (Modified non-linear Cauchy-Riemann equation) $u'_{\rm d}$, $u'_{\rm s}$, $U'_{\rm D}$ satisfy the equations:

$$\overline\partial u'_{\rm d}\in E_{\rm d}(u'_{\rm d}),\quad \overline\partial u'_{\rm s}\in E_{\rm s}(u'_{\rm s}),\quad \overline\partial U'_{\rm D}\in E_{\rm D}(U'_{\rm D}). \tag{4.30}$$

[7]The C^2 distance in part (2) of the definition are defined with respect to the metric g on $X\backslash\mathcal{D}$ and the metric on $\mathcal{N}_{\mathcal{D}}(X)\backslash\mathcal{D}$ which has the form in (3.2).

(4) (Transversal constraints) We also require:

$$u'(w_{\rm D}) \in \widehat{\mathcal N}_{\rm D}, \quad u'(w_{{\rm s},1}) \in \mathcal N_{{\rm s},1}, \quad u'(w_{{\rm s},2}) \in \mathcal N_{{\rm s},2}. \tag{4.31}$$

Here we use $I_{\rm D}$, $I_{\rm s}$ to regard $w_{\rm D}$, $w_{{\rm s},i}$ as elements of $\Sigma(\sigma_1, \sigma_2)$. In the case that one of the constants σ_1 and σ_2 vanishes, the other one is also zero, and u' is an element of the fiber product (4.19).

One might hope that the intersection of the space $\mathcal U_0$ with $(\sigma_1, \sigma_2) = (\sigma_1^0, \sigma_2^0)$ for each given (σ_1^0, σ_2^0) is cut down transversely by (4.30) and (4.31), and hence the space $\mathcal U_0$ could be used to define a Kuranishi neighborhood of $[\Sigma, z_0, u]$ in $\mathcal M_1^{\rm RGW}(L;\beta)$. However, this naive expectation does not hold. Roughly speaking, if that would hold, then one should obtain a solution for any element of the fiber product (4.19) close to $[\Sigma, z_0, u]$ and any small values of σ_1, σ_2. On the other hand, as a consequence of [DF18a, Remark 4.69], the stratum in (4.19) has real codimension 2 in our case, which is a contradiction. Note that this is in contrast with the stable map compactification, where a fiber product of the form (4.19) has codimension 4. To resolve this issue, we introduce a space $\mathcal U$ larger than $\mathcal U_0$ such that $\mathcal U$ is a smooth manifold and $\mathcal U_0$ is cut out from $\mathcal U$ by an equation of the following form:

$$\sigma_1^{p_1} = c\sigma_2^{p_2}. \tag{4.32}$$

Here c is a complex valued function on $\mathcal U$ which never vanishes. The space $\mathcal U$ is realized as the moduli space of *inconsistent solutions*, which will be defined in the next section. Note that the set of solutions of (4.32) has a singularity at the locus $\sigma_1 = \sigma_2 = 0$.

5. Inconsistent Solutions and the Main Analytical Result

In this section, we discuss the main step where the construction of the Kuranishi chart in our situation is different from the case of the stable map compactification.

Definition 5.1. For $\sigma_1, \sigma_2 \in D^2(\varepsilon)$, an *inconsistent solution* is a 7-tuple

$$(u'_{\rm d}, u'_{\rm s}, U'_{\rm D}, \sigma_1, \sigma_2, \rho_1, \rho_2)$$

satisfying the following properties.

(1) $u'_{\rm d} : \Sigma_{\rm d}(\sigma_1) \to X\backslash\mathcal D$, $u'_{\rm s} : \Sigma_{\rm s}(\sigma_2) \to X\backslash\mathcal D$, $U'_{\rm D} : \Sigma_{\rm D}(\sigma_1,\sigma_2) \to \mathcal N_{\mathcal D}(X)\backslash\mathcal D$. The C^2 distances of $u'_{\rm d}$, $u'_{\rm s}$ and $U'_{\rm D}$ with $u_{\rm d}$, $u_{\rm s}$ and $U_{\rm D}$ are less than ε.[8]

(2) The following equations are satisfied:

$$\overline\partial u'_{\rm d} \in E_{\rm d}(u'_{\rm d}), \quad \overline\partial u'_{\rm s} \in E_{\rm s}(u'_{\rm s}), \quad \overline\partial U'_{\rm D} \in E_{\rm D}(U'_{\rm D}). \tag{5.2}$$

Here $E_{\rm d}(u'_{\rm d})$, $E_{\rm d}(u'_{\rm s})$ and $E_{\rm d}(U'_{\rm D})$ are defined as in (4.25) and (4.27) using target parallel transport.

(3) We require the following transversal constraints:

$$\pi \circ U'_{\rm D}(w_{\rm D}) \in \mathcal N_{\rm D}, \quad u'_{\rm s}(w_{{\rm s},1}) \in \mathcal N_{{\rm s},1}, \quad u'_{\rm s}(w_{{\rm s},2}) \in \mathcal N_{{\rm s},2}. \tag{5.3}$$

(4) Let $z_1, z_2 \in D^2$.

(a) If $z_1 z_2 = \sigma_1$, then:

$$u'_{\rm d}(\varphi_{\rm d}(z_1)) = ({\rm Dil}_{\rho_1} \circ U'_{\rm D})(\varphi_{{\rm D},1}(z_2)).$$

In particular, we assume that the left hand side is contained in the open neighborhood $\mathfrak U$ of $\mathcal D$.

(b) If $z_1 z_2 = \sigma_2$, then:

$$u'_{\rm s}(\varphi_{\rm s}(z_1)) = ({\rm Dil}_{\rho_2} \circ U'_{\rm D})(\varphi_{{\rm D},2}(z_2)).$$

[8]Here we use the same convention as in Definition 4.29 to defined the C^2-distances.

We say two inconsistent solutions $(u_{\rm s}^{(j)}, u_{\rm d}^{(j)}, U_{\rm D}^{(j)}, \sigma_1^{(j)}, \sigma_2^{(j)}, \rho_1^{(j)}, \rho_2^{(j)})$, $j = 1, 2$, are *equivalent* if the following holds:

(i) $u_{\rm d}^{(1)} = u_{\rm d}^{(2)}$, $u_{\rm s}^{(1)} = u_{\rm s}^{(2)}$, $\sigma_1^{(1)} = \sigma_1^{(2)}$, $\sigma_2^{(1)} = \sigma_2^{(2)}$.
(ii) There exists a nonzero complex number c such that:

$$U_{\rm D}^{(2)} = {\rm Dil}_{1/c} \circ U_{\rm D}^{(1)}, \qquad \rho_1^{(2)} = c\rho_1^{(1)}, \qquad \rho_2^{(2)} = c\rho_2^{(1)}.$$

We will write $\mathcal{U}$ for the set of all equivalence classes of inconsistent solutions.

Remark 5.4. In the above definition, we include the case that σ_1 or σ_2 is 0 in the following way:

(1) If $\sigma_1 = 0$ (resp. $\sigma_2 = 0$), then the condition (4) (a) (resp. (b)) is replaced by the condition that $u'_{\rm d}(\varphi_{\rm d}(0)) = \pi \circ U'_{\rm D}(\varphi_{{\rm D},1}(0))$ (resp. $u'_{\rm s}(\varphi_{\rm s}(0)) = \pi \circ U'_{\rm D}(\varphi_{{\rm D},1}(0))$);
(2) If $\sigma_1 = 0$ (resp. $\sigma_2 = 0$), then $\rho_1 = 0$ (resp. $\rho_2 = 0$).

In the case that exactly one of σ_1 and σ_2 is zero, the source curve $\Sigma(\sigma_1, \sigma_2)$ has only one node. Such source curves do not appear in $\mathcal{U}_0$. However, there are elements of this form in $\mathcal{U}$.

Below we state our main analytic results about $\mathcal{U}$:

Proposition 5.5. *If ε is small enough, then the moduli space $\mathcal{U}$ is a smooth manifold diffeomorphic to*[9]*:*

$$(\mathcal{U}_{\rm d}\ {}_{\rm ev_d}\times_{\rm ev_{D,d}} \mathcal{U}_{\rm D}\ {}_{\rm ev_{D,s}}\times_{\rm ev_s} \mathcal{U}_{\rm s}) \times D^2(\varepsilon) \times D^2(\varepsilon). \tag{5.6}$$

The diffeomorphism has the following properties:

(1) *This diffeomorphism identifies the projection to the factor $D^2(\varepsilon) \times D^2(\varepsilon)$ with:*

$$[u'_{\rm s}, u'_{\rm d}, u'_{\rm D}, \sigma_1, \sigma_2, \rho_1, \rho_2] \mapsto (\sigma_1, \sigma_2).$$

(2) *There exists $\hat\rho_i : \mathcal{U} \to \mathbb{C}$ such that any element q of $\mathcal{U}$ has a representative whose ρ_i component is equal to $\hat\rho_i(q)$. The functions $\hat\rho_i$ are smooth. Moreover, there exists a homeomorphism:*

$$\mathcal{U}_0 \cong \{\mathfrak{y} \in \mathcal{U} \mid \hat\rho_1(\mathfrak{y}) = \hat\rho_2(\mathfrak{y})\}. \tag{5.7}$$

This homeomorphism is given as follows. Let:

$$\mathfrak{y} = [u'_{\rm d}, u'_{\rm D}, u'_{\rm s}, \sigma_1, \sigma_2, \hat\rho_1, \hat\rho_2]$$

be an element of the right hand side of (5.7) *with $c = \hat\rho_1 = \hat\rho_2$. Then we can glue the three maps $u'_{\rm d}, u'_{\rm s}, {\rm Dil}_c \circ U'_{\rm D}$ as in* (gl-i),(gl-ii) *to obtain a map $u' : \Sigma(\sigma_1, \sigma_2) \to X$. This gives the desired element of the left hand side of* (5.7).

Remark 5.8. We can take our diffeomorphism so that its restriction to $(\mathcal{U}_{\rm d}\ {}_{\rm ev_d}\times_{\rm ev_{D,d}} \mathcal{U}_{\rm D}\ {}_{\rm ev_{D,s}}\times_{\rm ev_s} \mathcal{U}_{\rm s}) \times \{(0, 0)\}$ is the obvious one. We can also specify the choice of $\hat\rho_i$ in (2) above by requiring

$$\hat\rho_1 = \sigma_1^{p_1}. \tag{5.9}$$

From now on, we will take this choice unless otherwise mentioned explicitly. The proof we will give implies that:

$$\hat\rho_2(\mathfrak{y}) = f(\mathfrak{y})\sigma_2^{p_2},$$

where f is a nonzero smooth function.

[9]See Remark 5.15 for the definition of the smooth structure of $D^2(\varepsilon)$.

The next proposition is the exponential decay estimate similar to those in the case of the stable map compactification. (See [FOOO16a] for the detail of the proof of this exponential decay estimate in the case of the stable map compactification.) To state our exponential decay estimate, we need to introduce some notations. We define $T_i \in [0,\infty)$, $\theta_i \in \mathbb{R}/2\pi\sqrt{-1}\mathbb{Z}$ by the formula:

$$\sigma_i = \exp(-(T_i + \sqrt{-1}\theta_i)). \tag{5.10}$$

The exponential decay estimate is stated in terms of T_i and θ_i.

Let $\xi = (u^{\xi}_{\rm d}, u^{\xi}_{\rm D}, u^{\xi}_{\rm s})$ be an element of the fiber product (4.19). The triple $(\xi, \sigma_1 = \exp(-(T_1+\sqrt{-1}\theta_1)), \sigma_2 = \exp(-(T_2+\sqrt{-1}\theta_2)))$ determines an element of $\mathcal{U}$, which is denoted by:

$$\begin{aligned} &\mathfrak{x}(\xi, T_1, T_2, \theta_1, \theta_2) \\ &= (u^{\xi}_{\rm d}(T_1,T_2,\theta_1,\theta_2;\cdot), u^{\xi}_{\rm D}(T_1,T_2,\theta_1,\theta_2;\cdot), u^{\xi}_{\rm s}(T_1,T_2,\theta_1,\theta_2;\cdot), \\ &\quad \sigma_1, \sigma_2, \rho_1(\xi,T_1,T_2,\theta_1,\theta_2), \rho_2(\xi,T_1,T_2,\theta_1,\theta_2)) \end{aligned} \tag{5.11}$$

Here we fix the representative by requiring (5.9), namely, $\rho_1(\xi,T_1,T_2,\theta_1,\theta_2) = \sigma_1^2$. Let $\mathfrak{R}_2 \in [0,\infty)$, $\eta_2 \in \mathbb{R}/2\pi\mathbb{Z}$ be functions of ξ, $T_1,T_2,\theta_1,\theta_2$ given by

$$\rho_2(\xi,T_1,T_2,\theta_1,\theta_2) = \exp(-(\mathfrak{R}_2 + \sqrt{-1}\eta_2)).$$

Proposition 5.12. (1) *Let $u^{\xi}_{\circ}$ be one of $u^{\xi}_{\rm d}$, $u^{\xi}_{\rm s}$, $u^{\xi}_{\rm D}$. Then for any compact subset K of $\Sigma_{\rm s}\backslash\{z_{\rm s}\}$ (resp. $\Sigma_{\rm d}\backslash\{z_{\rm d}\}$, $\Sigma_{\rm D}\backslash\{z_{\rm D,s}, z_{\rm D,d}\}$) we have the following exponential decay estimates:*

$$\left\| \frac{\partial^{m_1}}{\partial T_1^{m_1}} \frac{\partial^{m'_1}}{\partial \theta_1^{m'_1}} \frac{\partial^{m_2}}{\partial T_2^{m_2}} \frac{\partial^{m'_2}}{\partial \theta_2^{m'_2}} u^{\xi}_{\circ} \right\|_{L^2_\ell(K)} \leqslant C\exp(-c\upsilon_1 T_1 - c\upsilon_2 T_2). \tag{5.13}$$

Here $\upsilon_1 = 0$ if $m_1 = m'_1 = 0$. Otherwise $\upsilon_1 = 1$. Similarly, υ_2 equals to 0 if $m_2 = m'_2 = 0$ and is equal to 1 otherwise. Here C, c are positive constants depending on K, ℓ, m_1, m'_1, m_2, m'_2. The same estimate holds for the derivatives of $u^{\xi}_{\circ}$ with respect to ξ.

(2) *We also have the following estimates*

$$\begin{aligned} \left| \frac{\partial^{m_1}}{\partial T_1^{m_1}} \frac{\partial^{m'_1}}{\partial \theta_1^{m'_1}} \frac{\partial^{m_2}}{\partial T_2^{m_2}} \frac{\partial^{m'_2}}{\partial \theta_2^{m'_2}} (\mathfrak{R}_2 - p_2 T_2) \right| &\leqslant C\exp(-c\upsilon_1 T_1 - c\upsilon_2 T_2) \\ \left| \frac{\partial^{m_1}}{\partial T_1^{m_1}} \frac{\partial^{m'_1}}{\partial \theta_1^{m'_1}} \frac{\partial^{m_2}}{\partial T_2^{m_2}} \frac{\partial^{m'_2}}{\partial \theta_2^{m'_2}} (\eta_2 - p_2 \theta_2) \right| &\leqslant C\exp(-c\upsilon_1 T_1 - c\upsilon_2 T_2). \end{aligned} \tag{5.14}$$

Here υ_1 and υ_2 are defined as in the first part. The same estimate holds for the derivatives of $\mathfrak{R}_2$, η_2 with respect to ξ.

We will discuss the proofs of Propositions 5.5, 5.12 in Section 7.

Remark 5.15. We follow [FOOO09, Subsection A1.4], [FOOO16a, Section 8], [FOOO18, Subsection 9.1] to use a smooth structure on D^2 different from the standard one as follows. For $z \in D^2$, let T, θ be defined by the following identity:

$$z = \exp(-(T + \sqrt{-1}\theta)).$$

We define a homeomorhism from a neighborhood of the origin in D^2 to D^2 as follows:

$$z \mapsto w = \frac{1}{T}\exp(-\sqrt{-1}\theta).$$

We define a smooth structure on D^2, temporarily denoted by $D^2_{\rm new}$, such that $z \mapsto w$ becomes a diffeomorphism from $D^2_{\rm new}$ to D^2 with the standard smooth structure. This new smooth structure D^2_{new} is used to define a smooth structure on the factors D^2 in (5.6). (We drop the term 'new' from D^2_{new} hereafter.) The Proposition 5.12 implies smoothness of various maps at the origin of D^2 with respect to the new smooth structure. See for example [FOOO12, Lemma 22.6], [FOOO16a, Subsection 8.2], [FOOO18, Section 10] for further discussions related to this point.

6. Kuranishi Charts: a Special Case

In this section we use Propositions 5.5 and 5.12 to obtain a Kuranishi chart at the point $[\Sigma, z_0, u] \in \mathcal{M}_1^{\mathrm{RGW}}(L;\beta)$. By definition, a Kuranishi chart of a point p in a space M consists of $(V_p, \Gamma_p, \mathcal{E}_p, \mathfrak{s}_p, \psi_p)$ where V_p, the *Kuranishi neighborhood*, is a smooth manifold containing a distinguished point $\tilde{p}$, Γ_p, the *isotropy group*, is a finite group acting on V_p, $\mathcal{E}_p$, the *obstruction bundle* is a vector bundle over V_p and $\mathfrak{s}_p$, the *Kuranishi map*, is a section of $\mathcal{E}_p$ over V. Moreover, the action of Γ_p at $\tilde{p}$ is trivial and the action of this group on V_p is lifted to $\mathcal{E}_p$. The section $\mathfrak{s}_p$ is Γ_p-equivariant and vanishes at $\tilde{p}$. Finally, ψ_p is a homeomorphism from $\mathfrak{s}_p^{-1}(0)/\Gamma_p$ to a neighborhood of $[\Sigma, z_0, u]$ in $\mathcal{M}_1^{\mathrm{RGW}}(L;\beta)$, which maps $\tilde{p}$ to p.

In the present case, we define the Kuranishi neighborhood to be the manifold $\mathcal{U}$ in Proposition 5.5, and define the isotropy group to be the trivial one. The obstruction bundle $\mathcal{E}$ on $\mathcal{U}$ is a trivial bundle whose fiber is

$$E_{\mathrm{d}} \oplus E_{\mathrm{D}} \oplus E_{\mathrm{s}} \oplus \mathbb{C}. \tag{6.1}$$

The Kuranishi map

$$\mathfrak{s} = (\mathfrak{s}_{\mathrm{d}}, \mathfrak{s}_{\mathrm{D}}, \mathfrak{s}_{\mathrm{s}}, \mathfrak{s}_{\rho}) : \mathcal{U} \to E_{\mathrm{d}} \oplus E_{\mathrm{D}} \oplus E_{\mathrm{s}} \oplus \mathbb{C}$$

is defined by

$$\begin{aligned}
\mathfrak{s}_{\mathrm{d}}(\xi, T_1, T_2, \theta_1, \theta_2) &= \mathcal{P}_{\mathrm{d}}(\overline{\partial} u^{\xi}_{\mathrm{d}}(T_1, T_2, \theta_1, \theta_2; \cdot))\\
\mathfrak{s}_{\mathrm{D}}(\xi, T_1, T_2, \theta_1, \theta_2) &= \mathcal{P}_{\mathrm{D}}(\overline{\partial} u^{\xi}_{\mathrm{D}}(T_1, T_2, \theta_1, \theta_2; \cdot))\\
\mathfrak{s}_{\mathrm{s}}(\xi, T_1, T_2, \theta_1, \theta_2) &= \mathcal{P}_{\mathrm{s}}(\overline{\partial} u^{\xi}_{\mathrm{s}}(T_1, T_2, \theta_1, \theta_2; \cdot))\\
\mathfrak{s}_{\rho}(\xi, T_1, T_2, \theta_1, \theta_2) &= \sigma_1^{p_1} - \hat{\rho}_2(\xi, T_1, T_2, \theta_1, \theta_2).
\end{aligned} \tag{6.2}$$

Here $\mathcal{P}_{\mathrm{s}}, \mathcal{P}_{\mathrm{D}}, \mathcal{P}_{\mathrm{d}}$ are as in (4.28). The maps u^{ξ}_{s}, u^{ξ}_{D}, u^{ξ}_{d} are as in (5.11). Therefore, $\overline{\partial} u^{\xi}_{\mathrm{s}}(T_1, T_2, \theta_1, \theta_2; \cdot) \in E_{\mathrm{s}}(u^{\xi}_{\mathrm{s}}(T_1, T_2, \theta_1, \theta_2; \cdot))$ is a consequence of (4.30). Since $E_{\mathrm{s}}(u^{\xi}_{\mathrm{s}}(T_1, T_2, \theta_1, \theta_2; \cdot))$ is in the domain of $\mathcal{P}_{\mathrm{s}}$, the first map is well-defined. Similarly, we can show that the second and the third maps are also well-defined. The last map is equivalent to $\hat{\rho}_1 - \hat{\rho}_2$, because of (5.9).

Lemma 6.3. *The map $\mathfrak{s}$ is smooth.*

Proof. Proposition 5.5 (1) implies that σ_1 is a smooth function. Proposition 5.5 (2) implies that $\hat{\rho}_2$ is a smooth function. Therefore $\mathfrak{s}_{\rho}$ is smooth. Smoothness of the maps $\mathfrak{s}_{\mathrm{s}}, \mathfrak{s}_{\mathrm{d}}, \mathfrak{s}_{\mathrm{D}}$ for non-zero values of σ_1 and σ_2 is a consequence of standard elliptic regularity. Smoothness for $\sigma_i = 0$ follows from part (1) of Proposition 5.12. For similar results in the context of the stable map compactification, see [FOOO12, Lemma 22.6], [FOOO16a, Theorem 8.25], [FOOO18, Proposition 10.4], [FOOO12, Section 26] and [FOOO18, Section 12]. The first three references concerns the C^m property of the relevant maps whereas the last two discuss smoothness. □

We finally construct the parametrization map

$$\psi : \mathfrak{s}^{-1}(0) \to \mathcal{M}_1^{\mathrm{RGW}}(L;\beta).$$

Let $\mathfrak{x} = [u'_{\mathrm{d}}, u'_{\mathrm{D}}, u'_{\mathrm{s}}, \sigma_1, \sigma_2, \rho_1, \rho_2] \in \mathcal{U}$ be an element such that $\mathfrak{s}(\mathfrak{x}) = 0$. Firstly, let σ_1 and σ_2 be both non-zero. Equation $\mathfrak{s}_{\rho}(\mathfrak{x}) = 0$ implies that $\rho_1 = \rho_2$. Therefore, we can glue $u'_{\rm d}, u'_{\rm D}, u'_{\rm s}$, as in Proposition 5.5 (2), to obtain $u' : \Sigma(\sigma_1, \sigma_2) \to X\backslash\mathcal{D}$. We use $\mathfrak{s}_{\rm d}(\mathfrak{x}) = \mathfrak{s}_{\rm D}(\mathfrak{x}) = \mathfrak{s}_{\rm s}(\mathfrak{x}) = 0$ to conclude that u' is J-holomorphic. We define $\psi(\mathfrak{x}) \in \mathcal{M}_1^{\rm RGW}(L;\beta)$ to be the element determined by u' and $z_0 \in \partial D_{\rm d} \subset \partial\Sigma(\sigma_1, \sigma_2)$. In the case that $\sigma_1 = 0$, ρ_1 vanishes by definition. Equation $\mathfrak{s}(\mathfrak{x}) = 0$ implies that ρ_2 is also zero. We can also conclude from Definition 5.1 that $\sigma_2 = 0$. Finally the first three equations in (6.2) imply that $\mathfrak{x}$ determines an element of $\mathcal{M}_1^{\rm RGW}(L;\beta)$ in the stratum described in Section 2. The case that $\sigma_2 = 0$ can be treated similarly. It is easy to see that ψ is a homeomorphism to a neighborhood of $[\Sigma, z_0, u]$ in $\mathcal{M}_1^{\rm RGW}(L;\beta)$. Given Propositions 5.5 and 5.12, we thus proved the following result:

Proposition 6.4. *$(\mathcal{U}, \mathcal{E}, \mathfrak{s}, \psi)$ provides a Kuranshi chart for the moduli space $\mathcal{M}_1^{\rm RGW}(L;\beta)$ at $[\Sigma, z_0, u]$.*

7. Proof of the Main Analytical Result

The purpose of this section is to prove Proposition 5.5. The proofs are similar to the arguments in [FOOO16a]. However, there is one novel point, which is related to the fact that we need the notion of inconsistent solutions. In this section, we go through the construction of the required family of inconsistent solutions, emphasizing on this novel point. Then the estimates claimed in Proposition 5.12 can be proved in the same way as in [FOOO16a, Section 6].

Throughout this section, we use a different convention for our figures to sketch pseudo-holomorphic curves in X. In our figures in this section (e.g. Figure 4), we regard the divisor $\mathcal{D}$ as a vertical line on the right. This is in contrast with our convention in Figure 3 and [DF18a], where we regard the divisor as a horizontal line on the bottom. Our new convention is more consistent with the previous literature, especially [FOOO16a].

7.1. **Cylindrical Coordinates.** In (4.20), we fix coordinate charts on $\Sigma_{\rm d}$, $\Sigma_{\rm s}$, $\Sigma_{\rm D}$ near the nodal points and parametrized by the disc ${\rm Int}(D^2)$. In this section, it is convenient to use a cylindrical coordinates on the domain of these coordinate charts. Thus we modify the definition of the maps in (4.20) as follows:

$$\begin{aligned} \varphi_{\rm d} &: [0,\infty) \times S^1 \to \Sigma_{\rm d}, & \varphi_{\rm D,d} &: (-\infty, 0] \times S^1 \to \Sigma_{\rm D}, \\ \varphi_{\rm s} &: [0,\infty) \times S^1 \to \Sigma_{\rm s}, & \varphi_{\rm D,s} &: (-\infty, 0] \times S^1 \to \Sigma_{\rm D}, \end{aligned}$$

where

$$\begin{aligned} &\varphi_{\rm d}(r'_1, s'_1), & &\varphi_{\rm D,d}(r''_1, s''_1), \\ &\varphi_{\rm s}(r'_2, s'_2), & &\varphi_{\rm D,s}(r''_2, s''_2), \end{aligned}$$

for $(r'_i, s'_i) \in [0,\infty) \times S^1$, $(r''_i, s''_i) \in (-\infty, 0] \times S^1$, is defined to be what we denoted by

$$\begin{aligned} &\varphi_{\rm d}(\exp(-(r'_1 + \sqrt{-1}s'_1))), & &\varphi_{\rm D,d}(\exp(r''_1 + \sqrt{-1}s''_1)), \\ &\varphi_{\rm s}(\exp(-(r'_2 + \sqrt{-1}s'_2))), & &\varphi_{\rm D,s}(\exp(r''_2 + \sqrt{-1}s''_2)), \end{aligned} \tag{7.1}$$

in Section 4.

The equations $z_1 z_2 = \sigma_1$ or $z_1 z_2 = \sigma_2$ appearing in (gl-i) and (gl-ii)[10] can be rewritten as:

$$\begin{aligned} r''_1 &= r'_1 - 10T_1, & s''_1 &= s'_1 - \theta_1, \\ r''_2 &= r'_2 - 10T_2, & s''_2 &= s'_2 - \theta_2, \end{aligned} \tag{7.2}$$

where[11]

$$\sigma_i = \exp(-(10T_i + \sqrt{-1}\theta_i)). \tag{7.3}$$

[10] See the discussion about the construction of $\Sigma(\sigma_1, \sigma_2)$ around (4.22).

[11] We use the coefficient 10 here to be consistent with [FOOO16a]. Otherwise, they are not essential.

We define

$$r_i = r'_i - 5T_i = r''_i + 5T_i, \quad s_i = s'_i - \theta_i/2 = s''_i + \theta_i/2. \tag{7.4}$$

We also slightly change our convention for the polar coordinate of ρ_i of Definition 5.1 $(i = 1, 2)$ and define $\mathfrak{R}_i$, η_i as follows:

$$\rho_i = \exp(-(10\mathfrak{R}_i + \sqrt{-1}\eta_i)).$$

See Figure 4 below and compare with [FOOO16a, (6.2) and (6.3)][12].

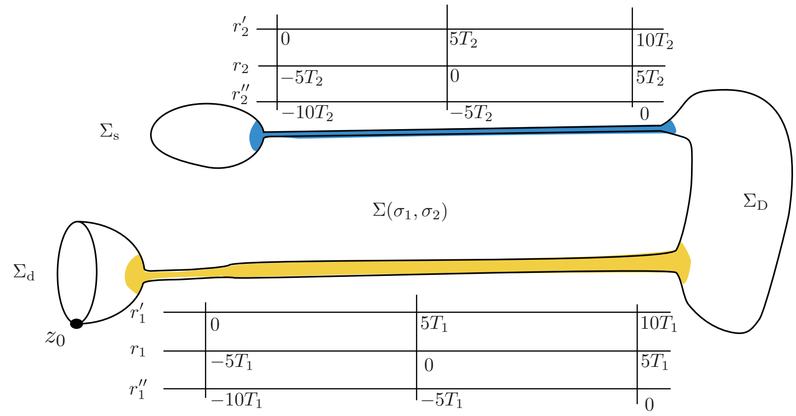

FIGURE 4. r_i, r'_i, r''_i

7.2. **Bump Functions.** For the purpose of constructing approximate solutions (pre-gluing) and for each step of the Newton's iteration used to solve our variant of non-linear Cauchy-Riemann equation, we use bump functions. Here we review various bump functions that we need. We may use the maps $\varphi_{\rm d}$, $\varphi_{\rm s}$, $\varphi_{\rm D,d}$ and $\varphi_{\rm D,s}$ to regard the following spaces as subspaces of $\Sigma(\sigma_1, \sigma_2)$:

$$\begin{aligned}
\mathcal{X}_{i,T_i} &= [-1,1]_{r_i} \times S^1_{s_i} = [5T_i - 1, 5T_i + 1]_{r'_i} \times S^1_{s'_i} \\
&\qquad\qquad\qquad\qquad = [-5T_i - 1, -5T_i + 1]_{r''_i} \times S^1_{s''_i}, \\
\mathcal{A}_{i,T_i} &= [-T_i - 1, -T_i + 1]_{r_i} \times S^1_{s_i} = [4T_i - 1, 4T_i + 1]_{r'_i} \times S^1_{s'_i} \\
&\qquad\qquad\qquad\qquad = [-6T_i - 1, -6T_i + 1]_{r''_i} \times S^1_{s''_i}, \\
\mathcal{B}_{i,T_i} &= [T_i - 1, T_i + 1]_{r_i} \times S^1_{s_i} = [6T_i - 1, 6T_i + 1]_{r'_i} \times S^1_{s'_i} \\
&\qquad\qquad\qquad\qquad = [-4T_i - 1, -4T_i + 1]_{r''_i} \times S^1_{s''_i}.
\end{aligned}$$

Using $\varphi_{\rm d}$ (resp. $\varphi_{\rm s}$), the spaces $\mathcal{X}_{1,T_1}$, $\mathcal{A}_{1,T_1}$, $\mathcal{B}_{1,T_1}$ (resp. $\mathcal{X}_{2,T_2}$, $\mathcal{A}_{2,T_2}$, $\mathcal{B}_{2,T_2}$) can be identified with subspaces of $\Sigma_{\rm d}\backslash\{z_{\rm d}\}$ (resp. $\Sigma_{\rm s}\backslash\{z_{\rm s}\}$). Similarly, the map $\varphi_{\rm D,d}$ (resp $\varphi_{\rm D,s}$) allows us to regard $\mathcal{X}_{1,T_1}$, $\mathcal{A}_{1,T_1}$, $\mathcal{B}_{1,T_1}$, $\mathcal{X}_{2,T_2}$, $\mathcal{A}_{2,T_2}$, $\mathcal{B}_{2,T_2}$ as subspaces of $\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}$. (See Figure 5 below.)

We fix a non-increasing smooth function $\chi : \mathbb{R} \to [0,1]$ such that

$$\chi(r) = \begin{cases} 1 & r < -1 \\ 0 & 1 < r, \end{cases}$$

[12] In [FOOO16a], the letter τ is used for the variables that we denote by r_i here. In this paper, we use τ to denote the $\mathbb{R}$ factor appearing in the target space.

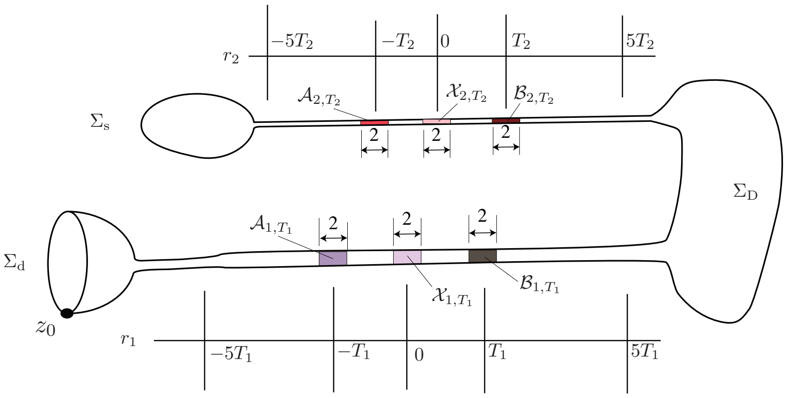

FIGURE 5. $\mathcal{X}_{i,T_i}$, $\mathcal{A}_{i,T_i}$, $\mathcal{B}_{i,T_i}$

and $\chi(-r) = 1 - \chi(r)$. We now define

$$\begin{aligned} \chi_{i,\mathcal{X}}^{\leftarrow}(r_i, s_i) &= \chi(r_i), & \chi_{i,\mathcal{X}}^{\rightarrow}(r_i, s_i) &= \chi(-r_i), \\ \chi_{i,\mathcal{A}}^{\leftarrow}(r_i, s_i) &= \chi(r_i + T_i), & \chi_{i,\mathcal{A}}^{\rightarrow}(r_i, s_i) &= \chi(-(r_i + T_i)), \\ \chi_{i,\mathcal{B}}^{\leftarrow}(r_i, s_i) &= \chi(r_i - T_i), & \chi_{i,\mathcal{B}}^{\rightarrow}(r_i, s_i) &= \chi(-(r_i - T_i)). \end{aligned} \tag{7.5}$$

The functions $\chi_{1,\mathcal{X}}^{\leftarrow}$, $\chi_{1,\mathcal{A}}^{\leftarrow}$ and $\chi_{1,\mathcal{B}}^{\leftarrow}$ can be extended to smooth functions on Σ_d which are locally constant outside of the spaces $\mathcal{X}_{1,T_1}$, $\mathcal{A}_{1,T_1}$ and $\mathcal{B}_{1,T_1}$, respectively. We use the same notations to denote these extensions. Similarly, we can define functions $\chi_{2,\mathcal{X}}^{\leftarrow}$, $\chi_{2,\mathcal{A}}^{\leftarrow}$ and $\chi_{2,\mathcal{B}}^{\leftarrow}$ on $\Sigma_{\rm s}$. These functions can be also regarded as functions defined on $\Sigma(\sigma_1, \sigma_2)$ in the obvious way.

We use $\chi_{i,\mathcal{X}}^{\rightarrow}$ (resp. $\chi_{i,\mathcal{A}}^{\rightarrow}$ and $\chi_{i,\mathcal{B}}^{\rightarrow}(r_i, s_i)$), for $i = 1, 2$, to define a smooth function $\chi_{\mathcal{X}}^{\rightarrow}$ (resp. $\chi_{\mathcal{A}}^{\rightarrow}$ and $\chi_{\mathcal{B}}^{\rightarrow}$) on $\Sigma(\sigma_1, \sigma_2)$ as follows. On the neck regions where the coordinate (r_i, s_i), for $i = 1$ or 2, is defined, we set $\chi_{\mathcal{X}}^{\rightarrow}$ (resp. $\chi_{\mathcal{A}}^{\rightarrow}$, $\chi_{\mathcal{B}}^{\rightarrow}$) to be the function $\chi_{i,\mathcal{X}}^{\rightarrow}(r_i, s_i)$ (resp. $\chi_{i,\mathcal{A}}^{\rightarrow}(r_i, s_i)$ and $\chi_{i,\mathcal{B}}^{\rightarrow}(r_i, s_i)$) given in (7.5). This function is defined to be locally constant on the complement of the above space. See Figures 6 and 7.

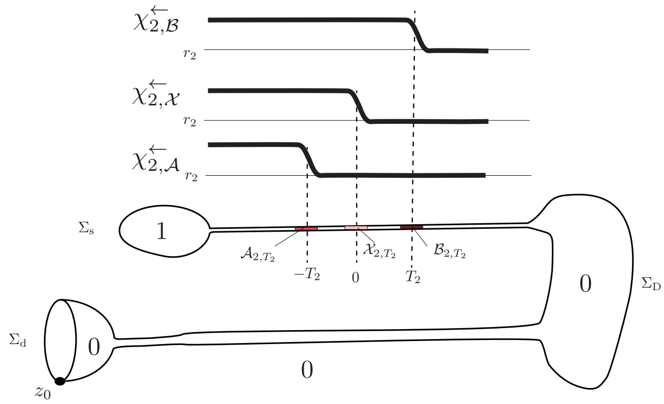

FIGURE 6. $\chi_{2,\mathcal{X}}^{\leftarrow}$, $\chi_{2,\mathcal{A}}^{\leftarrow}$, $\chi_{2,\mathcal{B}}^{\leftarrow}$

Note that the supports of the first derivatives of $\chi_{i,\mathcal{X}}^{\leftarrow}$, $\chi_{i,\mathcal{A}}^{\leftarrow}$, $\chi_{i,\mathcal{B}}^{\leftarrow}$ are subsets of $\mathcal{X}_{i,T_i}$, $\mathcal{A}_{i,T_i}$, $\mathcal{B}_{i,T_i}$, respectively. The supports of the first derivatives of $\chi_{\mathcal{X}}^{\rightarrow}$ $\chi_{\mathcal{A}}^{\rightarrow}$, $\chi_{\mathcal{B}}^{\rightarrow}$ are subsets of $\mathcal{X}_{1,T_1} \cup \mathcal{X}_{2,T_2}$, $\mathcal{A}_{1,T_1} \cup \mathcal{A}_{2,T_2}$, $\mathcal{B}_{1,T_1} \cup \mathcal{B}_{2,T_2}$, respectively.

7.3. **Weighted Sobolev Norms.** In Section 3, we define weighted Sobolev norms on several function spaces on $\Sigma_{\rm d}$, $\Sigma_{\rm s}$, $\Sigma_{\rm D}$. Here we use weighted Sobolev norms to define a

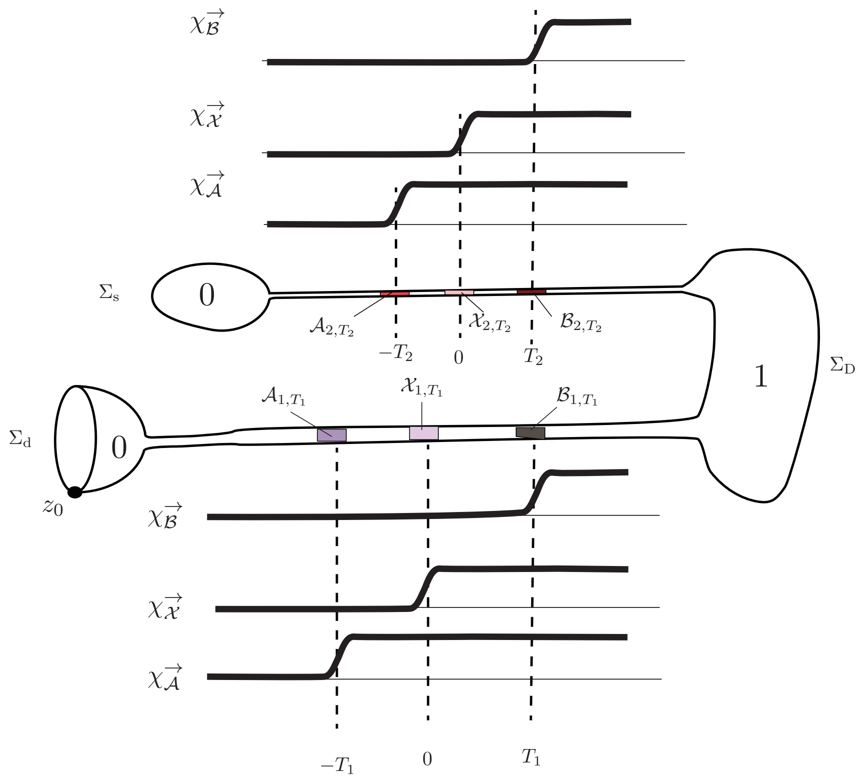

FIGURE 7. $\chi_{\vec{\mathcal{X}}}$ $\chi_{\vec{\mathcal{A}}}$, $\chi_{\vec{\mathcal{B}}}$

function space on $\Sigma(\sigma_1,\sigma_2)$. Since $\Sigma(\sigma_1,\sigma_2)$ is compact and the weight functions that we will define are smooth, the resulting weighted Sobolev norm is equivalent to the usual Sobolev norm. In other words, the ratio between the two norms is bounded as long as we fix σ_1,σ_2. However, the ratio depends on σ_1,σ_2 and is unbounded as σ_1,σ_2 goes to zero. Therefore, using weighted Sobolev norm is crucial to show that various estimates are independent of σ_1,σ_2.

We decompose $\Sigma(\sigma_1,\sigma_2)$ as follows:

$$\begin{aligned}\Sigma(\sigma_1,\sigma_2) =&(\Sigma_{\rm d}\backslash {\rm Im}\varphi_{\rm d}) \cup (\Sigma_{\rm s}\backslash {\rm Im}\varphi_{\rm s}) \cup (\Sigma_{\rm D}\backslash({\rm Im}\varphi_{\rm D,d} \cup {\rm Im}\varphi_{\rm D,s}))\\ &\cup ([-5T_1,5T_1]_{r_1}\times S^1_{s_1}) \cup ([-5T_2,5T_2]_{r_2}\times S^1_{s_2}).\end{aligned}$$

Here we identify $[-5T_1,5T_1]_{r_1}\times S^1_{s_1}$ and $[-5T_2,5T_2]_{r_2}\times S^1_{s_2}$ with their images in $\Sigma(\sigma_1,\sigma_2)$. We also introudce the following notations for various subspaces of $\Sigma(\sigma_1,\sigma_2)$: (See Figures 8, 9 and 10.)

$$\begin{aligned}\Sigma^-_{\rm d}(\sigma_1,\sigma_2) &= \Sigma_{\rm d}\backslash {\rm Im}\varphi_{\rm d},\\ \Sigma^+_{\rm d}(\sigma_1,\sigma_2) &= \Sigma_{\rm d}\backslash \varphi_{\rm d}(D^2(|\sigma_1|)),\\ \Sigma^-_{\rm s}(\sigma_1,\sigma_2) &= \Sigma_{\rm s}\backslash {\rm Im}\varphi_{\rm s},\\ \Sigma^+_{\rm s}(\sigma_1,\sigma_2) &= \Sigma_{\rm s}\backslash \varphi_{\rm s}(D^2(|\sigma_2|)),\\ \Sigma^-_{\rm D}(\sigma_1,\sigma_2) &= \Sigma_{\rm D}\backslash({\rm Im}\varphi_{\rm D,d} \cup {\rm Im}\varphi_{\rm D,s}),\\ \Sigma^+_{\rm D}(\sigma_1,\sigma_2) &= \Sigma_{\rm D}\backslash(\varphi_{\rm D,d}(|\sigma_1|) \cup {\rm Im}\varphi_{\rm D,s}(|\sigma_2|)).\end{aligned} \tag{7.6}$$

Note that $\Sigma^+_{\rm d}(\sigma_1,\sigma_2)$, $\Sigma^+_{\rm s}(\sigma_1,\sigma_2)$ and $\Sigma^+_{\rm D}(\sigma_1,\sigma_2)$ are respectively equal to the spaces $\Sigma_{\rm d}(\sigma_1)$, $\Sigma_{\rm s}(\sigma_2)$ and $\Sigma_{\rm D}(\sigma_1,\sigma_2)$ defined in (4.23).

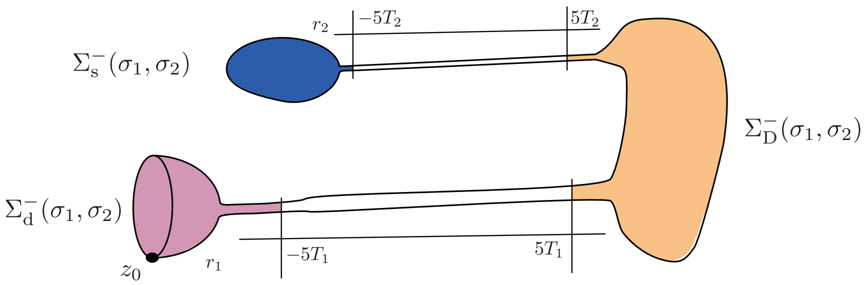

FIGURE 8. $\Sigma_{\mathrm{d}}^-(\sigma_1,\sigma_2)$, $\Sigma_{\mathrm{s}}^-(\sigma_1,\sigma_2)$, $\Sigma_{\mathrm{D}}^-(\sigma_1,\sigma_2)$

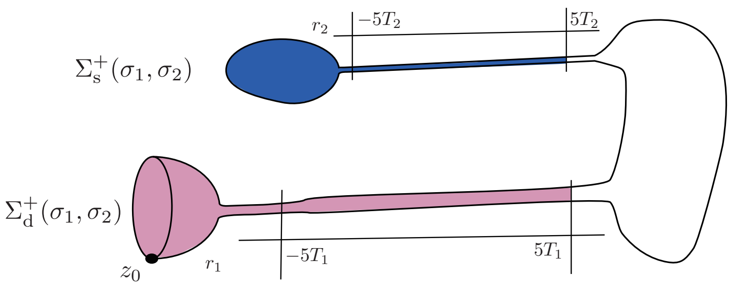

FIGURE 9. $\Sigma_{\mathrm{d}}^+(\sigma_1,\sigma_2)$, $\Sigma_{\mathrm{s}}^+(\sigma_1,\sigma_2)$

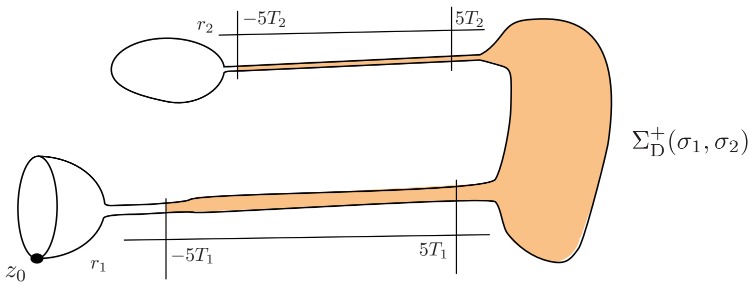

FIGURE 10. $\Sigma_{\mathrm{D}}^+(\sigma_1,\sigma_2)$

Lemma 7.7. *There exists a smooth function* $e_\delta^{\sigma_1,\sigma_2} : \Sigma(\sigma_1,\sigma_2) \to [0,\infty)$ *satisfying the following properties (see Figure 11):*

(i) *If* $x \in \Sigma_{\mathrm{d}}^-(\sigma_1,\sigma_2) \cup \Sigma_{\mathrm{s}}^-(\sigma_1,\sigma_2) \cup \Sigma_{\mathrm{D}}^-(\sigma_1,\sigma_2)$, *then* $e_\delta^{\sigma_1,\sigma_2}(x) = 1$;
(ii) *If* $r_i \in [1-5T_i,-1]$, *then* $e_\delta^{\sigma_1,\sigma_2}(r_i,s_i) = e^{\delta(r_i+5T_i)}$;
(iii) *If* $r_i \in [1,5T_i-1]$, *then* $e_\delta^{\sigma_1,\sigma_2}(r_i,s_i) = e^{\delta(-r_i+5T_i)}$;
(iv) *If* $||r_i|-5T_i| \leqslant 1$, *then* $e_\delta^{\sigma_1,\sigma_2}(r_i,s_i) \in [1,10]$;
(v) *If* $|r_i| \leqslant 1$, *then* $e_\delta^{\sigma_1,\sigma_2}(r_i,s_i) \in [e^{5T_i\delta}, 10e^{5T_i\delta}]$.

We fix a smooth map $u' : \Sigma(\sigma_1,\sigma_2) \to X\backslash\mathcal{D}$ and assume that the diameters of:

$$u'([-5T_1,5T_1]_{r_1} \times S^1_{s_1}) \qquad \text{and} \qquad u'([-5T_2,5T_2]_{r_2} \times S^1_{s_2}) \tag{7.8}$$

with respect to the metric g are less than a given positive real number κ. We require that the above sets are contained in $\mathfrak{U}$, introduced in the beginning of Section 2, where the partial $\mathbb{C}_*$-action is defined. Assuming κ is small enough, to any:

$$V \in L^2_m(\Sigma(\sigma_1,\sigma_2); u'^*TX).$$

we associate sections $\hat{v}_1$ and $\hat{v}_2$ of u'^*TX over the subspaces $[-5T_1,5T_1]_{r_1} \times S^1_{s_1}$ and $[-5T_2,5T_2]_{r_2} \times S^1_{s_2}$ in the following way.

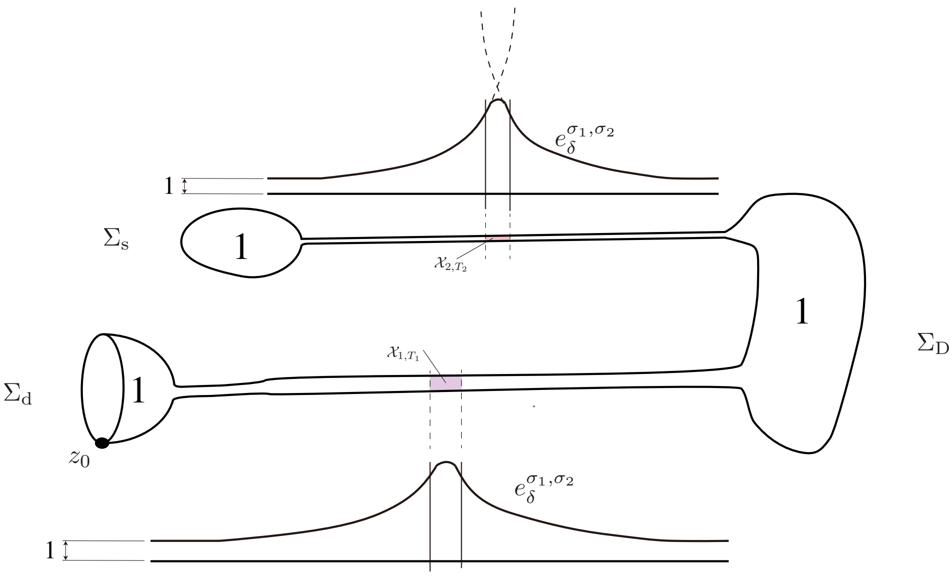

FIGURE 11. $e_\delta^{\sigma_1,\sigma_2}$

Let $(0,0)_i \in [-5T_i, 5T_i]_{r_i} \times S^1_{s_i}$ be the point whose r_i, s_i coordinates are 0. By choosing m to be greater than 1, the following vector is well-defined:

$$v_i = V((0,0)_i) \in T_{u'((0,0)_i)}X. \tag{7.9}$$

Suppose $v_i = v_{i,\mathbb{R}} + v_{i,S^1} + v_{i,\mathrm{D}}$ is the decomposition of this vector with respect to (3.5). If κ is small enough, we can assume that the distance between any two points of the projection of (7.8) to $\mathcal{D}$ is less than the injectivity radius of $\mathcal{D}$. In particular, we can extend $v_{i,\mathrm{D}}$ to a vector field $\hat{v}_{i,\mathrm{D}}$ in a neighborhood of $(0,0)_i$ using parallel transport along geodesics based at $u'((0,0)_i)$ with respect to the unitary connection on $T\mathcal{D}$, which we fixed before. Then the vector $\hat{v}_i$ is defined to be:

$$\hat{v}_i = v_{i,\mathbb{R}} + v_{i,S^1} + \hat{v}_{i,\mathrm{D}}. \tag{7.10}$$

Now we define

$$\begin{aligned}
&\|V\|^2_{W^2_{m,\delta}} \\
=&\|V\|^2_{L^2_m((\Sigma_{\mathrm{d}}\setminus \mathrm{Im}\varphi_{\mathrm{d}})\cup(\Sigma_{\mathrm{s}}\setminus \mathrm{Im}\varphi_{\mathrm{s}})\cup(\Sigma_{\mathrm{D}}\setminus(\mathrm{Im}\varphi_{\mathrm{D,d}}\cup \mathrm{Im}\varphi_{\mathrm{D,s}}))} \\
&+\sum_{i=1}^{2}\sum_{j=0}^{m}\int_{[-5T_i,5T_i]_{r_i}\times S^1_{s_i}} e^{\delta}_{\sigma_1,\sigma_2}(r_i,s_i)|\nabla^j(V-\hat{v}_i)|^2 dr_i ds_i \\
&+|v_1|^2+|v_2|^2.
\end{aligned} \tag{7.11}$$

We use the cylindrical metric on $\Sigma(\sigma_1,\sigma_2)$ and the metric g on $X\setminus\mathcal{D}$ to define norms in the first and the second lines of the right hand side of (7.11). This definition is analogous to (3.10). The space of all V as above with finite $\|\cdot\|_{W^2_{m,\delta}}$ norm which satisfies the boundary condition:

$$V(z) \in T_{u'(z)}L \qquad \forall z \in \partial\Sigma(\sigma_1,\sigma_2)$$

forms a Hilbert space, which we denoted by:

$$W^2_{m,\delta}((\Sigma(\sigma_1,\sigma_2),\partial\Sigma(\sigma_1,\sigma_2));(u'^*TX, u'|^*_\partial TL)). \tag{7.12}$$

Next, let:

$$V \in L^2_m(\Sigma(\sigma_1,\sigma_2); u'^*TX \otimes \Lambda^{0,1})$$

and define:

$$\|V\|^2_{L^2_{m,\delta}} = \sum_{j=0}^{m} \int_{\Sigma(\sigma_1,\sigma_2)} e^{\delta}_{\sigma_1,\sigma_2}(z)|\nabla^j V(z)|^2 \mathrm{vol}_{\Sigma(\sigma_1,\sigma_2)}. \tag{7.13}$$

We use the cylindrical metric on $\Sigma(\sigma_1, \sigma_2)$ and the metric g on $X\backslash\mathcal{D}$ to define the norm and the volume element $\mathrm{vol}_{\Sigma(\sigma_1,\sigma_2)}$. The set of all such V with $\|V\|_{L^2_{m,\delta}} < \infty$ forms a Hilbert space, which we denote by

$$L^2_{m,\delta}(\Sigma(\sigma_1, \sigma_2); u'^*TX \otimes \Lambda^{0,1}). \tag{7.14}$$

As a topological vector space, this is the same space as the standard space of Sobolev L^2_m sections. However, the ratio between the above $L^2_{m,\delta}$ norm and the standard Sobolev L^2_m norm is unbounded while σ_1, σ_2 go to 0.

Finally, we can use the above Sobolev spaces, to define the linearization of the non-linear Cauchy-Riemann equation at u', which is a Fredholm operator:

$$\begin{aligned} D_{u'}\overline{\partial} :& W^2_{m+1,\delta}((\Sigma(\sigma_1, \sigma_2), \partial\Sigma(\sigma_1, \sigma_2)); (u'^*TX, u'|^*_{\partial}TL)) \\ & \to L^2_{m,\delta}(\Sigma(\sigma_1, \sigma_2); u'^*TX \otimes \Lambda^{0,1}). \end{aligned} \tag{7.15}$$

7.4. **Pre-gluing.** Suppose $\xi = (u^{\xi}_{\mathrm{d}}, u^{\xi}_{\mathrm{D}}, u^{\xi}_{\mathrm{s}})$ is an element of the following space[13]:

$$\mathcal{U}_{\mathrm{d}} \ {}_{\mathrm{ev_d}}\times_{\mathrm{ev_{D,d}}} \mathcal{U}^+_{\mathrm{D}} \ {}_{\mathrm{ev_{D,s}}}\times_{\mathrm{ev_s}} \mathcal{U}^+_{\mathrm{s}} \tag{7.16}$$

In this subsection, for each choice of σ_1 and σ_2, we shall construct an approximate inconsistent solution and approximate the error for this approximate solution.

By assumption, the pull back bundle $(u^{\xi}_{\mathrm{D}})^*\mathcal{N}_{\mathcal{D}}(X)$ has a meromorphic section $\mathfrak{s}^{\xi}$ which has poles of order p_1 and p_2 at z_{d} and z_{s}, respectively. As in (2.1), $\mathfrak{s}^{\xi}$ gives rise to a map

$$U^{\xi}_{\mathrm{D}} : \Sigma_{\mathrm{D}}\backslash\{z_{\mathrm{d}}, z_{\mathrm{s}}\} \to \mathcal{N}_{\mathcal{D}}(X)\backslash\mathcal{D}. \tag{7.17}$$

A priori, the section $\mathfrak{s}^{\xi}$ is well-defined up to the action of $\mathbb{C}_*$ and for each ξ in (7.16), we fix one such section such that U^{ξ}_{D} depends smoothly on ξ. Later we will pin down the choice of sections such that (5.9) is satisfied. Recall that a neighborhood of the zero section in $\mathcal{N}_{\mathcal{D}}(X)$ is identified with the neighborhood $\mathfrak{U}$ of $\mathcal{D}$ in X. For now, we assume that the section $\mathfrak{s}^{\xi}$ is chosen such that the image of U^{ξ}_{D} on the domain $\Sigma^+_{\mathrm{D}}(\sigma_1, \sigma_2)$ belongs to this neighborhood of the zero section of $\mathcal{N}_{\mathcal{D}}(X)$. Recall that $\Sigma^+_{\mathrm{D}}(\sigma_1, \sigma_2)$ is defined in (7.6).

Next, we shall glue the three maps u^{ξ}_{d}, u^{ξ}_{s}, U^{ξ}_{D} by a partition of unity. One should beware that the output of this construction is an approximate inconsistent solution. In particular, it will not be a globally well-defined map from $\Sigma(\sigma_1, \sigma_2)$ to X. In order to describe this process, we need to fix an *exponential* map.

In the following, we need a map

$$\mathrm{Exp} : T(X\backslash\mathcal{D}) \to \mathfrak{N}(\Delta) \tag{7.18}$$

for a neighborhood $\mathfrak{N}(\Delta)$ of the diagonal Δ in $(X\backslash\mathcal{D}) \times (X\backslash\mathcal{D})$ that satisfies certain properties. Before stating the required properties for this map, we need to define partial $\mathbb{C}_*$ actions on a pair of a complex manifold and a submanifold. Recall that we defined partial $\mathbb{C}_*$ actions for a pair of an almost complex manifold Y and a submanifold D of (complex) codimension 1 in [DF18a, Subsection 3.2]. This notion can be generalized to the case of complex submanifolds of arbitrary codimension in an obvious way. For

[13] Recall that Σ_{d} together with the marked points z_0 and z_{d} is already source stable and we did not need to introduce auxiliary marked points on this space. This is the reason that the first factor is $\mathcal{U}_{\mathrm{d}}$, rather than $\mathcal{U}^+_{\mathrm{d}}$.

example, the derivative of the partial $\mathbb{C}_*$ action for the pair $(X, \mathcal{D})$ determines a partial $\mathbb{C}_*$ action for the pair $(TX, T\mathcal{D})$. Moreover, the product of two copies of partial $\mathbb{C}_*$ actions for the pair $(X, \mathcal{D})$ induces a partial $\mathbb{C}_*$ action on $(X \times X, \mathcal{D} \times \mathcal{D})$. Now we are ready to state the properties of Exp:

(i) For $p \in X\backslash\mathcal{D}$ and $V \in T_p(X\backslash\mathcal{D})$, the first component of $\mathrm{Exp}(p, V)$ is p.
(ii) Exp maps $(p, 0) \in T_p(X\backslash\mathcal{D})$ to $(p, p) \in (X\backslash\mathcal{D}) \times (X\backslash\mathcal{D})$. Moreover, at the point $(p, 0)$, the derivative of Exp in the fiber direction given by $T_p(X\backslash\mathcal{D}) \subset T(X\backslash\mathcal{D})$ is equal to $(0, \mathrm{id})$ where id is the identity map from $T_p(X\backslash\mathcal{D})$ to itself.
(iii) The map (7.18) is equivariant with respect to the partial $\mathbb{C}_*$ actions on the domain and the target defined above.
(iv) For a positive real number κ, let $D_\kappa TL$ denote the tangent vectors to L whose norms are smaller than κ. There is κ such that:

$$\mathrm{Exp}(D_\kappa TL) \subset L \times L.$$

Let exp be the exponential map with respect to the metric g. The map $(\mathrm{id}, \exp)$, defined on a neighborhood of the zero section of $T(X\backslash\mathcal{D})$, satisfies (i)-(iii). We can modify this map and extend it to a map on $T(X\backslash\mathcal{D})$ which satisfies (iv). We denote the inverse of (7.18) by

$$\mathrm{E} : \mathfrak{N}(\Delta) \to T(X\backslash\mathcal{D}).$$

We now define $\rho^{\xi}_{i,(0)} \in \mathbb{C}_*$ $(i = 1, 2)$ as follows. We define the compositions

$$u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}} : D^2 \to X\backslash\mathcal{D}, \qquad \pi \circ u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}} : D^2 \to \mathcal{D}.$$

We take a (holomorphic) trivialization $\Pi : (\pi \circ u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}})^* \mathcal{N}_{\mathcal{D}}(X) \to \mathbb{C}$ of the pullback of the normal bundle $\mathcal{N}_{\mathcal{D}}(X)$ in a neighborhood of z_{d}. Note that $u^{\xi}_{\mathrm{d}}(z_{\mathrm{d}}) \in \mathcal{D}$ is in a small neighborhood of $u^{0}_{\mathrm{d}}(z_{\mathrm{d}})$. Therefore, $u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}}$ induces a holomorphic function

$$\Pi \circ u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}} : D^2(o) \to \mathbb{C}$$

for a small $o > 0$. By assumption $\Pi \circ u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}}$ has a zero of order p_1 at z_{d}. We define $c^{\xi}_{\mathrm{d}} \in \mathbb{C}_*$ by

$$(\Pi \circ u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}})(z) = c^{\xi}_{\mathrm{d}} z^{p_1} + f(z) z^{p_1+1} \tag{7.19}$$

where $f(z)$ is holomorphic at 0.

Using the trivialization Π, we may regard the meromorphic section $\mathfrak{s}^{\xi} \circ \varphi_{\mathrm{d,D}}$ as a meromorphic *function* which has a pole of order 2 at z_{d}. In particular, there is a constant $c^{\xi}_{\mathrm{D,d}} \in \mathbb{C}_*$ such that $\Pi \circ \mathfrak{s}^{\xi} \circ \varphi_{\mathrm{d,D}} : D^2(o)\backslash\{0\} \to \mathbb{C}$ has the following form:

$$(\Pi \circ \mathfrak{s}^{\xi} \circ \varphi_{\mathrm{D,d}})(w) = c^{\xi}_{\mathrm{D,d}} w^{-p_1} + g(w) w^{-p_1+1}, \tag{7.20}$$

where g is holomorphic at 0. We now define:

$$\rho^{\xi}_{1,(0)}(\sigma_1, \sigma_2) = \frac{c^{\xi}_{\mathrm{d}} \sigma_1^{p_1}}{c^{\xi}_{\mathrm{D,d}}}. \tag{7.21}$$

Note that $\rho^{\xi}_{1,(0)}$ is independent of the choice of the trivialization of $\mathcal{N}_{\mathcal{D}}(X)$, because an alternative choice affects the numerator and the denominator of the right hand side by multiplying with the same number. The constant $\rho^{\xi}_{1,(0)}$ has the property that if $zw = \sigma_1$, then:

$$(u^{\xi}_{\mathrm{d}} \circ \varphi_{\mathrm{d}})(z) \sim (\mathrm{Dil}_{\rho^{\xi}_{1,(0)}} \circ U^{\xi}_{\mathrm{D}} \circ \varphi_{\mathrm{D,d}})(w) \tag{7.22}$$

where $\sim$ means the coincidence of the lowest order term.

We define $\rho^{\xi}_{2,(0)}$ in a similar way using the behavior of $u_{\rm s}$ and $u_{\rm D,s}$ in a neighborhood of $z_{\rm s}$. Namely, we replace (7.19) and (7.20) by:

$$(\Pi \circ u^{\xi}_{\rm s} \circ \varphi_{\rm s})(z) = c^{\xi}_{\rm s} z^{p_2} + f(z) z^{p_2+1}, \tag{7.23}$$

$$(\Pi \circ \mathfrak{s}^{\xi} \circ \varphi_{\rm D,s})(w) = c^{\xi}_{\rm D,s} w^{-p_2} + g(w) w^{-p_2+1}, \tag{7.24}$$

respectively and define:

$$\rho^{\xi}_{2,(0)}(\sigma_1,\sigma_2) = \frac{c^{\xi}_{\rm s}\sigma_2^{p_2}}{c^{\xi}_{\rm D,s}}. \tag{7.25}$$

Now we define a map

$$u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)} : \Sigma(\sigma_1,\sigma_2) \to X$$

as follows. Roughly speaking, $u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}$ is obtained by gluing the three maps $u^{\xi}_{\rm d}$, $u^{\xi}_{\rm s}$, $\mathrm{Dil}_{\rho^{\xi}_{i,(0)}} \circ U^{\xi}_{\rm D}$, using bump functions $\chi^{\leftarrow}_{i,\mathcal{X}}$, $\chi^{\rightarrow}_{\mathcal{X}}$. From now on, we write $\rho^{\xi}_{i,(0)}$ instead of $\rho^{\xi}_{i,(0)}(\sigma_1,\sigma_2)$ when the dependence on σ_i is clear.

Definition 7.26. (1) If $z \in \Sigma^{-}_{\rm d}(\sigma_1,\sigma_2)$, then:

$$u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)}(z) = u'^{,\xi,2}_{\sigma_1,\sigma_2,(0)}(z) = u^{\xi}_{\rm d}(z).$$

(2) If $z \in \Sigma^{-}_{\rm s}(\sigma_1,\sigma_2)$, then

$$u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)}(z) = u'^{,\xi,2}_{\sigma_1,\sigma_2,(0)}(z) = u^{\xi}_{\rm s}(z).$$

(3) If $z \in \Sigma^{-}_{\rm D}(\sigma_1,\sigma_2)$, then:

$$u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}(z) = (\mathrm{Dil}_{\rho^{\xi}_{i,(0)}} \circ U^{\xi}_{\rm D})(z)$$

for $i = 1,2$.

(4) Suppose $z = (r_1,s_1) \in [-5T_1,5T_1]_{r_1} \times S^1_{s_1}$. We define

$$u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}(z) = \mathrm{Exp}_2\left(u^{\xi}_{\rm d}(z), \chi^{\rightarrow}_{\mathcal{X}}(z)\mathrm{E}(u^{\xi}_{\rm d}(z), (\mathrm{Dil}_{\rho^{\xi}_{i,(0)}} \circ U^{\xi}_{\rm D})(z))\right).$$

Here Exp_2 denotes the composition of Exp and projection map from $(X\backslash\mathcal{D}) \times (X\backslash\mathcal{D})$ to the second factor.

(5) Suppose $z = (r_2,s_2) \in [-5T_2,5T_2]_{r_2} \times S^1_{s_2}$. We define

$$u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}(z) = \mathrm{Exp}_2\left(u^{\xi}_{\rm s}(z), \chi^{\rightarrow}_{\mathcal{X}}(z)\mathrm{E}(u^{\xi}_{\rm s}(z), (\mathrm{Dil}_{\rho^{\xi}_{i,(0)}} \circ U^{\xi}_{\rm D})(z))\right).$$

Remark 7.27. In part (4), if r_1 is close to $-5T_1$, then the right hand side is $u^{\xi}_{\rm d}$, and if r_1 is close to $5T_1$ then the right hand side is $\mathrm{Dil}_{\rho^{\xi}_{i,(0)}} \circ U^{\xi}_{\rm D}$. A similar property holds for the definition in part (5). In particular, our definition is well-defined.

(Step 0-3) (Error estimate) [14]

The next lemma provides an estimate of $\overline{\partial} u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}$ modulo the obstruction space

$$(E_{\rm d} \oplus E_{\rm s} \oplus E_{\rm D})(u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}).$$

[14] The enumeration of the steps of this paper is the same as those in [FOOO09, Section A1.4] and [FOOO16a].

In the case that $\rho^{\xi}_{1,(0)} \neq \rho^{\xi}_{2,(0)}$, we need to restrict the domain in the following way to obtain an appropriate estimate. We put

$$\Sigma(\sigma_1,\sigma_2)^-_i = \begin{cases} \Sigma(\sigma_1,\sigma_2)\backslash([-5T_2,5T_2]_{r_2}\times S^1_{s_2}) & \text{if } i=1,\\ \Sigma(\sigma_1,\sigma_2)\backslash([-5T_1,5T_1]_{r_1}\times S^1_{s_1}) & \text{if } i=2. \end{cases} \tag{7.28}$$

We consider the $L^2_{m,\delta}$ norm of the restriction of maps to $\Sigma(\sigma_1,\sigma_2)^-_i$ and denote it by $L^{2,i,-}_{m,\delta}$.

Lemma 7.29. *There exist constants δ_1, C_m (for any integer m) and vectors $\mathfrak{e}^{\xi}_{\mathrm{d},(0)} \in E_{\mathrm{d}}(u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)})$, $\mathfrak{e}^{\xi}_{\mathrm{s},(0)} \in E_{\mathrm{s}}(u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)})$, $\mathfrak{e}^{\xi,i}_{\mathrm{D},(0)} \in E_{\mathrm{D}}(u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)})$ such that δ_1, C_m are independent of σ_1, σ_2, ξ, and we have the following inequalities:*

(1)
$$\|\overline{\partial}u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{\mathrm{d},(0)} - \mathfrak{e}^{\xi}_{\mathrm{s},(0)} - \mathfrak{e}^{\xi}_{\mathrm{D},(0)}\|_{L^{2,1,-}_{m,\delta}} \leqslant C_m e^{-\delta_1 T_1}.$$

(2)
$$\|\overline{\partial}u'^{,\xi,2}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{\mathrm{d},(0)} - \mathfrak{e}^{\xi}_{\mathrm{s},(0)} - \mathfrak{e}^{\xi}_{\mathrm{D},(0)}\|_{L^{2,2,-}_{m,\delta}} \leqslant C_m e^{-\delta_1 T_2}.$$

We can be more specific about the value of the constant δ_1 as in (3.7) and (3.17). However, the actual choices do not matter for the details of our construction. So we do not give an exact value for this constant.

Proof. We define:

$$\begin{aligned} \mathfrak{e}^{\xi}_{\mathrm{d},(0)} &:= \overline{\partial}u^{\xi}_{\mathrm{d}} \in E_{\mathrm{d}}(u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}),\\ \mathfrak{e}^{\xi}_{\mathrm{s},(0)} &:= \overline{\partial}u^{\xi}_{\mathrm{s}} \in E_{\mathrm{s}}(u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}),\\ \mathfrak{e}^{\xi,i}_{\mathrm{D},(0)} &:= \overline{\partial}(\mathrm{Dil}_{\rho^{\xi}_{i,(0)}} \circ U^{\xi}_{\mathrm{D}}) \in E_{\mathrm{D}}(u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}). \end{aligned} \tag{7.30}$$

Then by construction the support of $\overline{\partial}u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{\mathrm{d},(0)} - \mathfrak{e}^{\xi}_{\mathrm{s},(0)} - \mathfrak{e}^{\xi}_{\mathrm{D},(0)}$ is contained in $([-5T_1,5T_1]_{r_1}\times S^1_{s_1}) \cup ([-5T_2,5T_2]_{r_2}\times S^1_{s_2})$. Therefore, it suffices to estimate $\overline{\partial}u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}$ on $[-5T_i,5T_i]_{r_i}\times S^i_{s_i}$. Below we discuss the case $i=1$. The other case is similar.

Let $z=\varphi_{\mathrm{d}}(r'_1,s'_1)$ be the coordinate on Σ_{d} used to denote points in a neighborhood of z_{d} and $w=\varphi_{\mathrm{D,d}}(r''_1,s''_1)$ be the coordinate on Σ_{D} used to denote points in a neighborhood of z_{d}. In order to obtain $\Sigma(\sigma_1,\sigma_2)$, the equation:

$$zw = \sigma_1$$

is used to glue Σ_{d} and Σ_{D}. Note that the supports of the derivatives of the bump functions $\chi^{\leftarrow}_{1,\mathcal{X}}$, $\chi^{\rightarrow}_{\mathcal{X}}$ are in $\mathcal{X}_{1,T_1}$. (Here we look at the restriction of the function $\chi^{\rightarrow}_{\mathcal{X}}$ to $\Sigma(\sigma_1,\sigma_2)^-_i$. Otherwise, part of the support of the derivate of this function is contained in $\mathcal{X}_{2,T_2}$.) Therefore, the support of $\overline{\partial}u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}$ is contained in the same subspace.

Firstly we wish to show that the maps $f_1 := u^{\xi}_{\mathrm{d}}$ and $f_2 := \mathrm{Dil}_{\rho^{\xi}_{1,(0)}} \circ U^{\xi}_{\mathrm{D}}$, as maps from from $\mathcal{X}_{1,T_1}$ to $\mathfrak{U}\backslash\mathcal{D} \subset X\backslash\mathcal{D}$, are close to each other in the C^m metric. In fact, analogues of the inequalities in (3.7) and (3.17) show that there are constants C'_m and δ_0 independent of σ_1, σ_2 and ξ such that:

$$d_{C^m}(f_1,f_2) \leqslant C'_m e^{-5\delta_0 T_1} \tag{7.31}$$

where d_{C^m} is computed with respect to the cylindrical metric g. To be a bit more detailed, this inequality holds because the leading terms of f_1 and f_2 agree with each other, and f_1 and f_2 are both holomorphic.

Let h_1, $h_2 : \mathcal{X}_{1,T_1} \to \mathfrak{U}$ be maps such that their C^0 distance is less than or equal to a constant κ. If κ is small enough, then the following map is well defined:

$$F(h_1, h_2) = \mathrm{Exp}_2\left(h_1, \chi_{\vec{\mathcal{X}}} \cdot \mathrm{E}(h_1, h_2)\right).$$

Clearly there is a constant K such that:

$$\|\overline{\partial} F(h_1, h_2) - \overline{\partial} F(h_1, h_2')\|_{L^2_m} \leqslant K \cdot d_{C^{m+1}}(h_2, h_2')$$

Since $F(f_1, f_1) = f_1$, the above inequality together with (7.31) implies that there is a constant C_m such that:

$$\|\overline{\partial} u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)}\|_{L^2_m(\mathcal{X}_{1,T_1})} \leqslant C_m e^{-5\delta_0 T_1}$$

Therefore, if we pick δ and δ_1 such that $\delta + \delta_1 < 5\delta_0$, then the desired inequality holds. □

7.5. **Why Inconsistent Solutions?** We already hinted at the necessity of inconsistent solutions at the end of Section 4. In this section we elaborate on this point with an eye toward modifying the approximate solution of the previous section to a solution. We firstly sketch our approach for this modification which is based on *Newton's iteration method*. Next, we explain the main point where the proof in the case of the RGW compactification diverges from the case of the stable map compactification. The discussion of this subsection is informal, and the actual proof will be carried out in the next two subsections.

Suppose $u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)}$ and $u'^{,\xi,2}_{\sigma_1,\sigma_2,(0)}$ are the approximate solutions of the previous subsection associated to the element $\xi = (u^\xi_{\mathrm{d}}, u^\xi_{\mathrm{D}}, u^\xi_{\mathrm{s}})$ of (7.16). We assume that σ_1 and σ_2 are chosen such that $\rho^\xi_{1,(0)}(\sigma_1, \sigma_2) = \rho^\xi_{2,(0)}(\sigma_1, \sigma_2)$. In particular, $u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)} = u'^{,\xi,2}_{\sigma_1,\sigma_2,(0)}$ and we denote these maps by u'. Lemma 7.29 gives the following estimate:

$$\|\overline{\partial} u'\|_{L^2_{m,\delta}(\Sigma(\sigma_1,\sigma_2))/E(u')} \leqslant C e^{-c\delta_1 \min\{T_1, T_2\}}.$$

Here $E(u') = E_{\mathrm{d}}(u') \oplus E_{\mathrm{s}}(u') \oplus E_{\mathrm{D}}(u')$, and the norm on the left hand side is the induced norm on the quotient space. The next step would be to find:

$$V \in W^2_{m+1,\delta}((\Sigma(\sigma_1, \sigma_2), \partial\Sigma(\sigma_1, \sigma_2)); (u'^*TX, u'|_\partial^* TL)).$$

which satisfies the equation:

$$(D_{u'}\overline{\partial})V + \overline{\partial} u' \equiv 0 \mod E(u'). \tag{7.32}$$

and

$$\|V\|_{W^2_{m+1,\delta}} \leqslant C \|\overline{\partial} u'\|_{L^2_m(\Sigma(\sigma_1,\sigma_2))/E(u')}.$$

Then we could define our first modified approximate solution as follows:

$$u''(z) = \mathrm{Exp}(u'(z), V(z))$$

This modified solution would satisfy the following inequality:

$$\|\overline{\partial} u''\|_{L^2_{m,\delta}(\Sigma(\sigma_1,\sigma_2))/E(u'')} \leqslant \mu \|\overline{\partial} u'\|_{L^2_{m,\delta}(\Sigma(\sigma_1,\sigma_2))/E(u')}$$

for a fixed $0 < \mu < 1$ if σ_1, σ_2 are sufficiently small (or equivalently, T_1, T_2 are sufficiently large).

We could then continue to obtain $u^{(i)}$ such that

$$\|\overline{\partial} u^{(i)}\|_{L^2_m(\Sigma(\sigma_1,\sigma_2))/E(u^{(i)}|)} \leqslant \mu^i \|\overline{\partial} u'\|_{L^2_m(\Sigma(\sigma_1,\sigma_2))/E(u')}$$

and for fixed constants C and c, the $W^2_{m+1,\delta}$-distance between $u^{(i)}$ and $u^{(i+1)}$ is bounded by $C\mu^i e^{-c\delta_1 \min\{T_1,T_2\}}$. Then $u^{(i)}$ would be convergent to a map u, and it would be the required solution of the equation:

$$\overline{\partial} u \equiv 0 \mod E(u). \tag{7.33}$$

This is the standard Newton's iteration method to solve a nonlinear equation using successive solutions to the linearized equation. However, the RGW compactification is singular at the starting point of our construction, the element ζ of (7.16). So we cannot expect the above Newton's iteration method works without some adjustments. We fix our approach by thickening the solution set of (7.33) to the set of inconsistent solutions.

The main reason that we will work with this larger moduli space lies in the step that we find the solution V of the equation (7.32). To solve this equation, we need to find a right inverse to the following operator modulo $E(u')$:

$$\begin{aligned} D_{u'}\overline{\partial} :& W^2_{m+1,\delta}((\Sigma(\sigma_1,\sigma_2),\partial\Sigma(\sigma_1,\sigma_2));(u'^*TX,u'|^*_{\partial}TL)) \\ &\to L^2_{m,\delta}(\Sigma(\sigma_1,\sigma_2);u'^*TX\otimes\Lambda^{0,1})/E(u'). \end{aligned}$$

The standard approach to construct this right inverse is to glue the right inverses of the linearized operators $D_{u_{\rm d}}\overline{\partial}$, $D_{u_{\rm s}}\overline{\partial}$ and $D_{U_{\rm D}}\overline{\partial}$. The linearized operator $D_{u'}\overline{\partial}$ over the cylinder $[-5T_i,5T_i]_{r_i}\times S^1_{s_i}$ is modeled by an operator of the form

$$\frac{\partial}{\partial r_i} + P_{r_i}.$$

The relevant operators P_{r_i} in our setup have non-trivial kernel and our gluing construction is of "Morse-Bott" type. As it was clarified by Mrowka's Mayer-Vietoris principle [Mro88], to have a well-behaved gluing problem we need to assume certain 'mapping transversality conditions'.

To be more specific, the zero eigenspace of the operator P_{r_i} can be identified with:

$$\begin{aligned} &(\mathbb{R}\oplus\mathbb{R})\oplus T_{u_{\rm d}(z_{\rm d})}\mathcal{D} \qquad \text{if } i=1,\\ &(\mathbb{R}\oplus\mathbb{R})\oplus T_{u_{\rm s}(z_{\rm s})}\mathcal{D} \qquad \text{if } i=2. \end{aligned} \tag{7.34}$$

Here $\mathbb{R}\oplus\mathbb{R}$ is the tangent space to $\mathbb{C}_*$. The mapping transversality condition we introduced in Definition 4.2 concerns the summand $T_{u_{\rm d}(z_{\rm d})}\mathcal{D}$. Therefore, it is *not* sufficient for the Mayer-Vietoris principle in our setup. However, working with inconsistent solutions allows us to enlarge the tangent spaces and obtain the required transversality condition. A byproduct of using inconsistent solutions is that we might end up with inconsistent solutions throughout Newton's iterations, even if the starting approximate solution has $\rho_1=\rho_2$.

7.6. **Inconsistent Maps and Linearized Equations.** In Section 5, the notion of holomorphic maps was extended to inconsistent solutions of the Cauchy-Riemann equation. It is also convenient to define generalizations of maps from $\Sigma(\sigma_1,\sigma_2)$ to $X\backslash\mathcal{D}$:

Definition 7.35. A 7-tuple $\mathfrak{u}'=(u'_{\rm d},u'_{\rm s},U'_{\rm D},\sigma_1,\sigma_2,\rho_1,\rho_2)$ is an *inconsistent map* if it satisfies only parts (1) and (4) of Definition 5.1. In other words, we do not require that the 7-tuple satisfies the Cauchy-Riemann equation in (5.2) and the constraint in (5.3). We define equivalence of inconsistent maps in the same way as in Definition 5.1.

An example of inconsistent maps can be constructed using the maps:

$$u'^{,\xi,i}_{\sigma_1,\sigma_2,(0)}:\Sigma(\sigma_1,\sigma_2)\to X \qquad i=1,\,2$$

of Subsection 7.4 which are associated to an element $\xi = (u^\xi_{\rm d}, u^\xi_{\rm D}, u^\xi_{\rm s})$ of (7.16). We use these two maps to define:

$$
\begin{aligned}
u^{\xi,\prime}_{{\rm d},\sigma_1,\sigma_2,(0)} &:= u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)}|_{\Sigma^+_{\rm d}(\sigma_1,\sigma_2)} \\
u^{\xi,\prime}_{{\rm s},\sigma_1,\sigma_2,(0)} &:= u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)}|_{\Sigma^+_{\rm s}(\sigma_1,\sigma_2)} \\
U^{\xi,\prime}_{{\rm D},\sigma_1,\sigma_2,(0)} &:= \begin{cases} {\rm Dil}_{1/\rho^\xi_{1,(0)}} \circ u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)} & \text{on } \Sigma^+_{\rm d}(\sigma_1,\sigma_2) \cap \Sigma^+_{\rm D}(\sigma_1,\sigma_2) \\ {\rm Dil}_{1/\rho^\xi_{2,(0)}} \circ u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)} & \text{on } \Sigma^+_{\rm s}(\sigma_1,\sigma_2) \cap \Sigma^+_{\rm D}(\sigma_1,\sigma_2) \\ {\rm Dil}_{1/\rho^\xi_{1,(0)}} \circ u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)} & \text{on } \Sigma^-_{\rm D}(\sigma_1,\sigma_2) \end{cases}
\end{aligned}
$$

Note that ${\rm Dil}_{1/\rho^\xi_{1,(0)}} \circ u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)} = {\rm Dil}_{1/\rho^\xi_{2,(0)}} \circ u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)}$ on $\Sigma^-_{\rm D}(\sigma_1,\sigma_2)$. The following lemma is obvious from the construction:

Lemma 7.36. *The 7-tuple:*

$$
\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)} := (u^{\xi,\prime}_{{\rm d},\sigma_1,\sigma_2,(0)}, u^{\xi,\prime}_{{\rm s},\sigma_1,\sigma_2,(0)}, U^{\xi,\prime}_{{\rm D},\sigma_1,\sigma_2,(0)}, \sigma_1, \sigma_2, \rho^\xi_{1,(0)}, \rho^\xi_{2,(0)})
$$

is an inconsistent map

The inconsistent map $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)}$ of Lemma 7.36 is the approximate solution at the 0-th step. In order to obtain an actual inconsistent solution, we keep modifying this approximate solution into better approximate solutions. To be more detailed, we firstly use $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)}$ and our bump functions to obtain a triple

$$
\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)} = (u^{\xi,\prime\prime}_{{\rm d},\sigma_1,\sigma_2,(0)}, u^{\xi,\prime\prime}_{{\rm s},\sigma_1,\sigma_2,(0)}, U^{\xi,\prime\prime}_{{\rm D},\sigma_1,\sigma_2,(0)})
$$

such that

$$
u^{\xi,\prime\prime}_{{\rm d},\sigma_1,\sigma_2,(0)} : \Sigma_{\rm d}\backslash\{z_{\rm d}\} \to X\backslash\mathcal{D}, \qquad u^{\xi,\prime\prime}_{{\rm s},\sigma_1,\sigma_2,(0)} : \Sigma_{\rm s}\backslash\{z_{\rm s}\} \to X\backslash\mathcal{D}, \tag{7.37}
$$

$$
U^{\xi,\prime\prime}_{{\rm D},\sigma_1,\sigma_2,(0)} : \Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\} \to \mathcal{N}_{\mathcal{D}}(X). \tag{7.38}
$$

are close to $(u^\xi_{\rm d}, u^\xi_{\rm s}, U^\xi_{\rm D})$. In fact, the smaller the values of σ_1 and σ_2 are, the closer $\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}$ is to ξ. Thus we can exploit this to conclude that an appropriate version of the Cauchy-Riemann operator associated to $\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}$ has a right inverse. (See Lemma 7.66.) This allows us to find a modified inconsistent map $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)}$. We repeat the same process to construct a sequence of inconsistent maps $\{\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(i)}\}$ which are approximate solutions and they converge to an inconsistent solution. This sequence of modified inconsistent solution is constructed using Newton's iteration, and it also has some components of the "alternating method".[15] In this method we solve the equation in various pieces and glue them together.

In order to carry out the above plan, we need to introduce norms to quantify the distance between two inconsistent maps and to measure how good an approximate solution is. Such norms are given in the following Definition 7.39:

[15]See, for example, [Fuk96, Sublemma 8.6]. Application of alternating method for gluing analysis of this kinds is initiated by Donaldson [Don86]. He applied alternating method directly to a nonlinear equation.

Let $\mathfrak{u}' = (u'_{\rm d}, u'_{\rm s}, U'_{\rm D}, \sigma_1, \sigma_2, \rho_1, \rho_2)$ be an inconsistent map. We consider a triple $\mathrm{V} = (V_{\rm d}, V_{\rm s}, V_{\rm D})$ with

$$\begin{aligned} &V_{\rm d} \in L^2_{m+1}(\Sigma^+_{\rm d}(\sigma_1,\sigma_2), (u'_{\rm d})^*TX),\\ &V_{\rm s} \in L^2_{m+1}(\Sigma^+_{\rm s}(\sigma_1,\sigma_2), (u'_{\rm s})^*TX),\\ &V_{\rm D} \in L^2_{m+1}(\Sigma^+_{\rm D}(\sigma_1,\sigma_2), (U'_{\rm D})^*(T\mathcal{N}_{\mathcal{D}}(X))). \end{aligned}$$

We assume $V_{\rm d}(\mathfrak{z}) \in T_{u'_{\rm d}(\mathfrak{z})}L$ if $\mathfrak{z} \in \partial\Sigma^+_{\rm d}(\sigma_1,\sigma_2)$. Moreover, we assume that there exist $(a_{\rm d}, b_{\rm d}), (a_{\rm s}, b_{\rm s}) \in \mathbb{R}\oplus\mathbb{R}$ such that

$$\begin{aligned} V_{\rm d} - V_{\rm D} &= (a_{\rm d}, b_{\rm d}) \qquad \text{on } [-5T_1, 5T_1]_{r_1} \times S^1_{s_1}\\ V_{\rm s} - V_{\rm D} &= (a_{\rm s}, b_{\rm s}) \qquad \text{on } [-5T_2, 5T_2]_{r_2} \times S^2_{s_2} \end{aligned}$$

Here we regard $\mathbb{R}\oplus\mathbb{R}$ as the vector field on the neighborhood $\mathfrak{U}$ of $\mathcal{D}$ given by the $\mathbb{C}_*$ action.

Definition 7.39. Define $v_i = V_{\rm D}((0,0)_i)$ where $(0,0)_i$ is the same as in (7.9). We then define $\hat v_i$ in the same way as in (7.10). We now define $\|\mathrm{V}\|^2_{W^{2,\sim}_{m,\delta}}$ as follows:

$$\begin{aligned} &\|V_{\rm d}\|^2_{L^2_m(\Sigma^-_{\rm d}(\sigma_1,\sigma_2))} + \|V_{\rm s}\|^2_{L^2_m(\Sigma^-_{\rm s}(\sigma_1,\sigma_2))} + \|V_{\rm D}\|^2_{L^2_m(\Sigma^-_{\rm D}(\sigma_1,\sigma_2))}\\ &+ \sum_{j=0}^m \int_{[-5T_1,5T_1]_{r_1}\times S^1_{s_1}} e^{\sigma_1,\sigma_2}_\delta \left|\nabla^j(V_{\rm D} - \hat v_1)\right|^2 dr_1 ds_1\\ &+ \sum_{j=0}^m \int_{[-5T_2,5T_2]_{r_2}\times S^1_{s_2}} e^{\sigma_1,\sigma_2}_\delta \left|\nabla^j(V_{\rm D} - \hat v_2)\right|^2 dr_2 ds_2\\ &+ |v_1|^2 + |v_2|^2. \end{aligned}$$

Let $\mathrm{V} = (V_{\rm d}, V_{\rm s}, V_{\rm D})$, $\mathrm{V}' = (V'_{\rm d}, V'_{\rm s}, V'_{\rm D})$ be as above. We say they are equivalent if $V_{\rm d} = V'_{\rm d}$, $V_{\rm s} = V'_{\rm s}$ and $V_{\rm D} - V'_{\rm D} \in \mathbb{R}\oplus\mathbb{R}$, where $\mathbb{R}\oplus\mathbb{R}$ is the set of vector fields generated by the $\mathbb{C}_*$ action. We put

$$\|\mathrm{V}\|^2_{W^2_{m,\delta}} = \inf\{\|\mathrm{V}'\|^2_{W^{2,\sim}_{m,\delta}} \mid \mathrm{V}' \text{ is equivalent to } \mathrm{V}.\}$$

Note in the above definition we take infimum over representatives. The ambiguity of the choice of representatives is determined by the shift of several components by real numbers. If those shifts are very big the norm $\|V\|$ becomes far from minimum. Therefore, the domain we need to take infimum is bounded. It implies that even though we take infimum this norm is positive, as long as $V \neq 0$.

Definition 7.40. For $j = 1, 2$, let $\mathfrak{u}'_{(j)}$ be an inconsistent map. We assume that there is a representative $(u'_{{\rm d},(j)}, u'_{{\rm s},(j)}, U'_{{\rm D},(j)}, \sigma_1, \sigma_2, \rho_{1,(j)}, \rho_{2,(j)})$ for $\mathfrak{u}'_{(j)}$ such that the triple $(u'_{{\rm d},(1)}, u'_{{\rm s},(1)}, U'_{{\rm D},(1)})$ is C^0-close to $(u'_{{\rm d},(2)}, u'_{{\rm s},(2)}, U'_{{\rm D},(2)})$. Define $V_{\rm d}$, $V_{\rm s}$, $V_{\rm D}$ by the following properties:

$$\begin{aligned} &\mathrm{Exp}(u'_{{\rm d},(1)}, V_{\rm d}) = u'_{{\rm d},(2)},\\ &\mathrm{Exp}(u'_{{\rm s},(1)}, V_{\rm s}) = u'_{{\rm s},(2)},\\ &\mathrm{Exp}(U'_{{\rm D},(1)}, V_{\rm D}) = U'_{{\rm D},(2)}. \end{aligned} \tag{7.41}$$

Let $\mathrm{V} = (V_{\rm d}, V_{\rm s}, V_{\rm D})$, and define:

$$d_{W^2_{m,\delta}}(\mathfrak{u}'_{(1)}, \mathfrak{u}'_{(2)}) = \inf\{\|\mathrm{V}\|_{W^2_{m,\delta}}\}.$$

where the infimum is taken over all representatives for $\mathfrak{u}'_{(1)}$ and $\mathfrak{u}'_{(2)}$ which are close enough to each other in the C^0 metric such that the vectors in (7.41) exist. Therefore,

$d_{W^2_{m,\delta}}$ is a well defined distance between two equivalence classes of inconsistent maps. Note that $d_{W^2_{m,\delta}}(\mathfrak{u}'_{(1)},\mathfrak{u}'_{(2)})$ is strictly positive if $\mathfrak{u}'_{(1)} \neq \mathfrak{u}'_{(2)}$. We can prove this fact in the same way as in Definition 7.39.

For any inconsistent map $\mathfrak{u}' = (u'_{\rm d}, u'_{\rm s}, U'_{\rm D}, \sigma_1, \sigma_2, \rho_1, \rho_2)$, we may use a similar parallel transport construction as in Definition 5.1 to define obstruction spaces for $\mathfrak{u}'$. That is to say, we define maps $\mathcal{PAL}$ as in (4.9) and (4.10). Then the images of $E_{\rm d}$ and $E_{\rm s}$ with respect to these maps give rise to the obstruction spaces $E_{\rm d}(u'_{\rm d})$ and $E_{\rm s}(u'_{\rm s})$. Similarly, we define $E_{\rm D}(U'_{\rm D})$ by replacing $u'_{\rm D}$ with $\pi \circ U'_{\rm D}$ in (4.14) and using the decomposition (3.5). We will write $E(\mathfrak{u}')$ for the direct sum of the vector spaces $E_{\rm d}(u'_{\rm d})$ and $E_{\rm s}(u'_{\rm s})$ and $E_{\rm D}(U'_{\rm D})$. Note that $E_{\rm d}(u'_{\rm d})$, $E_{\rm s}(u'_{\rm s})$ and $E_{\rm D}(U'_{\rm D})$ are identified with $E_{\rm d}$ and $E_{\rm s}$ and $E_{\rm D}$. Therefore, we drop $u'_{\rm d}$, $u'_{\rm s}$ and $U'_{\rm D}$ from our notation for these obstruction spaces if it does not make any confusion.

Definition 7.42. Let $\mathfrak{u}' = (u'_{\rm d}, u'_{\rm s}, U'_{\rm D}, \sigma_1, \sigma_2, \rho_1, \rho_2)$ be an inconsistent map and $\mathfrak{e} = (\mathfrak{e}_{\rm d}, \mathfrak{e}_{\rm s}, \mathfrak{e}_{\rm D}) \in E_{\rm d} \oplus E_{\rm s} \oplus E_{\rm D}$. Then we define $\|\overline{\partial}\mathfrak{u}' - \mathfrak{e}\|^2_{L^2_{m,\delta}}$ to be the following sum:

$$\begin{aligned}
&\|\overline{\partial} u'_{\rm d} - \mathfrak{e}_{\rm d}\|^2_{L^2_m(\Sigma^-_{\rm d}(\sigma_1,\sigma_2))} + \|\overline{\partial} u'_{\rm s} - \mathfrak{e}_{\rm s}\|^2_{L^2_m(\Sigma^-_{\rm s}(\sigma_1,\sigma_2))} \\
&+ \|\overline{\partial} U'_{\rm D} - \mathfrak{e}_{\rm D}\|^2_{L^2_m(\Sigma^-_{\rm D}(\sigma_1,\sigma_2))} \\
&+ \sum_{j=0}^{m} \int_{[-5T_1,5T_1]_{r_1}\times S^1_{s_1}} e^{\sigma_1,\sigma_2}_{\delta} \left|\nabla^j \overline{\partial} U'_{\rm D}\right|^2 dr_1 ds_1 \\
&+ \sum_{j=0}^{m} \int_{[-5T_2,5T_2]_{r_2}\times S^1_{s_2}} e^{\sigma_1,\sigma_2}_{\delta} \left|\nabla^j \overline{\partial} U'_{\rm D}\right|^2 dr_2 ds_2.
\end{aligned}$$

We remark that the first 3 terms in the above definition are the Sobolev norms of $\overline{\partial}\mathfrak{u}' - \mathfrak{e}$ in the *thick part*. The fourth and the fifth terms are its weighted Sobolev norms in the neck region. Because of our choice of cylindrical metrics on $\mathfrak{U}$, the partial $\mathbb{C}_*$-action induces isometries and preserves the almost complex structure. Therefore, the above sum is well-defined and only depends on the equivalence class of $\mathfrak{u}'$.

The process of the modifications of our approximate solutions are performed by finding solutions to the linearization of the modified Cauchy-Riemann equations in (5.2). Since our equation has terms induced by the obstruction bundle, the linearized operator has an extra term in addition to $D_{u'}\overline{\partial}$. The equations in (5.2) can be regarded as an equation for an inconsistent map $\mathfrak{u}' = (u'_{\rm d}, u'_{\rm s}, U'_{\rm D}, \sigma_1, \sigma_2, \rho_1, \rho_2)$ and $(\mathfrak{e}_{\rm d}, \mathfrak{e}_{\rm s}, \mathfrak{e}_{\rm D}) \in E_{\rm d} \oplus E_{\rm s} \oplus E_{\rm D}$:

$$\overline{\partial} u'_{\rm d} - \mathfrak{e}_{\rm d} = 0, \quad \overline{\partial} u'_{\rm s} - \mathfrak{e}_{\rm s} = 0, \quad \overline{\partial} U'_{\rm D} - \mathfrak{e}_{\rm D} = 0. \tag{7.43}$$

Suppose $\mathrm{V} = (V_{\rm d}, V_{\rm s}, V_{\rm D})$ is an element of the Hilbert space introduced in Definition 7.39. For each real number τ with $|\tau| < 1$, let $\mathfrak{u}^\tau$ be given by the triple $(u^\tau_{\rm d}, u^\tau_{\rm s}, U^\tau_{\rm D})$ defined as:

$$\begin{aligned}
&u^\tau_{\rm d} := \mathrm{Exp}(u'_{\rm d}, \tau V_{\rm d}), \qquad u^\tau_{\rm s} := \mathrm{Exp}(u'_{\rm s}, \tau V_{\rm s}), \\
&U^\tau_{\rm D} := \mathrm{Exp}(U'_{\rm D}, \tau V_{\rm D}).
\end{aligned} \tag{7.44}$$

We use parallel transport along minimal geodesics to obtain:

$$\mathcal{PAL}^\tau_{u'_{\rm d}} : L^2_{m,\delta}(\Sigma^+_{\rm d}(\sigma_1,\sigma_2); u'^*_{\rm d}TX \otimes \Lambda^{0,1}) \xrightarrow{\cong} L^2_{m,\delta}(\Sigma^+_{\rm d}(\sigma_1,\sigma_2); u^{\tau *}_{\rm d}TX \otimes \Lambda^{0,1}).$$

and maps $\mathcal{PAL}^{\tau}_{u'_{\rm s}}$ and $\mathcal{PAL}^{\tau}_{U'_{\rm D}}$. Then for $\mathfrak{e} = (\mathfrak{e}_{\rm d}, \mathfrak{e}_{\rm s}, \mathfrak{e}_{\rm D}) \in E_{\rm d} \oplus E_{\rm s} \oplus E_{\rm D}$, we define:

$$
\begin{aligned}
(D_{u'_{\rm d}}E)(\mathfrak{e}_{\rm d}, V_{\rm d}) &= \frac{d}{d\tau}\Big|_{\tau=0} ((\mathcal{PAL}^{\tau}_{u'_{\rm d}})^{-1}(\mathfrak{e}_{\rm d})),\\
(D_{u'_{\rm s}}E)(\mathfrak{e}_{\rm s}, V_{\rm s}) &= \frac{d}{d\tau}\Big|_{\tau=0} ((\mathcal{PAL}^{\tau}_{u'_{\rm s}})^{-1}(\mathfrak{e}_{\rm s})),\\
(D_{U'_{\rm D}}E)(\mathfrak{e}_{\rm D}, V_{\rm D}) &= \frac{d}{d\tau}\Big|_{\tau=0} ((\mathcal{PAL}^{\tau}_{U'_{\rm D}})^{-1}(\mathfrak{e}_{\rm D})).
\end{aligned}
\tag{7.45}
$$

We also reserve the following notation for the triple given by the above vectors:

$$
(D_{\mathfrak{u}'}E)(\mathfrak{e}, {\rm V}) = ((D_{u'_{\rm d}}E)(\mathfrak{e}_{\rm d}, V_{\rm d}), (D_{u'_{\rm s}}E)(\mathfrak{e}_{\rm s}, V_{\rm s}), (D_{U'_{\rm D}}E)(\mathfrak{e}_{\rm D}, V_{\rm D}))
\tag{7.46}
$$

The linearizations of the Cauchy-Riemann equations in (7.43) at $(\mathfrak{u}', \mathfrak{e})$ evaluated at V as above and $\mathfrak{f} \in E_{\rm d} \oplus E_{\rm s} \oplus E_{\rm D}$ have the following form:

$$
D_{\mathfrak{u}'}\overline{\partial}({\rm V}) - (D_{\mathfrak{u}'}E)(\mathfrak{e}, {\rm V}) - \mathfrak{f}.
\tag{7.47}
$$

where:

$$
D_{\mathfrak{u}'}\overline{\partial}({\rm V}) = (D_{u'_{\rm d}}\overline{\partial}(V_{\rm d}), D_{u'_{\rm s}}\overline{\partial}(V_{\rm s}), D_{U'_{\rm D}}\overline{\partial}(V_{\rm D})).
$$

7.7. **Newton's Iteration.** Now we are ready to carry out the strategy which is discussed in the previous subsection. In the following, we use the maps constructed in Subsection 7.4.

(Step 0-4) (Separating error terms into three parts)

We firstly fix notations for the error terms of our first approximation $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)}$:

$$
\begin{aligned}
{\rm Err}^{\xi}_{{\rm d},\sigma_1,\sigma_2,(0)} &= \chi^{\leftarrow}_{1,\mathcal{X}}(\overline{\partial}u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{{\rm d},(0)}),\\
{\rm Err}^{\xi}_{{\rm s},\sigma_1,\sigma_2,(0)} &= \chi^{\leftarrow}_{2,\mathcal{X}}(\overline{\partial}u'^{,\xi,2}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{{\rm s},(0)}),\\
{\rm Err}^{\xi}_{{\rm D},\sigma_1,\sigma_2,(0)} &= \chi^{\rightarrow}_{\mathcal{X}}(\overline{\partial}u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi,1}_{{\rm D},(0)}).
\end{aligned}
\tag{7.48}
$$

where $\mathfrak{e}^{\xi}_{{\rm d},(0)}$, $\mathfrak{e}^{\xi}_{{\rm s},(0)}$, $\mathfrak{e}^{\xi,i}_{{\rm D},(0)}$ are defined in (7.30).

(Step 1-1) (Approximate solution for linearization)

Next we define:

$$
\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)} = (u^{\xi,\prime\prime}_{{\rm d},\sigma_1,\sigma_2,(0)}, u^{\xi,\prime\prime}_{{\rm s},\sigma_1,\sigma_2,(0)}, U^{\xi,\prime\prime}_{{\rm D},\sigma_1,\sigma_2,(0)})
\tag{7.49}
$$

whose entries have the form given in (7.37) and (7.38). Let:

$$
\begin{aligned}
u^{\xi}_{\rm d}(z_{\rm d}) &= p^{\xi}_{{\rm d},\sigma_1,\sigma_2,(0)} = p_{{\rm D},{\rm d},\sigma_1,\sigma_2,(0)},\\
u^{\xi}_{\rm s}(z_{\rm s}) &= p^{\xi}_{{\rm s},\sigma_1,\sigma_2,(0)} = p_{{\rm D},{\rm s},\sigma_1,\sigma_2,(0)}.
\end{aligned}
\tag{7.50}
$$

We take $c^{\xi}_{\rm d}$, $c^{\xi}_{\rm s}$, $c^{\xi}_{\rm D,d}$, $c^{\xi}_{\rm D,s}$ as in (7.19), (7.23), (7.20), (7.24), respectively. We regard $c^{\xi}_{\rm d}z^{p_1}$ as an element of the fiber of $\mathcal{N}_{\mathcal{D}}(X)$ at $p^{\xi}_{{\rm d},\sigma_1,\sigma_2,(0)}$ and hence as an element of $X\backslash\mathcal{D}$. We define:

$$
\begin{aligned}
&u^{\xi,\prime\prime}_{{\rm d},\sigma_1,\sigma_2,(0)}(r_1, s_1)\\
&:= {\rm Exp}(c^{\xi}_{\rm d}z^{p_1}, \chi^{\leftarrow}_{1,\mathcal{B}}(r_1 - T_1, s_1){\rm E}(c^{\xi}_{\rm d}z^{p_1}, u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)}(r_1, s_1)))
\end{aligned}
\tag{7.51}
$$

if $(r_1, s_1) \in [-5T_1, \infty)_{r_1} \times S^1_{s_1} \subset \Sigma_{\rm d}\backslash\{z_{\rm d}\}$. If $\mathfrak{z} \in \Sigma_{\rm d}\backslash\{z_{\rm d}\}$ is an element in the complement of $[-5T_1, \infty)_{r_1} \times S^1_{s_1}$, then we define:

$$
u^{\xi,\prime\prime}_{{\rm d},\sigma_1,\sigma_2,(0)}(\mathfrak{z}) := u'^{,\xi,1}_{\sigma_1,\sigma_2,(0)}(\mathfrak{z}).
$$

This completes the definition of $u^{\xi,\prime\prime}_{{\rm d},\sigma_1,\sigma_2,(0)}$ as a map from $\Sigma_{\rm d}\backslash\{z_{\rm d}\}$ to $X\backslash\mathcal{D}$.

Similarly, we define:

$$\begin{aligned} & u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(0)}(r_1,s_1) \\ & := \mathrm{Exp}(c^{\xi}_{\mathrm{s}} z^{p_2}, \chi^{\leftarrow}_{2,\mathcal{B}}(r_2-T_2,s_2)\mathrm{E}(c^{\xi}_{\mathrm{s}} z^{p_2}, u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)}(r_2,s_2))) \end{aligned} \tag{7.52}$$

if $(r_2,s_2) \in [-5T_2,\infty)_{r_2} \times S^1_{s_2} \subset \Sigma_{\mathrm{s}}\backslash\{z_{\mathrm{s}}\}$. Here we regard $c^{\xi}_{\mathrm{s}} z^{p_2}$ as an element of the fiber of $\mathcal{N}_{\mathcal{D}}(X)$ at $p^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)}$ and hence as an element of $X\backslash\mathcal{D}$. If $\mathfrak{z} \in \Sigma_{\mathrm{s}}\backslash\{z_{\mathrm{s}}\}$ is an element in the complement of $[-5T_2,\infty)_{r_2} \times S^1_{s_2}$, then we define:

$$u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(0)}(\mathfrak{z}) := u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)}(\mathfrak{z}).$$

The next lemma is easy to prove.

Lemma 7.53. *If the constants T_1, T_2 are sufficiently large, then $u''_{\mathrm{d}} : (\Sigma_{\mathrm{d}}\backslash\{z_{\mathrm{d}}\}, \partial\Sigma_{\mathrm{d}}) \to (X\backslash\mathcal{D}, L)$ (resp. $u''_{\mathrm{s}} : \Sigma_{\mathrm{s}}\backslash\{z_{\mathrm{s}}\} \to X\backslash\mathcal{D}$) satisfies the following properties:*

(1) *u''_{d} (resp. u''_{s}) maps $[3T_1,\infty)_{r_1} \times S^1_{s_1}$ (resp. $[3T_2,\infty)_{r_2} \times S^1_{s_2}$) to $\mathfrak{U}$. There exist $\mathrm{p}_{\mathrm{d}}, \mathrm{p}_{\mathrm{s}} \in \mathcal{D}$ such that the restriction of $\pi \circ u''_{\mathrm{d}}$ (resp. $\pi \circ u''_{\mathrm{s}}$) to $[3T_1,\infty)_{r_1} \times S^1_{s_1}$ (resp. $[3T_2,\infty)_{r_2} \times S^1_{s_2}$) is a constant map to p_{d} (resp. p_{s}).*

(2) *After an appropriate trivialization of the pull back of the normal bundle $\mathcal{N}_{\mathcal{D}}(X)$ at the points $\mathrm{p}_{\mathrm{d}}, \mathrm{p}_{\mathrm{s}}$, there exist $c_{\mathrm{d}}, c_{\mathrm{s}} \in \mathbb{C}_*$ such that the restriction of $u''_{\mathrm{d}} \circ \varphi_{\mathrm{d}}$ to $[3T_1,\infty)_{r_1} \times S^1_{s_1}$ (resp. $u''_{\mathrm{s}} \circ \varphi_{\mathrm{s}}$ to $[3T_2,\infty)_{r_2} \times S^1_{s_2}$) is*

$$(u''_{\mathrm{d}} \circ \varphi_{\mathrm{d}})(z) = c_{\mathrm{d}} z^{p_1}, \quad (\textit{resp. } (u''_{\mathrm{s}} \circ \varphi_{\mathrm{s}})(z) = c_{\mathrm{s}} z^{p_2}). \tag{7.54}$$

Next, we define the map $U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)} : \Sigma_{\mathrm{D}}\backslash\{z_{\mathrm{d}}, z_{\mathrm{s}}\} \to \mathcal{N}_{\mathcal{D}}(X)$. A trivialization of the fibers of $\mathcal{N}_{\mathcal{D}}(X)$ at the points $p^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(0)}$ and $p^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)}$ allow us to identify $c^{\xi}_{\mathrm{D,d}} w^{-p_1}$ and $c^{\xi}_{\mathrm{D,s}} w^{-p_2}$ with elements of $\mathcal{N}_{\mathcal{D}}(X)\backslash\mathcal{D} = \mathbb{R}_{\tau} \times S\mathcal{N}_{\mathcal{D}}(X)$. We define:

$$\begin{aligned} & U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)}(r_1,s_1) \\ & = \mathrm{Exp}(c^{\xi}_{\mathrm{D,d}} w^{-p_1}, \chi^{\rightarrow}_{\mathcal{A}}(r_1+T_1,s_1) \cdot \\ & \qquad \mathrm{E}(c^{\xi}_{\mathrm{D,d}} w^{-p_1}, ((\mathrm{Dil}_{\rho^{\xi}_{1,(0)}})^{-1} \circ u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)})(r_1,s_1))) \end{aligned} \tag{7.55}$$

if $(r_1,s_1) \in (-\infty,5T_1]_{r_1} \times S^1_{s_1} \subset \Sigma_{\mathrm{D}}\backslash\{z_{\mathrm{d}}, z_{\mathrm{s}}\}$, and:

$$\begin{aligned} & U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)}(r_2,s_2) \\ & = \mathrm{Exp}(c^{\xi}_{\mathrm{D,s}} w^{-p_2}, \chi^{\rightarrow}_{\mathcal{A}}(r_2+T_2,s_2) \cdot \\ & \qquad \mathrm{E}(c^{\xi}_{\mathrm{D,s}} w^{-p_2}, ((\mathrm{Dil}_{\rho^{\xi}_{2,(0)}})^{-1} \circ u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)})(r_2,s_2))) \end{aligned} \tag{7.56}$$

if $(r_2,s_2) \in (-\infty,5T_2]_{r_2} \times S^1_{s_2} \subset \Sigma_{\mathrm{D}}\backslash\{z_{\mathrm{d}}, z_{\mathrm{s}}\}$. If $\mathfrak{z}$ is an element of $\Sigma_{\mathrm{D}}\backslash\{z_{\mathrm{d}}, z_{\mathrm{s}}\}$, that does not belong to the above cylinders, then we define:

$$U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)}(\mathfrak{z}) := (\mathrm{Dil}_{1/\rho^{\xi}_{1,(0)}} \circ u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)})(\mathfrak{z}).$$

Note that we can equivalently use the term $(\mathrm{Dil}_{1/\rho^{\xi}_{2,(0)}} \circ u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)})(\mathfrak{z})$ on the right hand side of the above definition. We remark that the 'highest order' terms of the maps $(\mathrm{Dil}_{\rho^{\xi}_{i,(0)}})^{-1} \circ u^{\prime,\xi,i}_{\sigma_1,\sigma_2,(0)}$ and U^{ξ}_{D} agree with each other on $[-5T_i,5T_i]_{r_i} \times S^2_{s_i}$. Similarly, $U^{\xi}_{\mathrm{D}}(\varphi_{\mathrm{D,d}}(w))$ (resp. $U^{\xi}_{\mathrm{D}}(\varphi_{\mathrm{D,s}}(w))$) and $c^{\xi}_{\mathrm{D,d}} w^{-2}$ (resp. $c^{\xi}_{\mathrm{D,s}} w^{-3}$) have the same highest order terms on $[-5T_i,5T_i]_{r_i} \times S^2_{s_i}$.

The following lemma can be verified in a straightforward way.

Lemma 7.57. *If the constants T_1, T_2 are sufficiently large, then $U''_{\rm D} : \Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\} \to X\backslash\mathcal{D}$ satisfies the following properties:*

(1) *There exist $p_{\rm D,d}, p_{\rm D,s} \in \mathcal{D}$ such that the restriction of $\pi\circ U''_D$ to $(-\infty, -3T_1]_{r_1} \times S^1_{s_1}$ (resp. $(-\infty, -3T_2]_{r_2} \times S^1_{s_2}$) is a constant map to $p_{\rm D,d}$ (rsep. $p_{\rm D,d}$).*

(2) *There exist $c_{\rm D,d}, c_{\rm D,s} \in \mathbb{C}_*$ such that the restriction of $U''_{\rm D}\circ\varphi_{\rm D,d}$ to $(-\infty, -3T_1]_{r_1} \times S^1_{s_1}$ (resp. $U''_{\rm D} \circ \varphi_{\rm D,s}$ to $(-\infty, -3T_2]_{r_2} \times S^1_{s_2}$) is*

$$(U''_{\rm D,d} \circ \varphi_{\rm D,d})(w) = c_{\rm D,d}w^{-p_1}, \quad \text{(resp. } (U''_{\rm D} \circ \varphi_{\rm D,s})(w) = c_{\rm D,s}w^{-p_2}\text{)}. \tag{7.58}$$

Let $\mathfrak{u}'' = (u''_{\rm d}, u''_{\rm s}, U''_{\rm D})$ be a triple of maps satisfying the properties in Lemmas 7.53, 7.57. We also assume:

$$p_{\rm d} = p_{\rm D,d}, \qquad p_{\rm s} = p_{\rm D,s}. \tag{7.59}$$

Definition 7.60. Let $W^{2\sim}_{m,\delta}(\mathfrak{u}'', U''_{\rm D}; TX)$ be the set of all $\mathbf{V} = (\mathbf{V}_{\rm d}, \mathbf{V}_{\rm s}, \mathbf{V}_{\rm D})$ satisfying the following properties:

(1) $\mathbf{V}_{\rm d} = (V_{\rm d}, (\mathfrak{r}_{\infty,\rm d}, \mathfrak{s}_{\infty,\rm d}), v_{\rm d}) \in W^2_{m,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\}; ((u''_{\rm d})^*TX, (u''_{\rm d})^*TL))$. (This function space is introduced in Definition 3.9.)

(2) $\mathbf{V}_{\rm s} = (V_{\rm s}, (\mathfrak{r}_{\infty,\rm s}, \mathfrak{s}_{\infty,\rm s}), v_{\rm s}) \in W^2_{m,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; (u''_{\rm s})^*TX)$. (This function space is introduced in Definition 3.18.)

(3) The tuple $\mathbf{V}_{\rm D} = (V_{\rm D}, (\mathfrak{r}_{\infty,\rm D,d}, \mathfrak{s}_{\infty,\rm D,d}), (\mathfrak{r}_{\infty,\rm D,s}, \mathfrak{s}_{\infty,\rm D,s})), v_{\rm D,d}, v_{\rm D,s})$ is an element of $W^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; (U''_{\rm D})^*T(\mathbb{R}_\sigma \times S\mathcal{N}_{\mathcal{D}}(X)))$. (This function space is introduced in Definition 3.23.)

(4) We assume
$$v_{\rm d} = v_{\rm D,d}, \qquad v_{\rm s} = v_{\rm D,s}.$$

The space $W^{2\sim}_{m,\delta}(\mathfrak{u}''; TX)$ is a linear subspace of finite codimension of the direct sum of three Hilbert spaces defined in Definitions 3.9, 3.18, 3.23. Therefore, it is also a Hilbert space.

We regard $\mathbb{R} \oplus \mathbb{R}$ as the subspace of $W^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; (U''_{\rm D})^*T(\mathbb{R} \times S\mathcal{N}_{\mathcal{D}}(X)))$ given by constant sections with values in $\mathbb{R} \oplus \mathbb{R} \subset T(\mathbb{R}_\sigma \times S\mathcal{N}_{\mathcal{D}}(X))$. Thus $\mathbb{R} \oplus \mathbb{R}$ can be also regarded as a subspace of $W^{2\sim}_{k,\delta}(\mathfrak{u}''; TX)$. We define $W^2_{m,\delta}(\mathfrak{u}''; TX)$ to be the quotient space of $W^{2\sim}_{m,\delta}(\mathfrak{u}''; TX)$ by this copy of $\mathbb{R} \oplus \mathbb{R}$.

Remark 7.61. We do *not* assume $\mathfrak{r}_{\infty,\rm d} = \mathfrak{r}_{\infty,\rm D,d}$ or $\mathfrak{r}_{\infty,\rm s} = \mathfrak{r}_{\infty,\rm D,s}$. The fact that we might have $\mathfrak{r}_{\infty,\rm d} \neq \mathfrak{r}_{\infty,\rm D,d}$ or $\mathfrak{r}_{\infty,\rm s} \neq \mathfrak{r}_{\infty,\rm D,s}$ is related to the shift of ρ_1, ρ_2, which we mentioned in Subsection 7.5.

Definition 7.62. Let $L^2_{m,\delta}(\mathfrak{u}''; TX \otimes \Lambda^{0,1})$ be the direct sum of the three Hilbert spaces:

$$\begin{aligned} &L^2_{m,\delta}(\Sigma_{\rm d}\backslash\{z_{\rm d}\}; (u''_{\rm d})^*TX \otimes \Lambda^{0,1}) \\ &\oplus L^2_{m,\delta}(\Sigma_{\rm s}\backslash\{z_{\rm s}\}; (u''_{\rm s})^*TX \otimes \Lambda^{0,1}) \\ &\oplus L^2_{m,\delta}(\Sigma_{\rm D}\backslash\{z_{\rm d}, z_{\rm s}\}; (U''_{\rm D})^*T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X)) \otimes \Lambda^{0,1}), \end{aligned}$$

introduced in Definitions 3.11, 3.18, 3.23. The three operators (3.14), (3.21), (3.26) together induce a Fredholm operator:

$$D_{\mathfrak{u}''}\overline{\partial} : W^2_{m+1,\delta}(\mathfrak{u}''; TX) \to L^2_{m,\delta}(\mathfrak{u}''; TX \otimes \Lambda^{0,1}). \tag{7.63}$$

Remark 7.64. If $u''_{\rm d}, u''_{\rm s}, U''_{\rm D}$ are C^1-close to $u^\xi_{\rm d}$, $u^\xi_{\rm s}$, $U^\xi_{\rm D}$, then the surjectivity of (3.14), (3.21), (3.26) (for $u^\xi_{\rm d}$, $u^\xi_{\rm s}$, $U^\xi_{\rm D}$) modulo $E_{\rm d}(u^\xi_{\rm d}) \oplus E_{\rm s}(u^\xi_{\rm s}) \oplus E_{\rm D}(u^\xi_{\rm D})$ and the mapping transversality condition of Definition 4.2 imply that (7.63) is surjective modulo the obstruction space $E_{\rm d}(u''_{\rm d}) \oplus E_{\rm s}(u''_{\rm s}) \oplus E_{\rm D}(u''_{\rm D})$. (See also Lemma 7.66.)

Lemma 7.65. *The triple:*

$$\mathrm{Err}^{\xi}_{\sigma_1,\sigma_2,(0)} := (\mathrm{Err}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(0)}, \mathrm{Err}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)}, \mathrm{Err}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(0)})$$

determines an element of $L^2_{m,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}; TX \otimes \Lambda^{0,1})$. *The terms above are defined in* (7.48).

Proof. It follows from the fact that the map $u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(0)}$ (resp. $u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(0)}$, $U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)}$) coincides with $u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)}$ (resp. $u^{\prime,\xi,2}_{\sigma_1,\sigma_2,(0)}$, $(\mathrm{Dil}_{\rho^{\xi}_{1,(0)}})^{-1} \circ u^{\prime,\xi,1}_{\sigma_1,\sigma_2,(0)}$) on the support of $\mathrm{Err}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(0)}$ (resp. $\mathrm{Err}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)}$, $\mathrm{Err}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(0)}$). □

Lemma 7.66. *Let the linear operator*

$$L : W^2_{m+1,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}; TX) \oplus E_{\mathrm{d}} \oplus E_{\mathrm{s}} \oplus E_{\mathrm{D}} \to L^2_{m,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}; TX \otimes \Lambda^{0,1})$$

be given as follows:

$$L(\mathbf{V}, \mathfrak{f}) = D_{\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}} \overline{\partial}(\mathbf{V}) - (D_{\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}} E)(\mathfrak{e}^{\xi}_{(0)}, \mathbf{V}) - \mathfrak{f}$$

where the compoents of $\mathfrak{e}^{\xi}_{(0)} := (\mathfrak{e}^{\xi}_{\mathrm{d},(0)}, \mathfrak{e}^{\xi}_{\mathrm{s},(0)}, \mathfrak{e}^{\xi,1}_{\mathrm{D},(0)})$ *are defined in* (7.30), *and the term* $(D_{\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}} E)(\mathfrak{e}^{\xi}_{(0)}, \mathbf{V})$ *is defined similar to the corresponding term in* (7.47). *If* σ_1 *and* σ_2 *are small enough, then there is a continuous operator*

$$Q : L^2_{m,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}; TX \otimes \Lambda^{0,1}) \to W^2_{m+1,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}; TX) \oplus E_{\mathrm{d}} \oplus E_{\mathrm{s}} \oplus E_{\mathrm{D}}$$

which is a right inverse to L. *Let* $(\overline{Q}, Q_{\mathrm{d}}, Q_{\mathrm{s}}, Q_{\mathrm{D}})$ *be the components of* Q *with respect to the decomposition of the target of* Q. *There is also constant* C, *independent of* σ_1, σ_2 *and* ξ, *such that for any* $z \in L^2_{m,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}; TX \otimes \Lambda^{0,1})$:

$$\|\overline{Q}(z)\|_{W^2_{m+1,\delta}} + |Q_{\mathrm{d}}(z)| + |Q_{\mathrm{s}}(z)| + |Q_{\mathrm{D}}(z)| \leqslant C\|z\|_{L^2_{m,\delta}}. \tag{7.67}$$

Moreover, we can make this choose of Q *unique by demanding that its image is* L^2-*orthogonal*[16] *to the subspace* $\ker(L)$.[17]

Proof. Using Definition 4.1 (2), (3), (4) and Definition 4.2, we can construct a continuous operator:

$$\begin{aligned} Q_0 = (Q_{0,\mathrm{d}}, Q_{0,\mathrm{s}}, Q_{0,\mathrm{D}}, Q_{0,E}) : & L^2_m(\Sigma_{\mathrm{d}} \backslash \{z_{\mathrm{d}}\}; u_{\mathrm{d}}^* TX \otimes \Lambda^{0,1}) \\ & \oplus L^2_m(\Sigma_{\mathrm{s}} \backslash \{z_{\mathrm{s}}\}; u_{\mathrm{s}}^* TX \otimes \Lambda^{0,1}) \\ & \oplus L^2_m(\Sigma_{\mathrm{D}} \backslash \{z_{\mathrm{d}}, z_{\mathrm{s}}\}; U_{\mathrm{D}}^* TX \otimes \Lambda^{0,1}) \\ \to & W^2_{m+1,\delta}(\Sigma_{\mathrm{d}} \backslash \{z_{\mathrm{d}}\}; (u_{\mathrm{d}}^* TX, u_{\mathrm{d}}^* TL)) \\ & \oplus W^2_{m+1,\delta}(\Sigma_{\mathrm{s}} \backslash \{z_{\mathrm{s}}\}; u_{\mathrm{s}}^* TX) \\ & \oplus W^2_{m+1,\delta}(\Sigma_{\mathrm{D}} \backslash \{z_{\mathrm{d}}, z_{\mathrm{s}}\}; U_{\mathrm{D}}^* T(\mathbb{R}_\tau \times S\mathcal{N}_{\mathcal{D}}(X))) \\ & \oplus E_{\mathrm{d}} \oplus E_{\mathrm{s}} \oplus E_{\mathrm{D}} \end{aligned}$$

such that:

$$\mathcal{EV}_{0,\mathrm{d}} \circ Q_{0,\mathrm{d}} = \mathcal{EV}_{0,\mathrm{d}} \circ Q_{0,\mathrm{D}}, \quad \mathcal{EV}_{0,\mathrm{s}} \circ Q_{0,\mathrm{s}} = \mathcal{EV}_{0,\mathrm{d}} \circ Q_{0,\mathrm{D}}.$$

and:

$$(D_{u_{\mathrm{d}}} \overline{\partial} Q_{0,\mathrm{d}}, D_{u_{\mathrm{s}}} \overline{\partial} Q_{0,\mathrm{s}}, D_{U_{\mathrm{D}}} \overline{\partial} Q_{0,\mathrm{D}}) = Q_{0,E}.$$

[16]We use the L^2 norm on the target of Q given by $W^2_{m,\delta}$ with $m = 0$ and $\delta = 0$.

[17]The last condition is similar to [FOOO16a, Definition 5.9].

Note that since $u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(0)}$, $u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(0)}$, $U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)}$ are respectively close to u_{d}, u_{s}, U_{D}, we can identify the function spaces $W^2_{m+1,\delta}(\Sigma_{\mathrm{d}}\backslash\{z_{\mathrm{d}}\};(u_{\mathrm{d}}^*TX,u_{\mathrm{d}}^*TL))$ etc. with $W^2_{m+1,\delta}(\Sigma_{\mathrm{d}}\backslash\{z_{\mathrm{d}}\};((u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(0)})^*TX,u_{\mathrm{d}}^*TL))$ etc. and $L^2_{m,\delta}(\Sigma_{\mathrm{d}}\backslash\{z_{\mathrm{d}}\};(u_{\mathrm{d}}^*TX,u_{\mathrm{d}}^*TL))$ etc. with $L^2_{m,\delta}(\Sigma_{\mathrm{d}}\backslash\{z_{\mathrm{d}}\};((u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(0)})^*TX,u_{\mathrm{d}}^*TL))$ etc. by target parallel transport.

Using this identification we obtain a map

$$Q_1 : L^2_{m,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)};TX\otimes\Lambda^{0,1}) \to W^2_{m+1,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)};TX)\oplus E_{\mathrm{d}}\oplus E_{\mathrm{s}}\oplus E_{\mathrm{D}}$$

such that

$$\|(L\circ Q_1)(z)-z\| \leqslant Ce^{-c\delta_1\min\{T_1,T_2\}}\|z\|, \qquad \|Q_1(z)\|\leqslant C'\|z\|. \tag{7.68}$$

Here $c,C,C'>0$ are constants independent of σ_1 and σ_2. Thus for σ_1 and σ_2 small enough, we may define

$$Q_2=\sum_{k=0}^{\infty}(-1)^kQ_1\circ(\mathrm{id}-L\circ Q_1)^k.$$

Then we have

$$(L\circ Q_2)(z)=\mathrm{id}, \qquad \|Q_2(z)\|\leqslant C''\|z\|. \tag{7.69}$$

(This formula is used there to estimate derivatives of the right inverse with respect to the gluing parameter.) The operator Q_2 has the required properties except the last one. To obtain the right inverse which also satisfies the last condition, we compose Q_2 with projection to the orthogonal complement of the finite dimensional space $\ker(L)$. □

Remark 7.70. The stable map compactification case of Lemma 7.66 is [FOOO16a, Lemma 5.7]. The function space $W^2_{m+1,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)};TX)$ is a subspace of the direct sum of the function spaces of the three irreducible components. By this reason the fact we are working on inconsistent maps does not affect the proof of Lemma 7.66.

Let $z:=\mathrm{Err}^{\xi}_{\sigma_1,\sigma_2,(0)}$. By Lemma 7.65, we know that z belongs to the target of L. Therefore, $\overline{Q}(z)$ determines a triple as follows:

$$\mathbf{V}^{\xi}_{\sigma_1,\sigma_2,(1)}=(\mathbf{V}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)},\mathbf{V}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)},\mathbf{V}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)})\in W^2_{m+1,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)};TX)$$

Moreover, we have:

$$\Delta\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)}:=Q_{\mathrm{d}}(z), \qquad \Delta\mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)}:=Q_{\mathrm{s}}(z), \qquad \Delta\mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)}:=Q_{\mathrm{D}}(z).$$

Lemmas 7.29 and 7.66 imply that:

$$\|\mathbf{V}^{\xi}_{\sigma_1,\sigma_2,(1)}\|_{W^2_{m+1,\delta}}\leqslant Ce^{-c\delta_1\min\{T_1,T_2\}}$$

and

$$|\Delta\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)}|,|\Delta\mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)}|,|\Delta\mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)}|\leqslant Ce^{-c\delta_1\min\{T_1,T_2\}}.$$

In summary, we obtained a solution of the linearized equation with appropriate decay properties.

(Step 1-2) (Gluing solutions)

In this step we will use $\mathbf{V}^{\xi}_{\sigma_1,\sigma_2,(1)}$ to obtain an improved approximate inconsistent solution. Suppose the entries of $\mathbf{V}^{\xi}_{\sigma_1,\sigma_2,(1)}$ are given as follows:

$$\begin{aligned}
\mathbf{V}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)} &= (V^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)}, (\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}), v^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}),\\
\mathbf{V}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)} &= (V^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)}, (\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}), v^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}),\\
\mathbf{V}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)} &= (V^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)}, (\mathfrak{r}^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)}),\\
&\qquad\qquad (\mathfrak{r}^{\xi}_{\infty,\mathrm{D},\mathrm{s},\sigma_1,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{D},\mathrm{s},\sigma_1,\sigma_2,(1)}),\\
&\qquad\qquad v^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)}, v^{\xi}_{\infty,\mathrm{D},\mathrm{s},\sigma_1,\sigma_2,(1)}).
\end{aligned}$$

We also have the following identities:

$$v^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)} = v^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)}, \qquad v^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)} = v^{\xi}_{\infty,\mathrm{D},\mathrm{s},\sigma_1,\sigma_2,(1)}.$$

by Definition 7.60 (4). We also define:

$$\begin{aligned}
\Delta\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)} &= \mathfrak{r}^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)} - \mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)},\\
\Delta\mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)} &= \mathfrak{s}^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)} - \mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)},
\end{aligned} \tag{7.71}$$

$$\begin{aligned}
\Delta\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)} &= \mathfrak{r}^{\xi}_{\infty,\mathrm{D},\mathrm{s},\sigma_1,\sigma_2,(1)} - \mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)},\\
\Delta\mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)} &= \mathfrak{s}^{\xi}_{\infty,\mathrm{D},\mathrm{s},\sigma_1,\sigma_2,(1)} - \mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}.
\end{aligned} \tag{7.72}$$

Definition 7.73. We define $u^{\xi,\prime}_{\mathrm{d},\sigma_1,\sigma_2,(1)} : \Sigma^{+}_{\mathrm{d}}(\sigma_1,\sigma_2) \to X\backslash\mathcal{D}$ as follows.

(1) If $\mathfrak{z} \in \Sigma^{-}_{\mathrm{d}}(\sigma_1,\sigma_2)$ then

$$u^{\xi,\prime}_{\mathrm{d},\sigma_1,\sigma_2,(1)}(\mathfrak{z}) = \mathrm{Exp}(u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(0)}(\mathfrak{z}), V^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)})$$

(2) If $\mathfrak{z} = (r_1,s_1) \in [-5T_1,5T_1]_{r_1} \times S^1_{s_1}$ then

$$u^{\xi,\prime}_{\mathrm{d},\sigma_1,\sigma_2,(1)}(\mathfrak{z}) = \mathrm{Exp}(u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(0)}(r_1,s_1), \diamondsuit)$$

where

$$\begin{aligned}
\diamondsuit =& \chi^{\leftarrow}_{1,\mathcal{B}}(r_1,s_1)(V^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)})\\
&+ \chi^{\rightarrow}_{\mathcal{A}}(r_1,s_1)(V^{\xi}_{\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{D},\mathrm{d},\sigma_1,\sigma_2,(1)})\\
&+ (\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}) + \hat{v}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}.
\end{aligned}$$

Here and in Items (4), (6), (7), we extend $v^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}$ to $\hat{v}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}$ in the same way as in Definition 3.9.

We define $u^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(1)} : \Sigma^{+}_{\mathrm{s}}(\sigma_1,\sigma_2) \to X\backslash\mathcal{D}$ as follows.

(3) If $\mathfrak{z} \in \Sigma^{-}_{\mathrm{s}}(\sigma_1,\sigma_2)$ then

$$u^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(1)}(\mathfrak{z}) = \mathrm{Exp}(u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(0)}(\mathfrak{z}), V^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)})$$

(4) If $\mathfrak{z} = (r_2,s_2) \in [-5T_2,5T_2]_{r_2} \times S^1_{s_2}$ then

$$u^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(1)}(\mathfrak{z}) = \mathrm{Exp}(u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(0)}(r_2,s_2), \clubsuit)$$

where

$$\begin{aligned}\clubsuit =&\chi_{2,\mathcal{B}}^{\leftarrow}(r_2,s_2)(V^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)})\\ &+\chi_{\mathcal{A}}^{\rightarrow}(r_2,s_2)(V^{\xi}_{\mathrm{D,s},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{D,s},\sigma_2,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{D,s},\sigma_2,\sigma_2,(1)})\\ &+(\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}) + \hat{v}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}.\end{aligned}$$

We next define $U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)} : \Sigma^{+}_{\mathrm{D}}(\sigma_1,\sigma_2) \to X\backslash\mathcal{D}$ as follows.

(5) If $\mathfrak{z} \in \Sigma^{-}_{\mathrm{D}}(\sigma_1,\sigma_2)$ then:

$$U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)}(\mathfrak{z}) = \mathrm{Exp}(U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)}(\mathfrak{z}), V^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)})$$

(6) If $\mathfrak{z} = (r_1,s_1) \in [-5T_1,5T_1]_{r_1} \times S^1_{s_1}$ then:

$$U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)}(\mathfrak{z}) = \mathrm{Exp}(U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)}(r_1,s_1), \heartsuit)$$

where:

$$\begin{aligned}\heartsuit =&\chi_{1,\mathcal{B}}^{\leftarrow}(r_1,s_1)(V^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)})\\ &+\chi_{\mathcal{A}}^{\rightarrow}(r_1,s_1)(V^{\xi}_{\mathrm{D,d},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)})\\ &+(\mathfrak{r}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)}) + \hat{v}^{\xi}_{\infty,\mathrm{Dd},\sigma_1,\sigma_2,(1)}.\end{aligned}$$

We remark that:

$$\heartsuit - \diamondsuit = (\Delta\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}, \Delta\mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}). \tag{7.74}$$

(7) If $\mathfrak{z} = (r_2,s_2) \in [-5T_2,5T_2]_{r_2} \times S^1_{s_2}$ then:

$$U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)}(\mathfrak{z}) = \mathrm{Exp}(U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(0)(r_2,s_2)}, \spadesuit)$$

where:

$$\begin{aligned}\spadesuit =&\chi_{2,\mathcal{B}}^{\leftarrow}(r_2,s_2)(V^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)})\\ &+\chi_{\mathcal{A}}^{\rightarrow}(r_2,s_2)(V^{\xi}_{\mathrm{D,s},\sigma_1,\sigma_2,(1)} - (\mathfrak{r}^{\xi}_{\infty,\mathrm{D,s},\sigma_2,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)}) - \hat{v}^{\xi}_{\infty,\mathrm{D,s},\sigma_2,\sigma_2,(1)})\\ &+(\mathfrak{r}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)},\mathfrak{s}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)}) + \hat{v}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)}.\end{aligned}$$

We remark that:

$$\spadesuit - \clubsuit = (\Delta\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}, \Delta\mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}). \tag{7.75}$$

Let:

$$\begin{aligned}\rho^{\xi,\Delta}_{1,(1)} &= \exp(-(\Delta\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)} + \sqrt{-1}\Delta\mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)})) \in \mathbb{C}_*\\ \rho^{\xi,\Delta}_{2,(1)} &= \exp(-(\Delta\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)} + \sqrt{-1}\Delta\mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)})) \in \mathbb{C}_*\end{aligned} \tag{7.76}$$

and

$$\rho^{\xi}_{i,(1)} = \rho^{\xi}_{i,(0)}\rho^{\xi,\Delta}_{i,(1)} \in \mathbb{C}_*, \tag{7.77}$$

for $i = 1,2$. Finally, we define:

$$\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)} := (u^{\xi,\prime}_{\mathrm{d},\sigma_1,\sigma_2,(1)}, u^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(1)}, U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)}, \sigma_1, \sigma_2, \rho^{\xi}_{1,(1)}, \rho^{\xi}_{2,(1)}). \tag{7.78}$$

Lemma 7.79. *The 7-tuple $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)}$ is an inconsistent map in the sense of Definition 7.35.*

Proof. This is a consequence of Lemma 7.36 and (7.74), (7.75). □

Remark 7.80. We remark that if we change $\mathbf{V}^{\xi}_{\mathrm{D,s},\sigma_1,\sigma_2,(1)}$ by an element of $\mathbb{R}\oplus\mathbb{R}$ (the tangent vector generated by the $\mathbb{C}_*$ action), then $V^{\xi}_{\mathrm{D,s},\sigma_1,\sigma_2,(1)}$, $(\mathfrak{r}^{\xi}_{\infty,\mathrm{D,d},\sigma_2,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)})$ and $(\mathfrak{r}^{\xi}_{\infty,\mathrm{D,s},\sigma_2,\sigma_2,(1)}, \mathfrak{s}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)})$ change by the same amount. Therefore, $\diamondsuit$ and $\clubsuit$ do not change. On the other hand, $\heartsuit$ and $\spadesuit$ change by the same element in $\mathbb{R}\oplus\mathbb{R}$. This implies that the equivalence class of $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)}$ is fixed among all representatives for $\mathbf{V}^{\xi}_{\mathrm{D,s},\sigma_1,\sigma_2,(1)}$.

(Step 1-3) (Error estimate)

$\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)}$ is our next approximate solution. Lemma 7.81 quantifies to what extent this inconsistent map improves the previous approximate solution $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)}$.

Lemma 7.81. *There is a constant C and for any positive number μ, there is a constant η such that the following holds. If σ_1,σ_2 are smaller than η, then there exists $\mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(1)} = (\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)}, \mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)}, \mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)})$ with $\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)} \in E_{\mathrm{d}}$, $\mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)} \in E_{\mathrm{s}}$, $\mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)} \in E_{\mathrm{D}}$ such that the following holds:*

(1)

$$\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(1)}\|_{L^2_{m,\delta}} \leqslant \mu\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{(0)}\|_{L^2_{m,\delta}} \tag{7.82}$$

where $\mathfrak{e}^{\xi}_{(0)} = (\mathfrak{e}^{\xi}_{\mathrm{d},(0)}, \mathfrak{e}^{\xi}_{\mathrm{s},(0)}, \mathfrak{e}^{\xi}_{\mathrm{D},(0)})$ is as in (7.30).

(2)

$$\|\mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(1)} - \mathfrak{e}^{\xi}_{(0)}\| \leqslant \mu C.$$

The square of the left hand side, by definition, is the sum of the squares of the factors associated to d, s, D.[18]

We define:

$$\begin{aligned} \mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)} &= \mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(0)} + \Delta\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(0)} \\ \mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)} &= \mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)} + \Delta\mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)} \\ \mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)} &= \mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(0)} + \Delta\mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(0)}. \end{aligned} \tag{7.83}$$

The proof of the estimates in the lemma is based on Lemma 7.66. That is to say, we use the estimate (7.67) of Lemma 7.66 and the fact that $\mathrm{V}^{\xi}_{\sigma_1,\sigma_2,(1)}$ is given by solving the linearized equation. The details of this estimate is similar to the proof of [FOOO16a, Proposition 5.17] and is omitted. In particular, to estimate the effect of the bump function appearing in Definition 7.73 (2), (4), (6) and (7), we use the 'drop of the weight' argument, which is explained in detail in [FOOO16a, right above Remark 5.21].

Lemma 7.84 below concerns the estimate of the difference between $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)}$ and $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)}$.

Lemma 7.84. *Let σ_1, σ_2 be small enough such that Lemma 7.66 holds. There is a fixed constant*[19] *C, independent of σ_1 and σ_2, such that:*

$$d_{W^2_{m+1,\delta}}(\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(1)}, \mathfrak{u}^{\xi,\prime}_{\sigma_2,\sigma_2,(0)}) \leqslant C\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(0)}\|_{L^2_{m,\delta}}.$$

[18] This estimate is provided for the first step of Newton's iteration. In the i-th step, a similar estimate appears where μC is replaced by $\mu^{\mathrm{i}}C$. It is important that C is independent of i.

[19] It is important that we can take the same constant for all the steps of inductive construction of Newton's iteration. The dependence of those constants to various choices are studied in detail in [FOOO16a]. So we do not repeat it here.

Proof. This is a consequence of Lemma 7.66 and definitions. ☐

(Step 1-4) (Separating error terms into three parts)
Define

$$
\begin{aligned}
\mathrm{Err}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)} &= \chi^{\leftarrow}_{1,\mathcal{X}}(\overline{\partial} u^{\xi,\prime}_{\mathrm{d},\sigma_1,\sigma_2,(1)} - \mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)}),\\
\mathrm{Err}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)} &= \chi^{\leftarrow}_{2,\mathcal{X}}(\overline{\partial} u^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(1)} - \mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)}),\\
\mathrm{Err}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)} &= \chi^{\rightarrow}_{\mathcal{X}}(\overline{\partial} u^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)} - \mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)}).
\end{aligned}
\tag{7.85}
$$

(Step 2-1) (Approximate solution for linearization)
We will next define

$$
\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(1)} = (u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(1)}, u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(1)}, U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)})
\tag{7.86}
$$

satisfying the properties in Lemmas 7.53, 7.57. This step is essentially the same as (Step 1-1). We mention a few points where the two steps slightly differ.

Let $v^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)} \in T_{p^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(0)}}\mathcal{D}$ and $v^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)} \in T_{p^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)}}\mathcal{D}$ be the element appearing at the beginning of Step 1-2. We put

$$
\begin{aligned}
p^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)} &= \mathrm{Exp}(p^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(0)}, v^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)})\\
p^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)} &= \mathrm{Exp}(p^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(0)}, v^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}).
\end{aligned}
\tag{7.87}
$$

We next define

$$
\begin{aligned}
c^{\xi}_{\mathrm{d},(1)} &= c^{\xi}_{\mathrm{d}} \exp(-(\mathfrak{r}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)} + \sqrt{-1}\mathfrak{s}^{\xi}_{\infty,\mathrm{d},\sigma_1,\sigma_2,(1)}))\\
c^{\xi}_{\mathrm{D,d},(1)} &= c^{\xi}_{\mathrm{D,d}} \exp(-(\mathfrak{r}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)} + \sqrt{-1}\mathfrak{s}^{\xi}_{\infty,\mathrm{D,d},\sigma_1,\sigma_2,(1)}))\\
c^{\xi}_{\mathrm{s},(1)} &= c^{\xi}_{\mathrm{s}} \exp(-(\mathfrak{r}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)} + \sqrt{-1}\mathfrak{s}^{\xi}_{\infty,\mathrm{s},\sigma_1,\sigma_2,(1)}))\\
c^{\xi}_{\mathrm{D,s},(1)} &= c^{\xi}_{\mathrm{D,s}} \exp(-(\mathfrak{r}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)} + \sqrt{-1}\mathfrak{s}^{\xi}_{\infty,\mathrm{D,s},\sigma_1,\sigma_2,(1)})).
\end{aligned}
$$

Then formulas similar to (7.19), (7.20), (7.23) and (7.24) hold.

In (7.51),(7.52),(7.55),(7.56), we replace c^{ξ}_{d} with $c^{\xi}_{\mathrm{d},(1)}$ and so on. In these formulas, we also replace (0) with (1). We thus define $u^{\xi,\prime\prime}_{\mathrm{d},\sigma_1,\sigma_2,(1)}$, $u^{\xi,\prime\prime}_{\mathrm{s},\sigma_1,\sigma_2,(1)}$, $U^{\xi,\prime\prime}_{\mathrm{D},\sigma_1,\sigma_2,(1)}$ and $\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(1)}$. Then $(\mathrm{Err}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(1)}, \mathrm{Err}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(1)}, \mathrm{Err}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(1)})$ determines an element of the space $L^2_{k,\delta}(\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(1)}; TX \otimes \Lambda^{0,1})$. (Lemma 7.65.)

We can then formulate an analogue of Lemma 7.66 where $\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(0)}$ is replaced by $\mathfrak{u}^{\xi,\prime\prime}_{\sigma_1,\sigma_2,(1)}$. Using this lemma, we can obtain $\mathrm{V}^{\xi}_{\sigma_1,\sigma_2,(2)}$, $\Delta\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(2)}$, $\Delta\mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(2)}$, $\Delta\mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(2)}$. The counterpart of the estimate in (7.67) can be used to give appropriate bounds for these four terms. This completes (Step 2-1). (Step 2-2) and (Step 2-3) can be carried out in the same way as in (Step 1-2) and (Step 1-3).

More generally, we can perform (Step i-1), (Step i-2) and (Step i-3) in the case that $|\sigma_1|, |\sigma_2|$ are smaller than a positive number ε_0 and obtain a sequence of inconsistent maps:

$$
\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\mathrm{i})} := (u^{\xi,\prime}_{\mathrm{d},\sigma_1,\sigma_2,(\mathrm{i})}, u^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(\mathrm{i})}, U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(\mathrm{i})}, \sigma_1, \sigma_2, \rho^{\xi}_{1,(\mathrm{i})}, \rho^{\xi}_{2,(\mathrm{i})}),
$$

and a triple:

$$
\mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\mathrm{i})} = (\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(\mathrm{i})}, \mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(\mathrm{i})}, \mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(\mathrm{i})}) \in E_{\mathrm{d}} \oplus E_{\mathrm{s}} \oplus E_{\mathrm{D}}
$$

such that

$$\begin{aligned}\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\mathrm{i}+1)} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\mathrm{i}+1)}\|^2_{L^2_{m,\delta}} &\leqslant \mu\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\mathrm{i})} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\mathrm{i})}\|^2_{L^2_{m,\delta}} \\ &\leqslant \mu^{\mathrm{i}+1}\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(0)} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(0)}\|^2_{L^2_{m,\delta}}\end{aligned} \tag{7.88}$$

and

$$\|\mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\mathrm{i}+1)} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\mathrm{i})}\| \leqslant \mu^{\mathrm{i}}C. \tag{7.89}$$

Moreover, we have:

$$d_{W^2_{m+1,\delta}}(\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\mathrm{i}+1)}, \mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\mathrm{i})}) \leqslant C\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\mathrm{i})} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\mathrm{i})}\|_{L^2_{m,\delta}}. \tag{7.90}$$

We make a remark that the constants ε_0 and C may be taken independent of i. But these constants might depend on m, the exponent in the weighted Sobolev space $L^2_{m,\delta}$.

The estimates in (7.88) and (7.90) imply that the sequence $\{\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\mathrm{i})}\}_{\mathrm{i}}$ converges in $W^2_{m+1,\delta}$. We denote the limit by:

$$\begin{aligned}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\infty)} := (&u^{\xi,\prime}_{\mathrm{d},\sigma_1,\sigma_2,(\infty)}, u^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(\infty)}, U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(\infty)}, \\ &\sigma_1, \sigma_2, \rho^{\xi}_{1,(\infty)}, \rho^{\xi}_{2,(\infty)}).\end{aligned} \tag{7.91}$$

The estimate in (7.89) implies that $\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(\mathrm{i})} \in E_{\mathrm{d}}$, $\mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(\mathrm{i})} \in E_{\mathrm{s}}$, $\mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(\mathrm{i})} \in E_{\mathrm{D}}$ converges as i goes to infinity. We denote the limit by:

$$\mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\infty)} = (\mathfrak{e}^{\xi}_{\mathrm{d},\sigma_1,\sigma_2,(\infty)}, \mathfrak{e}^{\xi}_{\mathrm{s},\sigma_1,\sigma_2,(\infty)}, \mathfrak{e}^{\xi}_{\mathrm{D},\sigma_1,\sigma_2,(\infty)}).$$

As a consequence of (7.88), we have:

$$\|\overline{\partial}\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\infty)} - \mathfrak{e}^{\xi}_{\sigma_1,\sigma_2,(\infty)}\|_{L^2_{m},\delta} = 0.$$

In other words, $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\infty)}$ satisfies (4.30). Thus $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\infty)}$ satisfies the requirements of an inconsistent solution (Definition 5.1) except possibly (4.31) (the transversal constraint).

7.8. **Completion of the Proof.** We are now in the position to complete the proof of Proposition 5.5. For any $\xi \in \mathcal{U}^+_{\mathrm{d}}\ {}_{\mathrm{ev}_{\mathrm{d}}}\times_{\mathrm{ev}^+_{\mathrm{D,d}}} \mathcal{U}^+_{\mathrm{D}}\ {}_{\mathrm{ev}_{\mathrm{D,s}}}\times_{\mathrm{ev}_{\mathrm{s}}} \mathcal{U}^+_{\mathrm{s}}$ and sufficiently small $\sigma_1, \sigma_2 \in \mathbb{C}$, we defined an inconsistent map $\mathfrak{u}^{\xi,\prime}_{\sigma_1,\sigma_2,(\infty)}$ in (7.91). For (σ_1, σ_2) we define

$$\mathrm{EVw}_{\sigma_1,\sigma_2} : \mathcal{U}^+_{\mathrm{d}}\ {}_{\mathrm{ev}_{\mathrm{d}}}\times_{\mathrm{ev}^+_{\mathrm{D,d}}} \mathcal{U}^+_{\mathrm{D}}\ {}_{\mathrm{ev}_{\mathrm{D,s}}}\times_{\mathrm{ev}_{\mathrm{s}}} \mathcal{U}^+_{\mathrm{s}} \to \mathcal{D} \times X^2$$

by:

$$\begin{aligned}&\mathrm{EVw}_{\sigma_1,\sigma_2}(\xi) = \\ &\qquad((\pi \circ U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(\infty)})(w_{\mathrm{D},1}), (\pi \circ U^{\xi,\prime}_{\mathrm{D},\sigma_1,\sigma_2,(\infty)})(w_{\mathrm{D},2}), U^{\xi,\prime}_{\mathrm{s},\sigma_1,\sigma_2,(\infty)}(w_{\mathrm{s}})).\end{aligned}$$

Here $w_{\mathrm{D},1}$, $w_{\mathrm{D},2}$, w_{s} are as in Definition 4.4.

Lemma 7.92. $\mathrm{EVw}_{\sigma_1,\sigma_2}$ *is transversal to* $\mathcal{N}_{\mathrm{D},1} \times \mathcal{N}_{\mathrm{D},2} \times \mathcal{N}_{\mathrm{s}}$ *for sufficiently small* σ_1, σ_2.

Proof. The map $\mathrm{EVw}_{\sigma_1,\sigma_2}$ converges to $\mathrm{EVw}_{0,0}$ in the C^1 sense as $\sigma_1, \sigma_2 \to 0$. Moreover, $\mathrm{EVw}_{0,0}$ is transversal to $\mathcal{N}_{\mathrm{D},1} \times \mathcal{N}_{\mathrm{D},2} \times \mathcal{N}_{\mathrm{s}}$ by assumption (Definition 4.2). The lemma follows from these observations. □

By definition

$$\bigcup_{\sigma_1,\sigma_2} (\mathrm{EVw}_{\sigma_1,\sigma_2})^{-1}(\mathcal{N}_{\mathrm{D},1} \times \mathcal{N}_{\mathrm{D},2} \times \mathcal{N}_{\mathrm{s}}) \times \{(\sigma_1, \sigma_2)\}$$

can be identified with $\mathcal{U}$, the set of inconsistent solutions. We also have:

$$(\mathrm{EVw}_{0,0})^{-1}(\mathcal{N}_{\mathrm{D},1} \times \mathcal{N}_{\mathrm{D},2} \times \mathcal{N}_{\mathrm{s}}) \cong \mathcal{U}_{\mathrm{d}} \ {}_{\mathrm{ev_d}} \times_{\mathrm{ev_{D,d}}} \mathcal{U}_{\mathrm{D}} \ {}_{\mathrm{ev_{D,s}}} \times_{\mathrm{ev_s}} \mathcal{U}_{\mathrm{s}}$$

by definition. Proposition 5.5 is a consequence of these facts.

Once Proposition 5.5 is proved the proof of Proposition 5.12 is similar to the proof of [FOOO16a, Theorem 6.4]. We have written the proof of Proposition 5.5 so that the construction of the inconsistent solutions are parallel to the gluing construction in [FOOO16a, Section 5]. Therefore, the proof of [FOOO16a, Section 6] can be applied with almost no change to prove Proposition 5.12. This completes the construction of the Kuranishi chart at the point $[\Sigma, z_0, u]$.

8. Kuranishi Charts: the General Case

Up to point, we constructed a Kuranishi chart of the space $\mathcal{M}_1^{\mathrm{RGW}}(L;\beta)$ at the particular point $[\Sigma, z_0, u]$ described in Section 2. In this section, we explain how this construction generalizes to an arbitrary point $\mathfrak{u}$ of $\mathcal{M}_{k+1}^{\mathrm{RGW}}(L;\beta)$. There is a DD-ribbon tree $\mathcal{R} = (R, c, \alpha, m, \lambda)$ such that $\mathfrak{u}$ belongs to $\mathcal{M}^0(\mathcal{R}) \subset \mathcal{M}_{k+1}^{\mathrm{RGW}}(L;\beta)$. (See [DF18a, Subsection 3.4]). Let $((\Sigma_v, \vec{z}_v, u_v); v \in C_0^{\mathrm{int}}(\mathcal{R}))$ be a representative for $\mathfrak{u}$. In the case that $c(v) = \mathrm{D}$, the image of u_v is contained in $\mathcal{D}$, and we are given a meromorphic section U_v of $u_v^*\mathcal{N}_{\mathcal{D}}(X)$. Recall that for each i, the set of all sections $\{U_v\}_{\lambda(v)=i}$ is well-defined up to an action of $\mathbb{C}_*$ [DF18a, Formula (3.78)]. We firstly, associate a combinatorial object to $\mathfrak{u}$ which is called a *very detailed DD-ribbon tree* and is the refinement of the notion of detailed DD-ribbon trees defined in [DF18a, Subsection 3.4].

Let $\hat{R}$ be the detailed tree associated to $\mathcal{R}$. Recall that each interior vertex v of $\hat{R}$ corresponds to a possibly nodal Riemann surface Σ_v. (See, for example, [DF18a, Figure 8].) We refine the detailed DD-ribbon tree $\hat{R}$ further to the very detailed DD-ribbon tree $\check{R}$ so that each vertex of $\check{R}$ corresponds to an irreducible component of Σ. To be more detailed, for each $v \in C_0^{\mathrm{int}}(\check{R})$, we form a tree $\mathcal{Q}_v$ such that the following holds.

(1) Each vertex corresponds to either an irreducible component of Σ_v or a marked point on it. The latter corresponds to an edge of $\hat{R}$, which contains v. We call any such vertex an exterior vertex.
(2) There are two types of edges in $\mathcal{Q}_v$. An edge of the first type joins two edges such that the corresponding irreducible components intersect. An edge of the second type is called an exterior edge and connects a vertex corresponding to a marked point to the vertex corresponding to the irreducible component containing the marked point.

We replace each interior vertex v of the detailed tree $\hat{R}$ with $\mathcal{Q}_v$ and identify exterior edges of $\mathcal{Q}_v$ with the corresponding edges of $\hat{R}$ containing v. We thus obtain a tree $\check{R}$, called the very detailed DD-ribbon tree associated to $\mathfrak{u}$, or the very detailed tree associated to $\mathfrak{u}$ for short. Figure 12 sketches an element $\mathfrak{u}$ of our moduli space. The associated detailed DD-ribbon tree $\hat{R}$ and the very detailed DD-ribbon tree $\check{R}$ are given in Figures 13 and 14.

We say an edge of $\check{R}$ is a *fine edge* if it does not correspond to an edge of the detailed DD-ribbon tree $\hat{R}$. In Figure 14, the fine edges are illustrated by narrow lines and level 0 edges are illustrated by dotted lines. An edge of $\check{R}$, which is not fine, is called a *thick edge*. We denote by $C_{\mathrm{fi}}^{\mathrm{int}}(\check{R})$ and $C_{\mathrm{th}}^{\mathrm{int}}(\check{R})$ the set of all fine and thick edges of $\check{R}$, respectively. The level of a vertex of $\check{R}$ induced by a vertex of $\mathcal{Q}_v$ is defined to be $\lambda(v)$. We do not associate a multiplicity number to a fine edge. Homology class of a vertex is the homology class of the map u on this component. The color of an interior vertex v of $\check{R}$, denoted by $c(v)$, is D if its level is positive. If this vertex has level 0, then its color is either s or d depending on whether Σ_v is a sphere or a disk.

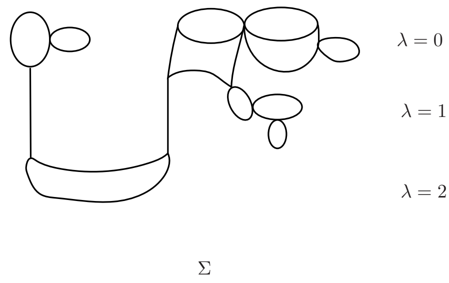

FIGURE 12. An element of the moduli space $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$

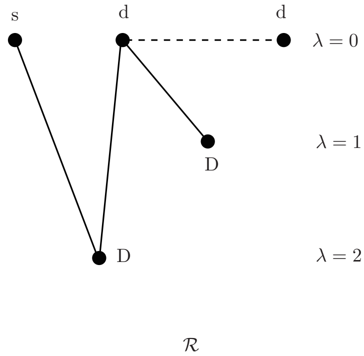

FIGURE 13. The detailed DD-ribbon tree $\hat R$

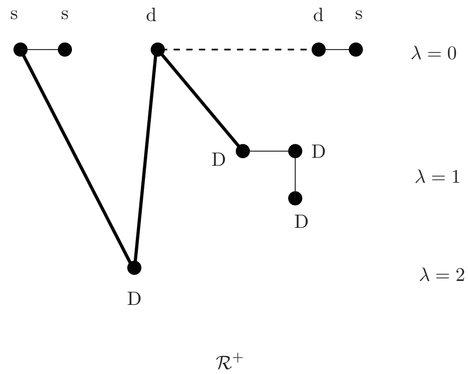

FIGURE 14. The very detailed DD-ribbon tree $\check R$

The notion of level shrinking and level 0 edge shrinking for very detailed DD-ribbon trees can be defined as in the case of detailed DD-ribbon trees. We define a *fine edge shrinking* as follows. We remove a fine edge e and identify the two vertices connected to each other by e. For two very detailed DD-ribbon trees $\check R$, $\check R'$, we say $\check R' \leqslant \check R$ if $\check R$ is obtained from $\check R'$ by a sequence of level shrinkings, level 0 edge shrinkings and fine edge shrinkings. Note that there might be a fine edge joining two vertices of level 0. We do *not* call any such edge a level 0 edge. The level 0 edges are limited to those joining vertices of color d.

We can stratify the moduli space $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ using very detailed DD-ribbon trees $\check R$. Namely, we define $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)(\check R)$ to be the subset of $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ consists of elements $\mathfrak{u}$ whose associated very detailed DD-ribbon tree is $\check R$. If $\check R' \leqslant \check R$, then the closure of $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)(\check R)$ contains $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)(\check R')$.

Let $\mathfrak{u} = ((\Sigma_v, \vec{z}_v, u_v); v \in C_0^{\rm int}(\check{R}))$ be an element of $\mathcal{M}_{k+1}^{\rm RGW}(L;\beta)(\check{R})$ as above. For an interior vertex v of $\check{R}$, the triple $\mathfrak{u}_v = (\Sigma_v, \vec{z}_v, u_v)$ is stable by definition. If $(\Sigma_v, \vec{z}_v)$ is not stable, then we may add auxiliary interior marked points $\vec{w}_v$ so that $(\Sigma_v, \vec{z}_v \cup \vec{w}_v)$ is stable. We assume that $\vec{w}_v$ is chosen such that the following symmetry assumption holds. Suppose $\Gamma_{\mathfrak{u}}$ denotes the group of automorphisms of $\mathfrak{u}$. Given $\gamma \in \Gamma_{\mathfrak{u}}$, for each interior vertex v, there exists a vertex $\gamma(v)$ and a bi-holomorphic map $\gamma_v : (\Sigma_v, \vec{z}_v) \to (\Sigma_{\gamma(v)}, \vec{z}_{\gamma(v)})$ such that $u_{\gamma(v)} \circ \gamma_v = u_v$. We assume $\vec{w}_v$ is mapped to $\vec{w}_{\gamma(v)}$ via γ_v. Note that the case $\gamma(v) = v$ is also included. For each member $w_{v,i}$ of $\vec{w}_v$, we take a codimension 2 submanifold $\mathcal{N}_{v,i}$ of X (resp. $\mathcal{D}$) if $c(v) = $ d or s (resp. if $c(v) = $ D.) We assume that the same condition as Condition 4.4 holds for these choices of *transversals*. If $\gamma \in \Gamma_{\mathfrak{u}}$ and $\gamma_v(w_{v,i}) = w_{v',i'}$, then we require that $\mathcal{N}_{v,i} = \mathcal{N}_{v',i'}$.

In order to define Cauchy-Riemann operators, we introduce function spaces similar to those of Section 3. For an interior vertex v of $\check{R}$, if the color of v is d, s or D, the Hilbert space $W^2_{m,\delta}(\mathfrak{u}_v; T)$ is respectively defined as in Definition 3.9, Definition 3.18 or Definition 3.23. Here T is a placeholder for the pull-back of the tangent bundle of X (if $c(v) = $ d, s) or $\mathcal{N}_{\mathcal{D}}X\backslash\mathcal{D}$ (if $c(v) = $ D). Similarly, for each v, we define the Weighted Sobolev spaces $L^2_{m,\delta}(\mathfrak{u}_v; T \otimes \Lambda^{0,1})$ as in Section 3.

Remark 8.1. In the case that $c(v) = $ d, the space Σ_v may have boundary nodes. In that case we take cylindrical coordinates on a neighborhood of each boundary node and use a cylindrical metric on this neighborhood. The approach here is very similar to the case of interior nodes which is discussed in Section 3. See [FOOO16a] for the case of boundary nodes in the context of the stable map compactification.

We also need to fix cylindrical coordinates for nodes corresponding to fine edges. In this case the target of the corresponding cylindrical end is contained in a compact subset of $X\backslash\mathcal{D}$ (for fine edges connecting level 0 vertices) or $\mathcal{N}_{\mathcal{D}}X\backslash\mathcal{D}$ (for fine edges connecting positive level vertices). In the first case we use the metric g given in Section 3. In the latter case, we use the metric on $\mathcal{N}_{\mathcal{D}}X\backslash\mathcal{D}$ with the form given in (3.2).

Let $W^{2,\sim}_{m,\delta}(\mathfrak{u}; T)$ be the subspace of the direct sum:

$$\bigoplus_{v \in C_0^{\rm int}(\check{R})} W^2_{m,\delta}(\mathfrak{u}_v; T) \tag{8.2}$$

consisting of elements $(V_v; v \in C_0^{\rm int}(\check{R}))$ with the following properties. Let e be an interior edge of $\check{R}$ joining v_1 and v_2. The source curve of the element $\mathfrak{u}_{v_i}$ contains a nodal point $z_{v_i,e}$ corresponding to the edge e. Suppose e is not a level 0 edge or a fine edge. By definition V_{v_i} has an asymptotic value $(\mathfrak{r}_{v_i,e}, \mathfrak{s}_{v_i,e}, v_{v_i,e}) \in \mathbb{R} \oplus \mathbb{R} \oplus T_{p_{v_i,e}}\mathcal{D}$ where $p_{v_i,e}$ is the point of $\mathcal{D}$ such that $u_v(z_{v_i,e}) = p_{v_i,e}$ and $\mathbb{R} \oplus \mathbb{R}$ corresponds to the tangent space of the partial $\mathbb{C}_*$-action. (See Definition 3.9.) We require:

$$v_{v_1,e} = v_{v_2,e}.$$

This condition is the counterpart of part (4) of Definition 7.60. In the case of a level 0 edge (resp. a fine edge), the corresponding asymptotic values are tangent vectors of L (resp. tangent vectors of X or $\mathcal{N}_X\mathcal{D}\backslash\mathcal{D}$) and we require that these two tangent vectors agree with each other. (See [FOOO16a, Definition 3.4].)

Analogous to Definition 7.60, there is an action of $(\mathbb{R} \oplus \mathbb{R})^{|\lambda|}$ on $W^{2,\sim}_{m,\delta}(\mathfrak{u}; T)$ with $|\lambda|$ being the number of levels of $\check{R}$. We define $W^2_{m,\delta}(\mathfrak{u}; T)$ to be the quotient space with respect to this action. We also write $L^2_{m,\delta}(\mathfrak{u}; T \otimes \Lambda^{0,1})$ for the direct sum of $L^2_{m,\delta}(\mathfrak{u}_v; T \otimes \Lambda^{0,1})$ for $v \in C_0^{\rm int}(\check{R})$.

The linearization of the Cauchy-Riemann equation associated to each vertex v of the very detailed tree $\check{R}$, defines the linear operator:

$$D_{\mathfrak{u}_v}\overline{\partial} : W^2_{m+1,\delta}(\mathfrak{u}_v; T) \to L^2_{m,\delta}(\mathfrak{u}_v; T \otimes \Lambda^{0,1}).$$

The direct sum of these operators together determines a Fredholm operator:

$$D_{\mathfrak{u}}\overline{\partial} : W^2_{m+1,\delta}(\mathfrak{u}; T) \to L^2_{m,\delta}(\mathfrak{u}; T \otimes \Lambda^{0,1}). \tag{8.3}$$

In the case that this operator is not surjective, we need to introduce obstruction spaces as in Section 4.

Definition 8.4. *Obstraction data* $E := \{E_v\}$ assign, for each interior vertex v, a vector space E_v such that the following conditions are satisfied:

(1) E_v is a finite dimensional subspace of $L^2_{m,\delta}(\Sigma_v \backslash \vec{z}_v; u_v^* T(X\backslash\mathcal{D}) \otimes \Lambda^{0,1})$ if $c(v) = \mathrm{d}$ or s, and is a finite dimensional subspace of $L^2_m(\Sigma_v; u_v^* T\mathcal{D} \otimes \Lambda^{0,1})$ if $c(v) = \mathrm{D}$. Moreover, E_v consists of smooth sections. Using the decomposition in (3.5), we can also regard E_v as a subspace of $L^2_{m,\delta}(\mathfrak{u}_v; T \otimes \Lambda^{0,1})$.
(2) Elements of E_v have compact supports away from nodal points and boundary.
(3) If u_v is a constant map, then E_v is 0.
(4) If $\gamma \in \Gamma_{\mathfrak{u}}$, then
$$(\gamma_v)_* E_v = E_{\gamma(v)}.$$
Here $(\gamma_v)_* : L^2_{m,\delta}(\mathfrak{u}_v; T \otimes \Lambda^{0,1}) \to L^2_{m,\delta}(\mathfrak{u}_{\gamma(v)}; T \otimes \Lambda^{0,1})$ is the map induced by $\gamma_v : \Sigma_v \to \Sigma_{\gamma(v)}$. (Recall that $u_{\gamma(v)} \circ \gamma_v = u_v$.)
(5) The operator $D_{\mathfrak{u}}\overline{\partial}$ in (8.3) is transversal to:

$$E_0 = \bigoplus_{v \in C_0^{\rm int}(\check{R})} E_v \subset L^2_{m,\delta}(\mathfrak{u}; T \otimes \Lambda^{0,1}). \tag{8.5}$$

It is straightforward to see that there are obstruction data satisfying the conditions in Definition 8.4. Since each operator $D_{\mathfrak{u}_v}\overline{\partial}$ is Fredholm, we can fix E_v, which satisfies part (1). This choice also would imply the required transversality in part (5). In the case that u_v is constant, we can pick E_v to be the trivial vector space because Σ_v has genus 0. (It is either a disk or a sphere.) Unique continuation implies that we can assume that the supports of the elements of E_v is contained in a compact subset of Σ_v away from the nodal points and boundary. By taking direct sums over the action of $\Gamma_{\mathfrak{u}}$ if necessary, we may also assume that (4) holds. Using E_0, we can define a thickened moduli space which gives a Kuranishi neighborhood of $\mathfrak{u}$ in a stratum of $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ which contains $\mathfrak{u}$. (See Definition 8.17.) In the upcoming sections, we give a systematic construction of the obstruction spaces E_0 which satisfy further compatibility assumptions.

We next discuss the process of gluing the irreducible components of $\mathfrak{u}$. We firstly need to explain how the deformation of source curves is parametrized. The mathematical content here is classical and we follow the approaches in [FOOO16a, Section 8] and [FOOO18, Section 3].

For an interior vertex v, we consider $(\Sigma_v, \vec{z}_v \cup \vec{w}_v)$. This is a disk or a sphere with marked points, which is stable. We may regard it as an element of the moduli space $\mathcal{M}_v^{\rm source}$. The space $\mathcal{M}_v^{\rm source}$ is metrizable and we fix one metric on it for our purposes later. The moduli space $\mathcal{M}_v^{\rm source}$ comes with a universal family:

$$\pi : \mathcal{C}_v^{\rm source} \to \mathcal{M}_v^{\rm source} \tag{8.6}$$

and $\#\vec{z}_v + \#\vec{w}_v$ sections which are in correspondence with the marked points. (See, for example, [FOOO18, Section 2].) For $\mathfrak{x} \in \mathcal{M}_v^{\rm source}$, the fiber $\pi^{-1}(\mathfrak{x})$ together with the values of the sections at $\mathfrak{x}$ determines a representative for $\mathfrak{x}$, which we denote it by $(\Sigma_{\mathfrak{x},v}, \vec{z}_{\mathfrak{x},v} \cup \vec{w}_{\mathfrak{x},v})$.

Since Σ_v has no singularity, (8.6) is a C^∞-fiber bundle near the point $(\Sigma_v, \vec{z}_v \cup \vec{w}_v)$. We fix a neighborhood $\mathcal{V}_v^{\rm source}$ of $[\Sigma_v, \vec{z}_v \cup \vec{w}_v]$ and a trivialization

$$\phi_v : \mathcal{V}_v^{\rm source} \times \Sigma_v \to \mathcal{C}_v^{\rm source}. \tag{8.7}$$

of (8.6) over this neighborhood. We assume that these trivializations are compatible with the automorphisms of $\mathfrak{u}$. For $\mathfrak{x}_v \in \mathcal{V}_v^{\rm source}$, we define a complex structure $j_{\mathfrak{x}_v}$ on Σ_v such that the restriction of the trivialization (8.7) to $\{\mathfrak{x}_v\} \times \Sigma_v$ defines a bi-holomorphic map $(\Sigma_v, j_{\mathfrak{x}_v}) \to \pi^{-1}(\mathfrak{x}_v)$.

Let e be an interior edge of $\check R$ containing v. There is a nodal point of $(\Sigma_v, \vec{z}_v)$ associated to e. Let $\mathfrak{s}_{v,e}$ be the section of (8.6) corresponding to this marked point. In the case of an interior node, an *analytic family of coordinates* at this nodal point is a holomorphic map

$$\varphi_{v,e} : \mathcal{V}_v^{\rm source} \times {\rm Int}(D^2) \to \mathcal{C}_v^{\rm source} \tag{8.8}$$

such that for each $\mathfrak{x} \in \mathcal{V}_v^{\rm source}$, we have $\varphi_{v,e}(\mathfrak{x}, 0) = \mathfrak{s}_{v,e}(\mathfrak{x})$ and the restriction of $\varphi_{v,e}$ to $\{\mathfrak{x}\} \times {\rm Int} D^2$ determines a holomorphic coordinate for $\Sigma_{\mathfrak{x}} = \pi^{-1}(\mathfrak{x})$ around $\varphi_{v,e}(\mathfrak{x}, 0) = \mathfrak{s}_{v,e}(\mathfrak{x})$. Thus $\varphi_{v,e}$ commutes with the projection map to $\mathcal{M}_v^{\rm source}$ and $\varphi_{v,e}$ is a bi-holomorphic map onto an open subset of $\mathcal{C}_v^{\rm source}$. When $z_{v,e}$ is a boundary node, we replace D^2 by $D^2_+ = \{z \in D^2 \mid {\rm Im} z \geqslant 0\}$ and define the notion of analytic family of coordinates in a similar way. (See [FOOO18, Section 3] for more details.) We require that the images of the maps $\varphi_{v,e}$ are disjoint and away from the image of the sections of $\mathcal{C}_v^{\rm source}$ corresponding to the auxiliary marked points $\vec{w}_v$. We also assume that the chosen analytic families are compatibile with automorphisms of $\mathfrak{u}$.

We use analytic family of coordinates $\varphi_{v,e}$ to desingularize the nodal points as follows. Fix an element:

$$\vec{\sigma} = (\sigma_e; e \in C_1^{\rm int}(\check R)) \in \prod_{e \in C_1^{\rm int}(\check R)} \mathcal{V}_e^{\rm deform}. \tag{8.9}$$

Here $\sigma_e \in D^2 =: \mathcal{V}_e^{\rm deform}$ if $z_{v,e}$ is an interior node, and $\sigma_e \in [0,1] =: \mathcal{V}_e^{\rm deform}$ if $z_{v,e}$ is a boundary node.[20] Let

$$\vec{\mathfrak{x}} = (\mathfrak{x}_v; v \in C_0^{\rm int}(\check R)) \in \prod_{v \in C_0^{\rm int}(\check R)} \mathcal{V}_v^{\rm source}. \tag{8.10}$$

We put

$$\begin{aligned} \Sigma_v^+(\vec{\mathfrak{x}}, \vec{\sigma}) = \Sigma_{\mathfrak{x}_v, v} \setminus & \bigcup_{(v,e): e \text{ is not a level 0 edge}} \varphi_{v,e}(\mathfrak{x}_v, D^2(|\sigma_e|)) \\ \setminus & \bigcup_{(v,e): e \text{ is a level 0 edge}} \varphi_{v,e}(\mathfrak{x}_v, D^2_+(\sigma_e)) \\ \Sigma_v^-(\vec{\mathfrak{x}}, \vec{\sigma}) = \Sigma_{\mathfrak{x}_v, v} \setminus & \bigcup_{(v,e): e \text{ is not a level 0 edge}} \varphi_{v,e}(\mathfrak{x}_v, D^2(1)) \\ \setminus & \bigcup_{(v,e): e \text{ is a level 0 edge}} \varphi_{v,e}(\mathfrak{x}_v, D^2_+(1)). \end{aligned} \tag{8.11}$$

Recall that a fine edge e connecting two level 0 vertices is not a level 0 edge by definition.The auxiliary marked points $\vec{w}_v$ determine a set of marked points on $\Sigma_v^-(\vec{\mathfrak{x}}, \vec{\sigma})$, which is also denoted by $\vec{w}_v$.

We define an equivalence relation $\sim$ on:

$$\bigcup_{v \in C_0^{\rm int}(\check R)} \Sigma_v^+(\vec{\mathfrak{x}}, \vec{\sigma}) \tag{8.12}$$

[20] Note that $z_{v,e}$ is a boundary node if and only if e is a level 0 edge.

as follows. Let e be an edge which is not a level 0 edge and connects the vertices v_1, v_2. Suppose $z_1, z_2 \in \operatorname{Int}(D^2)$ with $z_1 z_2 = \sigma_e$. Then:

$$\varphi_{v_1,e}(\mathfrak{x}_{v_1}, z_1) \sim \varphi_{v_2,e}(\mathfrak{x}_{v_2}, z_2).$$

Let e be a level 0 edge connecting the vertices v_1, v_2. Suppose $z_1, z_2 \in \operatorname{Int}(D^2_+)$ with $z_1 z_2 = -\sigma_e$. Then:

$$\varphi_{v_1,e}(\mathfrak{x}_{v_1}, z_1) \sim \varphi_{v_2,e}(\mathfrak{x}_{v_2}, z_2).$$

We divide the space (8.12) by the equivalence relation $\sim$ and denote the quotient space by

$$\Sigma(\vec{\mathfrak{x}}, \vec{\sigma}). \tag{8.13}$$

Let:

$$\sigma_e = \exp(-(10T_e + \theta_e\sqrt{-1})) \qquad e \text{ is not a level 0 edge,}$$
$$\sigma_e = \exp(-10T_e) \qquad e \text{ is a level 0 edge.}$$

For each $e \in C_1^{\rm int}(\check R)$, there is a corresponding neck region in $\Sigma(\vec{\mathfrak{x}}, \vec{\sigma})$. We define coordinates r_e, s_e on this region as follows. Suppose e is not a level 0 edge. We choose v_1, v_2 so that v_1 and the root of $\check R$ (corresponding to the zero-th exterior marked point z_0 of $\mathfrak{u}$) are in the same connected component of $\check R \backslash e$. Let:

$$\Sigma_v^+(\vec{\mathfrak{x}}, \vec{\sigma}) \backslash \Sigma_v^-(\vec{\mathfrak{x}}, \vec{\sigma}) = [-5T_e, 5T_e]_{r_e} \times S^1_{s_e}$$

where

$$(-5T_e, s_e) \in \operatorname{Closure}(\Sigma_{v_1}^-(\vec{\mathfrak{x}}, \vec{\sigma})) \qquad \forall s_e \in S^1.$$

The coordinate r_e, s_e is defined in the same way as in (7.1), (7.2).

If e is a level 0 edge, then we take v_1, v_2 so that v_1 and the root of $\check R$ are in the same connected component of $\check R \backslash e$. Then $\Sigma_{v_1}^+(\vec{\mathfrak{x}}, \vec{\sigma}) \backslash \Sigma_{v_1}^-(\vec{\mathfrak{x}}, \vec{\sigma})$ has a connected component corresponding to each edge which is incident to v_1. We identify the connected component corresponding to the edge e with:

$$[-5T_e, 5T_e]_{r_e} \times [0, \pi]_{s_e} \tag{8.14}$$

where the point $\varphi_{v_1,e}(\mathfrak{x}_v, \exp(-(r_e + 5T_e) - \sqrt{-1}s_e))$ is identified with (r_e, s_e) in (8.14). Similarly, $\Sigma_{v_2}^+(\vec{\mathfrak{x}}, \vec{\sigma}) \backslash \Sigma_{v_2}^-(\vec{\mathfrak{x}}, \vec{\sigma})$ has a connected component corresponding to each edge which is incident to v_2. We identify the connected component corresponding to the edge e with (8.14) where the point $\varphi_{v_2,e}(\mathfrak{x}_v, \exp((r_e - 5T_e) + \sqrt{-1}s_e))$ is identified with (r_e, s_e). These identifications are compatible with the equivalence relation $\sim$.

We thus have the decomposition:

$$\begin{aligned}\Sigma(\vec{\mathfrak{x}}, \vec{\sigma}) = &\bigcup_{v \in C_0^{\rm int}(\check R)} \Sigma_v^-(\vec{\mathfrak{x}}, \vec{\sigma}) \\ &\cup \bigcup_{e \in C_1^{\rm int}(\check R), e \text{ is not a level 0 edge}} [-5T_e, 5T_e]_{r_e} \times S^1_{s_e} \\ &\cup \bigcup_{e \in C_1^{\rm int}(\check R), e \text{ is a level 0 edge}} [-5T_e, 5T_e]_{r_e} \times [0, \pi]_{s_e}.\end{aligned} \tag{8.15}$$

This is the thick and thin decomposition which is used frequently in various kinds of Gromov-Witten theory. The inclusion of $\vec w_v$ in $\Sigma_v^-(\vec{\mathfrak{x}}, \vec{\sigma})$ induces a set of marked points in $\Sigma(\vec{\mathfrak{x}}, \vec{\sigma})$, which is also denoted by $\vec w_v$.

Definition 8.16. We call $\Xi = (\vec w_v, (\mathcal N_{v,i}), (\phi_v), (\varphi_{v,e}), \kappa)$ a choice of *trivialization and stabilization data* (*TSD*) for $\mathfrak{u}$. Here $w_{v,i}$ is the choice of the additional marked points and $\mathcal N_{v,i}$ the set of transversals, ϕ_v the trivializations of the universal family and analytic family of coordinates $\varphi_{v,e}$. The size of Ξ is the sum of a small number κ, the diameters of $\mathcal V_v^{\rm source}$ and the images of the maps $\varphi_{v,e}$. When we say Ξ is *small enough*, we mean that the size of Ξ is small enough. The way they are used is explained in Definition 8.17 below.

We call a pair (Ξ, E) of trivialization and stabilization data Ξ and obstraction data $E := \{E_v\}$ (as in Definition 8.4), trivialization, stalibization and obstruction data. (TSO).

We remark in Section 4 we introduced the notions of stabilization data and of stabilization and obstruction data. In the situation of Section 4 there is an obvious choice of trivializations and coordinates at the nodes.

Now we introduce several thickened moduli spaces which are used in the definition of our Kuranishi structures. We firstly define the stratum corresponding to $\check R$:

Definition 8.17. Given a TSO $\Upsilon = (\Xi, E)$ $(\Xi = (\vec w_v, (\mathcal N_{v,i}), (\phi_v), (\varphi_{v,e}), \kappa))$, the space $\widetilde{\mathcal U}(\mathfrak u, \check R, \Upsilon)$ consists of triples $(\vec{\mathfrak x}, u', U')$ with the following properties:

(1) $\vec{\mathfrak x} = (\mathfrak x_v; v \in C_0^{\rm int}(\check R)) \in \prod_{v\in C_0^{\rm int}(\check R)} \mathcal V_v^{\rm source}$. For each interior vertex v, $\mathfrak x_v$ belongs to the κ-neighborhood of the point of $\mathcal V_v^{\rm source}$ induced by $\mathfrak u$. A representative $(\Sigma_v, \vec z_{\mathfrak x,v} \cup \vec w_{\mathfrak x,v})$ of $\mathfrak x_v$ is also given where the irreducible component Σ_v is equipped with an almost complex structure $j_{\mathfrak x_v}$.

(2) $u' : \Sigma \to X$ is a continuous map, whose restriction u'_v to Σ_v is smooth. If $c(v) = {\rm D}$, then we require that $u'(\Sigma_v) \subset \mathcal D$. Moreover, a meromorphic[21] section U'_v of $(u'_v)^*(\mathcal N_{\mathcal D}(X))$ is also fixed such that the data of the zeros and poles of U'_v is determined by the multiplicity of the thick edges connected to v. If $c(v) = {\rm d}$, then the restriction of u'_v to the boundary of the disc Σ_v is mapped to L.

(3) The C^2-distance[22] between u'_v and u_v is less than κ. If $c(v) = {\rm D}$, then the C^2-distance[23] between U'_v and U_v is less than κ.

(4) We require

$$\overline\partial_{j_{\mathfrak x_v}} u'_v \in E_v(u'_v). \tag{8.18}$$

Here $j_{\mathfrak x_v}$ is the complex structure of Σ_v corresponding to $\mathfrak x_v$. (See the discussion proceeding (8.7).) Using the complex structure $j_{\mathfrak x_v}$ on Σ_v, we may define the target parallel transportation in the same way as in Section 4, and obtain $E_v(u'_v)$ from E_v. (We will explain the definition of $E_v(u'_v)$ more after Definition 8.17.) We also require:

$$\overline\partial_{j_{\mathfrak x_v}} U'_v \in E_v(u'_v). \tag{8.19}$$

if $c(v) = {\rm D}$. Here we use (3.5) to regard $E_v(u'_v)$ as a subspace of the function space $L^2_{m,\delta}(\Sigma_v; (U'_v)^* T\mathcal N_{\mathcal D}(X) \otimes \Lambda^{0,1})$.

(5) We require

$$u'_v(w_{v,i}) \in \mathcal N_{v,i}. \tag{8.20}$$

(6) If e is a fine edge connecting vertices v_1 and v_2 with color d or s, then the values of u'_{v_1} and u'_{v_2} at the node $\Sigma_{v_1} \cap \Sigma_{v_2}$ are equal to each other. If e is a fine edge connecting vertices v_1 and v_2 with color D, then the values of U'_{v_1} and U'_{v_2} at the node $\Sigma_{v_1} \cap \Sigma_{v_2}$ are equal to each other.

---

[21] The complex line bundle $(u'_v)^*(\mathcal N_{\mathcal D}(X))$ becomes a holomorphic line bundle by using the $(0,1)$ part of the (given) $U(1)$ connection. This is because the base space is complex one dimensional.

[22] If $c(v) = {\rm d}$ or s, then the C^2-distance is defined using the metric g on X, and if $c(v) = {\rm D}$, then the C^2-distance is defined using the metric g' on $\mathcal D$.

[23] The C^2-distance is defined with respect to a metric which has the form given in (3.2). Note that the set of sections $\{U_v\}_v$ is defined up to action of $\mathbb C_*^{|\lambda|}$, and here we mean than there is a representative for $\{U_v\}_v$ such that the distance between U'_v and U_v is less than κ.

We define an equivalence relation $\sim$ on $\widetilde{\mathcal{U}}(\mathfrak{u}, \check{R}, \Upsilon)$ as follows. Let $|\lambda|$ be the number of levels of the DD-ribbon tree associated to $\mathfrak{u}$. For $i = 1, \dots, |\lambda|$, we take $a_i \in \mathbb{C}_*$. We define $(\vec{\mathfrak{x}}, u', U'_{(0)}) \sim (\vec{\mathfrak{x}}, u', U'_{(1)})$

$$U'_{(1),v} = \mathrm{Dil}_{a_{\lambda(v)}} \circ U'_{(0),v}. \tag{8.21}$$

We denote the quotient space with respect to this equivalence relation by $\widehat{\mathcal{U}}(\mathfrak{u}, \check{R}, \Upsilon)$. The group of automorphisms $\Gamma_{\mathfrak{u}}$ acts on $\widehat{\mathcal{U}}(\mathfrak{u}, \check{R}, \Upsilon)$ in an obvious way. We write $\mathcal{U}(\mathfrak{u}, \check{R}, \Upsilon)$ for the quotient space $\widehat{\mathcal{U}}(\mathfrak{u}, \check{R}, \Upsilon)/\Gamma_{\mathfrak{u}}$.

We replace the condition (4) to

$$\overline{\partial}_{j_{\mathfrak{x}_v}} u'_v = 0, \qquad \overline{\partial}_{j_{\mathfrak{x}_v}} U'_v = 0 \tag{8.22}$$

to define $\widetilde{\mathfrak{U}}(\mathfrak{u}, \check{R}, \Xi)$, $\widehat{\mathfrak{U}}(\mathfrak{u}, \check{R}, \Xi)$ and $\mathfrak{U}(\mathfrak{u}, \check{R}, \Xi)$. Note that they depend only on TSD Ξ and is independent of the choice of E.

We now elaborate on the definition of $E_v(u'_v)$. Let $z \in \Sigma_v$ be a point in the support of E_v. There exists a map $I^0_v : \mathrm{Supp}E_v \to \Sigma'_v$ since Σ'_v is close to Σ_v. The map I^0_v is not much canonical. However we can modify it to more canonical map $I^t_v : \mathrm{Supp}E_v \to \Sigma$ as follows. There exists a unique point $I^t_v(z)$ in Σ which is close to I^0_v and that the minimal geodesic joining $u_v(z)$ to $u'_v(I_v(z))$ is perpendicular to the image of u_v. Note that this condition is a generalization of Conditions (1)(2) of Definition 4.8, where the identity map (that is not canonical) played the role of I^0_v. Now using I^t_v we define $E_v(u'_v)$ in the same way as (4.15).

The space $\mathcal{U}(\mathfrak{u}, \check{R}, \Upsilon)$ is a generalization of $\mathcal{U}_{\mathrm{d}}\ {}_{\mathrm{ev}_{\mathrm{d}}}\times_{\mathrm{ev}_{\mathrm{D,d}}} \mathcal{U}_{\mathrm{D}}\ {}_{\mathrm{ev}_{\mathrm{D,s}}}\times_{\mathrm{ev}_{\mathrm{s}}} \mathcal{U}_{\mathrm{s}}$ appearing in (5.6) and is a thickened version of a neighborhood of $\mathfrak{u}$ in the stratum $\mathcal{M}^0(\mathcal{R})$ of $\mathcal{M}^{\mathrm{RGW}}_{k+1}(L;\beta)$ defined in [DF18a, (3.78)]. The following lemma is a consequence of Definition 8.4 and the implicit function theorem.

Lemma 8.23. *If κ is small enough, then $\widehat{\mathcal{U}}(\mathfrak{u}, \check{R}, \Upsilon)$ is a smooth manifold and $\mathcal{U}(\mathfrak{u}, \check{R}, \Upsilon)$ is a smooth orbifold.*

We next introduce the generalization of the space $\mathcal{U}_0$ in Definition 4.29.

Definition 8.24. Let $\Xi = (\vec{w}_v, (\mathcal{N}_{v,i}), (\phi_v), (\varphi_{v,e}), \kappa)$ be a TSD at the element $\mathfrak{u}$ of $\mathcal{M}^{\mathrm{RGW}}_{k+1}(L,\beta)$ and $\Upsilon = (\Xi, E)$ a TSO. The space $\widehat{\mathcal{U}}_0(\mathfrak{u}, \Upsilon)$ consists of $(\vec{\mathfrak{x}}, \vec{\sigma}, u')$ with the following properties:

1. $\vec{\mathfrak{x}}(\mathfrak{x}_v; v \in C^{\mathrm{int}}_0(\check{R})) \in \prod_{v \in C^{\mathrm{int}}_0(\check{R})} \mathcal{V}^{\mathrm{source}}_v$ and $\vec{\sigma} = (\sigma_e; e \in C^{\mathrm{int}}_1(\check{R})) \in \prod_{e \in C^{\mathrm{int}}_1(\check{R})} \mathcal{V}^{\mathrm{deform}}_e$. Furthermore, for each interior vertex v, $\mathfrak{x}_v$ belongs to the κ-neighborhood of the point of $\mathcal{V}^{\mathrm{source}}_v$ induced by $\mathfrak{u}$. Similarly, for each e, we have $|\sigma_e| < \kappa$.
2. $u' : (\Sigma(\vec{\mathfrak{x}}, \vec{\sigma}), \partial(\Sigma(\vec{\mathfrak{x}}, \vec{\sigma}))) \to (X \backslash \mathcal{D}, L)$ is a continuous map and is smooth on each irreducible component.
3. If e is (resp. is not) a level 0 edge, then the image of the restriction to u' to $[-5T_e, 5T_e]_{r_e} \times [0,\pi]_{s_e}$ (resp. $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e}$) has a diameter[24] less than κ. If $c(v) = \mathrm{D}$, then the restriction of u' to $\Sigma^+_v(\vec{\mathfrak{x}}, \vec{\sigma})$ is included in the open neighborhood $\mathfrak{U}$ of $\mathcal{D}$.
4. If $c(v) = \mathrm{d}$ or s, then the C^2-distance between the restrictions of u' and u_v to $\Sigma^-_v(\vec{\mathfrak{x}}, \vec{\sigma})$ is less than κ. If $c(v) = \mathrm{D}$, then the previous part implies that the restriction of u' to $\Sigma^-_v(\vec{\mathfrak{x}}, \vec{\sigma})$ may be regarded as a map to $\mathcal{N}_{\mathcal{D}}(X)\backslash\mathcal{D}$. We also demand that the C^2-distance between this map and U_v is less than κ.[25]

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[24]The diameter is defined with respect to the metric g_{NC}.

[25]Here again we use the convention that the distance between an object and $\{U_v\}_{\lambda(v)>0}$, is defined to be the minimum of the relevant distance between that object and all representatives of $\{U_v\}_{\lambda(v)>0}$.

(5) We require

$$\overline{\partial}_{j_{\vec{\mathfrak{x}},\vec{\sigma}}} u' \in E_0(u'). \tag{8.25}$$

Here $j_{\vec{\mathfrak{x}},\vec{\sigma}}$ is the complex structure of $\Sigma(\vec{\mathfrak{x}},\vec{\sigma})$, and $E_0(u')$ is defined from E_v by target parallel transportation in the same way as in Section 4.

(6) We require

$$u'(w_{v,i}) \in \mathcal{N}_{v,i}. \tag{8.26}$$

The group of automorphisms $\Gamma_{\mathfrak{u}}$ acts on $\widehat{\mathcal{U}}_0(\mathfrak{u},\Upsilon)$ in the obvious way. We write $\mathcal{U}_0(\mathfrak{u},\Upsilon)$ for the quotient space $\widehat{\mathcal{U}}_0(\mathfrak{u},\Upsilon)/\Gamma_{\mathfrak{u}}$.

Remark 8.27. The above definition needs to be slightly modified if some of the components of $\vec{\sigma}$ are zero. Let e be an edge connecting a vertex of level i to a vertex of level $i+1$ such that $\sigma_e = 0$. If e' is another edge that connects a vertex of level i to a vertex of level $i+1$, then $\sigma_{e'} = 0$. Next, we decompose $\check{R}$ into several blocks such that $\sigma_e = 0$ for the edges e joining two different blocks and $\sigma_e \neq 0$ for an edge e, which is inside a block and is not a fine edge. In each block, we use Definition 8.24 and join spaces associated to various blocks in the same way as in Definition 8.17. We omit the details of this process because the actual space we use for the definition of our Kuranishi structure is not $\mathcal{U}_0(\mathfrak{u},\Upsilon)$ but $\mathcal{U}(\mathfrak{u},\Upsilon)$, introduced in Definition 8.28. We can also define $\mathcal{U}_0(\mathfrak{u},\Upsilon)$ as a subspace of $\mathcal{U}(\mathfrak{u},\Upsilon)$. We brought firstly Definition 8.24 because its geometric meaning is more clear.

The space $\mathcal{U}_0(\mathfrak{u},\Upsilon)$ in general is singular (not an orbifold). We introduce the notion of inconsistent solutions to thicken $\mathcal{U}_0(\mathfrak{u},\Upsilon)$ into an orbifold.

Definition 8.28. Let $\Xi = (\vec{w}_v, (\mathcal{N}_{v,i}), (\phi_v), (\varphi_{v,e}), \kappa)$ be a TSD at the element $\mathfrak{u}$ of $\mathcal{M}^{\rm RGW}_{k+1}(L,\beta)$ and $\Upsilon = (\Xi, E)$ a TSO. We say $(\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i))$ is an *inconsistent solution near* $\mathfrak{u}$ *with respect to* Υ if it satisfies the following properties:

(1) $\vec{\mathfrak{x}}(\mathfrak{x}_v; v \in C^{\rm int}_0(\check{R})) \in \prod_{v\in C^{\rm int}_0(\check{R})} \mathcal{V}^{\rm source}_v$, $\vec{\sigma} = (\sigma_e; e \in C^{\rm int}_1(\check{R})) \in \prod_{e\in C^{\rm int}_1(\check{R})} \mathcal{V}^{\rm deform}_e$ and $\rho_e \in \mathbb{C}$ for each edge $e \in C^{\rm int}_{\rm th}(\check{R})$ that is not a level 0 edge, and $\rho_i \in D^2$ for each level $i = 1,\dots,|\lambda|$. Furthermore, for each interior vertex v, $\mathfrak{x}_v$ belongs to the κ-neighborhood of the point of $\mathcal{V}^{\rm source}_v$ induced by $\mathfrak{u}$. Similarly, for each e, we have $|\sigma_e| < \kappa$.

(2) If $c(v) = {\rm d}$ (resp. s), then $u'_v : (\Sigma^+_v(\vec{\mathfrak{x}},\vec{\sigma}), \partial\Sigma^+_v(\vec{\mathfrak{x}},\vec{\sigma})) \to (X\backslash\mathcal{D}, L)$ (resp. $u'_v : \Sigma^+_v(\vec{\mathfrak{x}},\vec{\sigma}) \to X\backslash\mathcal{D}$) is a smooth map.

(3) If $c(v) = {\rm D}$, then $U'_v : \Sigma^+_v(\vec{\mathfrak{x}},\vec{\sigma}) \to \mathcal{N}_{\mathcal{D}}X\backslash\mathcal{D}$ is a smooth map, and $u'_v = \pi\circ U'_v$.

(4) $\rho_e = 0$ if and only if $\sigma_e = 0$.

(5) Suppose e is an edge connecting vertices v_0 and v_1 such that $\lambda(v_0) = 0$ and $\lambda(v_1) \geqslant 1$. Then we require:

$$u'_{v_0} = {\rm Dil}_{\rho_e}\circ U'_{v_1} \tag{8.29}$$

on $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e} = \Sigma^+_{v_1}(\vec{\mathfrak{x}},\vec{\sigma}) \cap \Sigma^+_{v_2}(\vec{\mathfrak{x}},\vec{\sigma})$ if $\sigma_e \neq 0$. In particular, we assume that the restriction of u'_{v_0} to $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e}$ is contained in the open neighborhood $\mathfrak{U}$ of $\mathcal{D}$. If $\sigma_e = 0$, then the values of u'_{v_0} and $\pi\circ U'_{v_1}$ at the nodal points corresponding to e are equal to each other.

(6) Suppose e is an edge connecting vertices v_1 and v_2 such that $\lambda(v_1) = i > 0$ and $\lambda(v_2) \geqslant i+1$. We require

$$U'_{v_1} = {\rm Dil}_{\rho_e}\circ U'_{v_2} \tag{8.30}$$

on $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e} = \Sigma^+_{v_1}(\vec{\mathfrak{x}},\vec{\sigma}) \cap \Sigma^+_{v_2}(\vec{\mathfrak{x}},\vec{\sigma})$ if $\sigma_e \neq 0$. If $\sigma_e = 0$, then the values of U'_{v_1} and $\pi\circ U'_{v_2}$ at the nodal points corresponding to e are equal.

(7) Suppose e is a level 0 edge connecting the vertices v_1 and v_2. If $\sigma_e \neq 0$, then we require:

$$u'_{v_1} = u'_{v_2} \tag{8.31}$$

on $[-5T_e, 5T_e]_{r_e} \times [0,1]_{s_e} = \Sigma^+_{v_1}(\vec{\mathfrak{x}}, \vec{\sigma}) \cap \Sigma^+_{v_2}(\vec{\mathfrak{x}}, \vec{\sigma})$. If $\sigma_e = 0$, then (8.31) holds at the nodal point corresponding to e.

(8) Suppose e is a fine edge connecting the vertices v_1 and v_2 with level zero (resp. with the same positive level). If $\sigma_e \neq 0$, then we require:

$$u'_{v_1} = u'_{v_2} \quad (\text{resp.} \quad U'_{v_1} = U'_{v_2}) \tag{8.32}$$

on $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e} = \Sigma^+_{v_1}(\vec{\mathfrak{x}}, \vec{\sigma}) \cap \Sigma^+_{v_2}(\vec{\mathfrak{x}}, \vec{\sigma})$. If $\sigma_e = 0$, then (8.32) holds at the nodal point corresponding to e.

(9) If e is (resp. is not) a level 0 edge, then the image of the restriction to u'_v to $[-5T_e, 5T_e]_{r_e} \times [0,\pi]_{s_e}$ (resp. $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e}$) has a diameter[26] less than κ.

(10) If $c(v) = $ d or s, then the C^2-distance between the restrictions of u'_v and u_v to $\Sigma^-_v(\vec{\mathfrak{x}}, \vec{\sigma})$ is less than κ. If $c(v) = $ D, then we demand that the C^2-distance between the restrictions of U'_v and U_v to $\Sigma^-_v(\vec{\mathfrak{x}}, \vec{\sigma})$ is less than κ.[27]

(11) If $c(v) = $ d or s, then we require:

$$\overline{\partial}_{j_{\vec{\mathfrak{x}},\vec{\sigma}}} u'_v \in E_v(u'_v). \tag{8.33}$$

Here $j_{\vec{\mathfrak{x}},\vec{\sigma}}$ is the complex structure of $\Sigma(\vec{\mathfrak{x}}, \vec{\sigma})$, and $E_v(u'_v)$ is defined from E_v by target parallel transportation in the same way as in Section 5.

(12) If $c(v) = $ D, then we require:

$$\overline{\partial}_{j_{\mathfrak{x}_v}} U'_v \in E_v(U'_v). \tag{8.34}$$

Here $j_{\vec{\mathfrak{x}},\vec{\sigma}}$ is the complex structure of $\Sigma(\vec{\mathfrak{x}}, \vec{\sigma})$, and we use (3.5) to obtain $E_v(U'_v)$ from $E_v(u'_v)$ as a subspace of $L^2_{m,\delta}(\Sigma_v; (U'_v)^* T\mathcal{N}_{\mathcal{D}}(X) \otimes \Lambda^{0,1})$.

(13) We have:

$$u'_v(w_{v,i}) \in \mathcal{N}_{v,i}. \tag{8.35}$$

We denote by $\widetilde{\mathcal{U}}(\mathfrak{u}, \Upsilon)$ the set of all $(\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i))$ satisfying the above properties. We define an equivalence relation $\sim$ on $\widetilde{\mathcal{U}}(\mathfrak{u}, \Upsilon)$ in the following way. Let $\mathbf{x}_j = (\vec{\mathfrak{x}}_{(j)}, \vec{\sigma}_{(j)}, (u'_{v,(j)}), (U'_{v,(j)}), (\rho_{e,(j)}), (\rho_{i,(j)}))$ be elements of $\widetilde{\mathcal{U}}(\mathfrak{u}, \Upsilon)$ for $j = 1, 2$. We say that $\mathbf{x}_1 \sim \mathbf{x}_2$ if there exists $a_i \in \mathbb{C}_*$ $(i = 1, \dots, |\lambda|)$ with the following properties. Let:

$$b_i = a_1 \cdots a_i \in \mathbb{C}_*.$$

(i) $\vec{\mathfrak{x}}_{(1)} = \vec{\mathfrak{x}}_{(2)}$, $\vec{\sigma}_{(1)} = \vec{\sigma}_{(2)}$, $u'_{v,(1)} = u'_{v,(2)}$.

(ii) $\rho_{i,(2)} = a_i \rho_{i,(1)}$.

(iii) $U'_{v,(1)} = \mathrm{Dil}_{b_{\lambda(v)}} \circ U'_{v,(2)}$

(iv) Suppose e is an edge connecting a vertex v_0 with $\lambda(v_0) = 0$ to a vertex v_1 with $\lambda(v_1) \geqslant 1$. Then we require:

$$\rho_{e,(2)} = b_{\lambda(v_2)} \rho_{e,(1)}.$$

(v) Suppose e is an edge connecting a vertex v_1 with $\lambda(v_1) \geqslant 1$ to a vertex v_2 with $\lambda(v_2) \geqslant 2$. Then we require:

$$\rho_{e,(2)} = a_{\lambda(v_1)+1} \cdots a_{\lambda(v_2)} \rho_{e,(1)}.$$

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[26] The diameter is defined with respect to the metric g_{NC}.

[27] Here again we use the convention that the distance between an object and $\{U_v\}_{\lambda(v)>0}$, is defined to be the minimum of the relevant distance between that object and all representatives of $\{U_v\}_{\lambda(v)>0}$.

We denote by $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$ the quotient space $\widetilde{\mathcal{U}}(\mathfrak{u},\Upsilon)/\sim$. The group $\Gamma_{\mathfrak{u}}$ acts on $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$ in an obvious way. We denote by $\mathcal{U}(\mathfrak{u},\Upsilon)$ the quotient space $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)/\Gamma_{\mathfrak{u}}$. We say an element of $\mathcal{U}(\mathfrak{u},\Upsilon)$ is an *inconsistent solution* near $\mathfrak{u}$ with respect to Υ. When it does not make any confusion, the elements of $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$ or $\widetilde{\mathcal{U}}(\mathfrak{u},\Upsilon)$ are also called inconsistent solutions near $\mathfrak{u}$ with respect to Υ.

We replace Conditions (11) and (12) by pseudo-holomorphicity to define $\widehat{\mathfrak{U}}(\mathfrak{u},\Xi)$, $\widetilde{\mathfrak{U}}(\mathfrak{u},\Xi)$ and $\mathfrak{U}(\mathfrak{u},\Xi)$.

Remark 8.36. Initially it might seem that the complex numbers ρ_i do not play any role in the definition of the elements of $\mathcal{U}(\mathfrak{u},\Upsilon)$. However, later they make it slightly easier for us to define the obstruction maps.

Our generalization of Proposition 5.5 claims that $\mathcal{U}(\mathfrak{u},\Upsilon)$ is a smooth orbifold. Before stating this result, we elaborate on the relationship between $\mathcal{U}(\mathfrak{u},\Upsilon)$ and $\mathcal{U}_0(\mathfrak{u},\Upsilon)$.

Definition 8.37. In the situation of Definition 8.28, we say $(\vec{\mathfrak{x}},\vec{\sigma},(u'_v),(U'_v),(\rho_e),(\rho_i))$ is an *inconsistent map near* $\mathfrak{u}$ *with respect to TSD* Ξ if Conditions (1)-(10), (13) of Definition 8.28 is satisfied. (We do not require (11) and (12) here.)

Definition 8.38. Let $(\vec{\mathfrak{x}},\vec{\sigma},(u'_v),(U'_v),(\rho_e),(\rho_i))$ be an inconsistent solution near $\mathfrak{u}$ with respect to Υ. We say that it satisfies *consistency equation* if for each edge e connecting vertices v_1 and v_2 with $0 \leqslant \lambda(v_1) < \lambda(v_2)$, we have:

$$\rho_e = \rho_{\lambda(v_1)+1}\cdots\rho_{\lambda(v_2)}. \tag{8.39}$$

It is easy to see that the consistency equation (8.39) is independent of the choice of the representative with respect to the relations given by $\sim$ and the action of $\Gamma_{\mathfrak{u}}$.

Lemma 8.40. *The set of inconsistent solutions near* $\mathfrak{u}$ *satisfying consistency equation can be identified with* $\mathcal{U}_0(\mathfrak{u},\Upsilon)$.

Proof. Let $(\vec{\mathfrak{x}},\vec{\sigma},(u'_v),(U'_v),(\rho_e),(\rho_i))$ be an inconsistent solution near $\mathfrak{u}$ satisfying consistency equations. For the simplicity of exposition, we consider the case that all the components of $\vec{\sigma}$ are nonzero.[28] Define τ_i to be the product $\rho_1\cdot\rho_2\dots\rho_i$. For each vertex v with $c(v) = D$, we also define:

$$U_v^{\mathfrak{m}} = \mathrm{Dil}_{\tau_{\lambda(v)}} \circ U'_v.$$

Then the maps $U_v^{\mathfrak{m}}$ for $c(v) = \mathrm{D}$ and u'_v for $c(v) = \mathrm{d}$ or s are compatible on the overlaps and by gluing them together, we obtain an element of $\mathcal{U}_0(\mathfrak{u},\Upsilon)$. The reverse direction is clear. □

Example 8.41. We consider the case of detailed DD-ribbon tree in Figure 1. This tree has two edges (whose multiplicities are p_1 and p_2, respectively). We denote them by e_{d} and e_{s}, respectively. Two parameters ρ_{d} and ρ_{s} are associated to these edges. (In Section 7, ρ_{d} and ρ_{s} are denoted by ρ_1 and ρ_2, respectively). The total number of levels is 1. So there is a parameter ρ associated to this level. The consistency equation (8.39) implies that:

$$\rho_{\mathrm{d}} = \rho = \rho_{\mathrm{s}}.$$

which is the same as the equation in (5.7).

For any $\ell \leqslant m-2$, we fix a C^ℓ structure on $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$ in the following way. For an interior vertex v of Σ_v, let Σ_v^- be the space $\Sigma_v^-(\vec{\mathfrak{x}},\vec{\sigma})$ in the case that $\vec{\sigma}=0$ and $\vec{\mathfrak{x}}$ is

[28] This is the case that we gave a detailed definition of $\mathcal{U}_0(\mathfrak{u},\Upsilon)$, after all. For other cases, this lemma can be used as the definition.

induced by $\mathfrak{u}$. The trivialization of the universal family allows us also to identify Σ_v^- with $\Sigma_v^-(\vec{\mathfrak{x}},\vec{\sigma})$ for different choices of $\vec{\mathfrak{x}}$, $\vec{\sigma}$. Define maps:

$$\mathrm{Res}_v : \widetilde{\mathcal{U}}(\mathfrak{u},\Upsilon) \to L^2_{m+1}(\Sigma_v^-, X\backslash\mathcal{D}) \qquad \text{if } \lambda(v)=0, \tag{8.42}$$

$$\mathrm{Res}_v : \widetilde{\mathcal{U}}(\mathfrak{u},\Upsilon) \to L^2_{m+1}(\Sigma_v^-, \mathcal{N}_{\mathcal{D}}X\backslash\mathcal{D}) \qquad \text{if } \lambda(v)>0, \tag{8.43}$$

such that $\mathrm{Res}_v(\vec{\mathfrak{x}},\vec{\sigma},(u'_v),(U'_v),(\rho_e),(\rho_i))$ is the restriction of u'_v or U'_v to $\Sigma_v^-(\vec{\mathfrak{x}},\vec{\sigma}) \cong \Sigma_v^-$. By unique continuation, Res_v and the obvious projection maps induce an embedding:

$$\begin{aligned}\widetilde{\mathcal{U}}(\mathfrak{u},\Upsilon) \to &\prod_{v\in C_0^{\mathrm{int}}(\check{R})} \mathcal{V}_v^{\mathrm{source}} \times \prod_{e\in C_1^{\mathrm{int}}(\check{R})} \mathcal{V}_e^{\mathrm{deform}} \times (D^2)^{|\lambda|}\\ &\times \prod_{v\in C_0^{\mathrm{int}}(\check{R}),\lambda(v)=0} L^2_{m+1}(\Sigma_v^-, X\backslash\mathcal{D})\\ &\times \prod_{v\in C_0^{\mathrm{int}}(\check{R}),\lambda(v)>0} L^2_{m+1}(\Sigma_v^-, \mathcal{N}_{\mathcal{D}}X\backslash\mathcal{D})\end{aligned} \tag{8.44}$$

We use this embedding to fix a C^ℓ-structure on $\widetilde{\mathcal{U}}(\mathfrak{u},\Upsilon)$. The group $\mathbb{C}_*^{|\lambda|}$ acts freely on the target and the domain of (8.44), and the above embedding is equivariant with respect to this action. We use the induced map at the level of the quotients to define a C^ℓ-structure on $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$. Note that we can define a slice for $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$ using the following idea. For each $1 \leqslant i \leqslant |\lambda|$, we fix an interior vertex v_i with $\lambda(v_i) = i$ and a base point $x_i \in \Sigma_{v_i}^-$. We also trivialize the bundle $\mathcal{N}_{\mathcal{D}}(X)$ in a neighborhood of $U_{v_i}(x_i)$. Each element of $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$ has a unique representative $(\vec{\mathfrak{x}},\vec{\sigma},(u'_v),(U'_v),(\rho_e),(\rho_i))$ such that $U'_{v_i}(x_i) = 1 \in \mathbb{C}$. Here we assume that κ is small enough such that $U'_{v_i}(x_i)$ belongs to the neighborhood of $U_{v_i}(x_i)$ that the pull back of $\mathcal{N}_{\mathcal{D}}(X)$ is trivialized.

Proposition 8.45. *The space $\widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$ is a C^ℓ-manifold and $\mathcal{U}(\mathfrak{u},\Upsilon)$ is a C^ℓ-orbifold. There exists a $\Gamma_{\mathfrak{u}}$-invariant open C^ℓ-embedding for $\ell \leqslant m-2$:*

$$\Phi : \prod_{e\in C_1^{\mathrm{int}}(\check{R})} \mathcal{V}_e^{\mathrm{deform}} \times \widehat{\mathcal{U}}(\mathfrak{u},\check{R},\Upsilon) \times D^2(\varepsilon)^{|\lambda|} \to \widehat{\mathcal{U}}(\mathfrak{u},\Upsilon)$$

with the following properties:

(1)
$$\Phi(\vec{\sigma},\xi,(\rho_i)) = [\vec{\mathfrak{x}},\vec{\sigma},(u'_{\vec{\sigma},\xi,v}),(U'_{\vec{\sigma},\xi,v}),(\rho_e(\vec{\sigma},\xi)),(\rho_i)]$$
Namely, the gluing parameters $\vec{\sigma}$ are preserved by the map Φ. Moreover, the deformation parameter $\vec{\mathfrak{x}}$ is the same as the one for the source curve of ξ.

(2) *For each edge $e \in C_{\mathrm{th}}^{\mathrm{int}}(\check{R})$ that is not a level 0 edge, there exists a nonzero smooth function f_e such that:*
$$\rho_e(\vec{\sigma},\xi) = f_e(\vec{\sigma},\xi)\sigma_e^{m(e)}$$
where $m(e)$ is the multiplicity of the edge e.

(3) *Let $\xi = (\vec{\mathfrak{x}},u',U') \in \widehat{\mathcal{U}}(\mathfrak{u},\check{R},\Upsilon)$ and $\vec{\sigma}_0$ be the vector that σ_e are all zero. Then we have:*
$$\Phi(\vec{\sigma}_0,\xi,(\rho_i)) = [\vec{\mathfrak{x}},\vec{\sigma}_0,(u'_{\vec{\sigma}_0,\xi,v}),(U'_{\vec{\sigma}_0,\xi,v}),(\rho_e(\vec{\sigma}_0,\xi)),(\rho_i)],$$
where $u'_{\vec{\sigma}_0,\xi,v}$ is the restriction u'_v of u', $U'_{\vec{\sigma}_0,\xi,v}$ is the restriction of U'_v and $\rho_e(\vec{\sigma}_0,\xi) = 0$.

The proof of Proposition 8.45 is essentially the same as the proof of Proposition 5.5, and it is only notationally more involved.

We next state a generalization of Proposition 5.12. For a thick edge e which is not of level 0, we define T_e, θ_e, $\mathfrak{R}_e$, η_e using the following identities:

$$\sigma_e = \exp(-(T_e + \sqrt{-1}\theta_e)),\quad \rho_e = \exp(-(\mathfrak{R}_e + \sqrt{-1}\eta_e)). \tag{8.46}$$

If e is a level 0 edge, then we define T_e using:

$$\sigma_e = \exp(-T_e). \tag{8.47}$$

We may also define T_e and θ_e for a fine edge as in (8.46). Using Φ, we regard $\mathfrak{R}_e$, η_e as functions of $T_{e'}$, $\theta_{e'}$ and ξ. We again use the trivialization of the universal family to identify $\Sigma_v^-(\vec{\mathfrak{x}}_0, \vec{\sigma})$ (see (8.11).) for various choices of $\vec{\mathfrak{x}}_0$, $\vec{\sigma}$. For the purpose of the next proposition, we also regard $u'_{\vec{\sigma},\xi,v}$, $U'_{\vec{\sigma},\xi,v}$ as maps

$$u'_{\vec{\sigma},\xi,v} : \Sigma_v^-(\vec{\sigma}) \to X\backslash\mathcal{D}$$
$$U'_{\vec{\sigma},\xi,v} : \Sigma_v^-(\vec{\sigma}) \to \mathcal{N}_{\mathcal{D}}X\backslash\mathcal{D}.$$

In particular, the domain of these maps are independent of T_e, θ_e and ξ.

Proposition 8.48. *Let ℓ be an arbitrary positive integer and k_e, k'_e be non-negative integers. Let $\upsilon_e = 0$ if $k_e, k'_e = 0$. Otherwise we define $\upsilon_e = 1$.*

(1) *We have the following exponential decay estimates:*

$$\left\| \prod_e \frac{\partial^{k_e}}{\partial^{k_e} T_e}\frac{\partial^{k'_e}}{\partial^{k'_e}\theta_e} u'_{\vec{\sigma},\xi,v} \right\|_{L^2_\ell(\Sigma_v^-(\vec{\sigma}))} \leqslant C\exp(-c\sum \upsilon_e T_e).$$
$$\left\| \prod_e \frac{\partial^{k_e}}{\partial^{k_e} T_e}\frac{\partial^{k'_e}}{\partial^{k'_e}\theta_e} U'_{\vec{\sigma},\xi,v} \right\|_{L^2_\ell(\Sigma_v^-(\vec{\sigma}))} \leqslant C\exp(-c\sum \upsilon_e T_e). \tag{8.49}$$

Here C, c are positive constants depending on ℓ, k_e, k'_e. The same estimate holds for the ξ derivatives of $u'_{v,\vec{\sigma},\xi}$, $U'_{v,\vec{\sigma},\xi}$.

(2) *For any thick edge e_0 which is not a level 0 edge, we also have the following exponential decay estimates:*

$$\left| \prod_e \frac{\partial^{k_e}}{\partial^{k_e} T_e}\frac{\partial^{k'_e}}{\partial^{k'_e}\theta_e} (\mathfrak{R}_{e_0} - m(e_0)T_{e_0}) \right| \leqslant C\exp(-c\sum \upsilon_e T_e)$$
$$\left| \prod_e \frac{\partial^{k_e}}{\partial^{k_e} T_e}\frac{\partial^{k'_e}}{\partial^{k'_e}\theta_e} (\eta_{e_0} - m(e_0)\theta_{e_0}) \right| \leqslant C\exp(-c\sum \upsilon_e T_e). \tag{8.50}$$

Here C, c are positive constants depending on ℓ, k_e, k'_e. The same estimate holds for the ξ derivatives of $\mathfrak{R}_{e_0}$, η_{e_0}.

Similar to Proposition 5.12, Proposition 8.48 can be verified using the same argument as in the proof of [FOOO16a, Section 6].

We now use Propositions 8.45 and 8.48 to produce a Kuranishi chart at $\mathfrak{u}$. Let $\mathfrak{y} = (\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i))$ be a representative of an element of $\widehat{\mathcal{U}}(\mathfrak{u}, \Upsilon)$. Recall that we fix vector spaces E_v for each $\mathfrak{u}$, and use target parallel transportation to obtain the vector spaces $E_v(u'_v)$ and $E_v(U'_v)$. We define:

$$\mathcal{E}_{0,\mathfrak{u}}(\mathfrak{y}) = \bigoplus_{v\in C_0^{\rm int}(\check R),\, \lambda(v)=0} E_v(u'_v) \oplus \bigoplus_{v\in C_0^{\rm int}(\check R),\, \lambda(v)>0} E_v(U'_v). \tag{8.51}$$

Using Proposition 8.48, it is easy to see that (8.51) defines a $\Gamma_{\mathfrak{u}}$-equivariant C^ℓ vector bundle on $\widehat{\mathcal{U}}(\mathfrak{u}, \Upsilon)$.

We define the other part of the obstruction bundle as follows. Let e be a thick edge which connects vertices v_1 and v_2 with $0 \leqslant \lambda(v_1) < \lambda(v_2)$. We fix the trivial line bundle $\mathbb{C}_e$ on $\widetilde{\mathcal{U}}(\mathfrak{u}, \Upsilon)$. Let $\mathbf{x}_1 \sim \mathbf{x}_2$ and $a_i \in \mathbb{C}_*$ $(i = 1, \dots, |\lambda|)$ be as in Definition 8.28 (i)-(v). Then define an equivalence relation on $\mathbb{C}_e$ where $(\mathbf{x}_1, V_1) \sim (\mathbf{x}_2, V_2)$ if:

$$V_2 = a_{\lambda(v_1)+1} \dots a_{\lambda(v_2)} V_1.$$

We thus obtain a line bundle $\mathscr{L}_e$ on $\widehat{\mathcal{U}}(\mathfrak{u}, \Upsilon)$. The group $\Gamma_{\mathfrak{u}}$ acts on $\bigoplus_e \mathscr{L}_e$ in an obvious way. Our obstruction bundle $\mathcal{E}_{\mathfrak{u}}$ on $\widehat{\mathcal{U}}(\mathfrak{u}, \Upsilon)$ is defined to be:

$$\mathcal{E}_{\mathfrak{u}} = \mathcal{E}_{0,\mathfrak{u}} \oplus \bigoplus_{e \in C^{\rm int}_{\rm th}(\check{R}),\, \lambda(e)>0} \mathscr{L}_e. \tag{8.52}$$

It induces an orbi-bundle on $\mathcal{U}(\mathfrak{u}, \Upsilon)$. By an abuse of notation, this orbi-bundle is also denoted by $\mathcal{E}_{\mathfrak{u}}$.

Next, we define Kuranishi maps. If v is a vertex with $\lambda(v) = 0$, then we define:

$$\mathfrak{s}_{\mathfrak{u},v}(\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) = \overline{\partial} u'_v \in E_v(u'_v). \tag{8.53}$$

If v is a vertex with $\lambda(v) > 0$, then we define:

$$\mathfrak{s}_{\mathfrak{u},v}(\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) = \overline{\partial} U'_v \in E_v(U'_v). \tag{8.54}$$

If e is a thick edge connecting vertices v_1 and v_2 with $0 \leqslant \lambda(v_1) < \lambda(v_2)$, then we define:

$$\mathfrak{s}_{\mathfrak{u},e}(\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) = \rho_e - \rho_{\lambda(v_1)+1} \cdots \rho_{\lambda(v_2)}. \tag{8.55}$$

We define:

$$\mathfrak{s}_{\mathfrak{u}} = ((\mathfrak{s}_{\mathfrak{u},v}; v \in C^{\rm int}_0(\check{R})), (\mathfrak{s}_{\mathfrak{u},e}; e \in C^{\rm int}_{\rm th}(\check{R}), \lambda(e) > 0)).$$

It is easy to see that $\mathfrak{s}_{\mathfrak{u}}$ induces a $\Gamma_{\mathfrak{u}}$-invariant section of $\mathcal{E}_{\mathfrak{u}}$. Using Proposition 8.48, we can show that the section $\mathfrak{s}_{\mathfrak{u}}$ is smooth.

Suppose $\mathfrak{s}_{\mathfrak{u}}(\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i)) = 0$. By (8.55) and Lemma 8.40, it induces an element $(\vec{\mathfrak{x}}, \vec{\sigma}, u')$ of $\widehat{\mathcal{U}}_0(\mathfrak{u}, \Upsilon)$. By (8.53) and (8.54) the map u' is pseudo-holomorphic. Therefore, $(\Sigma(\vec{\mathfrak{x}}, \vec{\sigma}), u')$ and marked points on $\Sigma_{\vec{\mathfrak{x}}}$ determine an element of $\mathcal{M}^{\rm RGW}_{k+1}(L; \beta)$. This element does not change if we change $(\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i))$ by the $\Gamma_{\mathfrak{u}}$-action. We thus obtained:

$$\psi_{\mathfrak{u}} : \mathfrak{s}_{\mathfrak{u}}^{-1}(0)/\Gamma_{\mathfrak{u}} \to \mathcal{M}^{\rm RGW}_{k+1}(L; \beta), \tag{8.56}$$

which is a homeomorphism onto an open neighborhood of $\mathfrak{u}$. We thus proved:

Theorem 8.57. $\mathcal{U}_{\mathfrak{u}} = (\mathcal{U}(\mathfrak{u}), \mathcal{E}_{\mathfrak{u}}, \Gamma_{\mathfrak{u}}, \mathfrak{s}_{\mathfrak{u}}, \psi_{\mathfrak{u}})$ *givecs a Kuranshi chart for the moduli space* $\mathcal{M}^{\rm RGW}_{k+1}(L; \beta)$ *at* $\mathfrak{u}$.

9. Construction of Kuranishi Structures

So far, we constructed a Kuranishi chart at each point of $\mathcal{M}^{\rm RGW}_{k+1}(L; \beta)$. In this section we construct a global Kuranishi structure. We follow similar arguments as in [FOOO12, FOOO18]. However, there are certain points that our treatment is different. We discuss the construction emphasizing on those differences.

9.1. **Compatible Trivialization and Stabilization Data.** Throughout this subsection, we fix:

$$\mathfrak{u}_{(j)} = ((\Sigma_{(j),v}, \vec{z}_{(j),v}, u_{(j),v}); v \in C^{\rm int}_0(\check{R}_{(j)})) \qquad j = 1, 2$$

an element of $\mathcal{M}^{\rm RGW}_{k+1}(L; \beta)$ which is contained in the stratum corresponding to the very detailed DD-ribbon trees $\check{R}_{(j)} = (c_{(j)}, \alpha_{(j)}, m_{(j)}, \lambda_{(j)})$. We denote the union of the irreducible components $\Sigma_{(j),v}$ by $\Sigma_{(j)}$. The map $u_{(j)}$ are also defined similarly, and $\vec{z}_{(j)}$ is the set of boundary marked points of $\Sigma_{(j)}$. We use a similar convention several times in this section. We assume that $\mathfrak{u}_{(2)}$ belongs to a small neighborhood of $\mathfrak{u}_{(1)}$ in the RGW-topology. To be more precise, for $\mathfrak{u}_{(j)}$, let $\Xi_{(j)} = (\vec{w}_{(j)}, (\mathcal{N}_{(j),v,i}), (\phi_{(j),v}), (\varphi_{(j),v,e}), \kappa_{(j)})$ be a fixed TSD. We assume that $\mathfrak{u}_{(2)}$ is represented by an element of the space $\mathfrak{U}(\mathfrak{u}_{(1)}, \Xi_{(1)})$.

This assumption implies that $\check{R}_{(2)}$ is obtained from $\check{R}_{(1)}$ by level shrinkings, level 0 edge shrinkings and fine edge shrinkings. In particular, we may regard:

$$C_1^{\rm int}(\check{R}_{(2)}) \subseteq C_1^{\rm int}(\check{R}_{(1)}).$$

There also exists a surjective map $\pi : \check{R}_{(1)} \to \check{R}_{(2)}$ inducing:

$$\pi : C_0^{\rm int}(\check{R}_{(1)}) \to C_0^{\rm int}(\check{R}_{(2)})$$

such that the irreducible component corresponding to $v \in C_0^{\rm int}(\check{R}_{(2)})$ is obtained by gluing the irreducible components corresponding to $\hat{v} \in \pi^{-1}(v) \subset C_0^{\rm int}(\check{R}_{(1)})$. There also exists a surjective map:

$$\nu : \{0, 1, \dots, |\lambda_{(1)}|\} \to \{0, 1, \dots, |\lambda_{(2)}|\}$$

such that $i \leqslant j$ implies $\nu(i) \leqslant \nu(j)$, and $\lambda_{(2)}(\pi(\hat{v})) = \nu(\lambda_{(1)}(\hat{v}))$ for inside vertices $\hat{v}$ of $\check{R}_{(1)}$. The maps π and ν are the analogue of treesh and levsh in [DF18a, Lemma 4.47] defined for detailed trees.

To describe the coordinate change, it is convenient to start with the case that the TSDs $\Xi_{(1)}$ and $\Xi_{(2)}$ satisfy some compatibility conditions. In this subsection, we discuss these compatibility conditions and in Subsection 9.3, we explain how a coordinate change can be constructed assuming these conditions. In Subsection 9.4, we consider the case that $\Xi_{(1)}$ and $\Xi_{(2)}$ are two (not necessarily compatible) TSDs associated to the same element of the moduli space. We combine the results of Subsections 9.3 and 9.4 in Subsection 9.5 to define coordinate changes in the general case and verify the co-cycle condition for these coordinate changes.

The assumption that $\mathfrak{u}_{(2)}$ belongs to a small neighborhood of $\mathfrak{u}_{(1)}$ implies that we can find:

$$\vec{\sigma}_0 = (\sigma_{0,e}; e \in C_1^{\rm int}(\check{R}_{(1)})) \in \prod_{e \in C_1^{\rm int}(\check{R}_{(1)})} \mathcal{V}^{\rm deform}_{(1),e}$$

and

$$\vec{\mathfrak{x}}_0 = (\mathfrak{x}_{0,v}; v \in C_0^{\rm int}(\check{R}_{(1)})) \in \prod_{v \in C_0^{\rm int}(\check{R}_{(1)})} \mathcal{V}^{\rm source}_{(1),v}$$

such that the inconsistent map:

$$(\Sigma_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0), \vec{z}_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0), u_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0)) \tag{9.1}$$

is isomorphic to $(\Sigma_{(2)}, \vec{z}_{(2)}, u_{(2)})$. Although it is not clear from the notation, the map $u_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0)$ in (9.1) depends on $\mathfrak{u}_{(2)}$ and not just on $\vec{\mathfrak{x}}_0$ and $\vec{\sigma}_0$. We assume that the additional marked points and transversals in $\Xi_{(2)}$ satisfy the following conditions:

Condition 9.2. *Since (9.1) is induced by an element of $\mathfrak{U}(\mathfrak{u}_{(1)}, \Xi_{(1)})$, there is a set of marked points $\vec{w}_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0) \subset \Sigma^-_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0)$ determined by $\vec{w}_{(1)}$, $\vec{\mathfrak{x}}_0$ and $\vec{\sigma}_0$. Then we require that the marked points $\vec{w}_{(2)}$ of $\Xi_{(2)}$ are chosen such that:*

$$(\Sigma_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0), \vec{z}_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0) \cup \vec{w}_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0)) \cong (\Sigma_{(2)}, \vec{z}_{(2)} \cup \vec{w}_{(2)}). \tag{9.3}$$

Furthermore, if $w_{(2),v,i}$ corresponds to $w_{(1),\hat{v},\hat{i}}$, then we require:[29]

$$\mathcal{N}_{(2),v',i'} = \mathcal{N}_{(1),v,i}. \tag{9.4}$$

[29]Here we use the correspondence between $\vec{w}_{(1)}$ $\vec{w}_{(2)}$ given by the identification in (9.3) and the correspondence between the elements of $\vec{w}_{(1)}$ and $\vec{w}_{(1)}(\vec{\mathfrak{x}}_0, \vec{\sigma}_0)$.

Next, we impose some constraints on the choices of the maps $\phi_{(2),v}$ and $\varphi_{(2),v,e}$. Let v be an interior vertex of $\check R_{(2)}$. We consider the moduli space $\mathcal{M}_v^{\rm source}$ of deformation of the irreducible component $(\Sigma_{(2),v}, \vec z_{(2),v} \cup \vec w_{(2),v})$. We firstly fix a neighborhood $\mathcal{V}^{\rm source}_{(2),v}$ of $[\Sigma_{(2),v}, \vec z_{(2),v} \cup \vec w_{(2),v}]$ in $\mathcal{M}_v^{\rm source}$ as follows. The Riemann surface $\Sigma_{(2),v}$ is obtained by gluing spaces $\Sigma_{(1),\hat v}$ for $\hat v \in \pi^{-1}(v)$. Here the complex structure on $\Sigma_{(1),\hat v}$ is given by $\mathfrak{x}_{0,\hat v}$ and the gluing parameters are $\sigma_{(0),\hat e}$ for edges $\hat e$ in $\pi^{-1}(v)$. There is a neighborhood $\mathcal{U}^{\rm source}_{(1),\hat v}$ of $\mathfrak{x}_{0,\hat v}$ in $\mathcal{V}^{\rm source}_{(1),\hat v}$ and a neighborhood $\mathcal{V}^{\rm deform}_{(1),(2),e}$ of $\sigma_{(0),\hat e} \in \mathcal{V}^{\rm deform}_{(1),e}$ such that the following map:

$$\prod_{\hat v \in C_0^{\rm int}(\check R_{(1)}),\, \pi(\hat v)=v} \mathcal{U}^{\rm source}_{(1),\hat v} \times \prod_{e \in C_1^{\rm int}(\check R_{(1)}),\, \pi(e)=v} \mathcal{V}^{\rm deform}_{(1),(2),e} \to \mathcal{M}_v^{\rm source}.$$

is an isomorphism onto an open neighborhood of the point determined by $(\Sigma_{(2),v}, \vec z_{(2),v} \cup \vec w_{(2),v})$. Therefore, we may define:

$$\mathcal{V}^{\rm source}_{(2),v} := \prod_{\hat v \in C_0^{\rm int}(\check R_{(1)}),\, \pi(\hat v)=v} \mathcal{U}^{\rm source}_{(1),\hat v} \times \prod_{\hat e \in C_1^{\rm int}(\check R_{(1)}),\, \pi(\hat e)=v} \mathcal{V}^{\rm deform}_{(1),\hat e} \tag{9.5}$$

Let $\mathfrak{x}_{2,v} = ((\mathfrak{x}_{1,\hat v}), (\sigma_{1,\hat e}))$ be an element of (9.5). Then $\Sigma_{(2),v}(\mathfrak{x}_{2,v})$, the Riemann surface $\Sigma_{(2),v}$ with the complex structure induced by $\mathfrak{x}_{2,v}$, has the following decomposition:

$$\Sigma_{(2),v}(\mathfrak{x}_{2,v}) = \coprod_{\hat v \in C_0^{\rm int}(\check R_{(1)}),\, \pi(\hat v)=v} \Sigma^-_{(1),\hat v}(\mathfrak{x}_{1,\hat v}) \tag{9.6}$$

$$\cup \coprod_{e \in C_1^{\rm int}(\check R_{(2)}),\, v \in \partial e} D^2 \tag{9.7}$$

$$\cup \coprod_{\hat e \in C_1^{\rm int}(\check R_{(1)}),\, \pi(\hat e)=v} [-5T_{\hat e}, 5T_{\hat e}] \times S^1. \tag{9.8}$$

The following comments about the above decomposition is in order. In (9.6), $\Sigma^-_{(1),\hat v}(\mathfrak{x}_{(1),\hat v})$ denotes the subspace of $\Sigma_{(1),\hat v}(\mathfrak{x}_{(1),\hat v})$ given by the complements of $\varphi_{(1),\hat v,\hat e}(\mathfrak{x}_{(1),\hat v}, D^2(1))$ where $\hat e$ runs among the edges of $\check R_{(1)}$ which are connected to $\hat v$. For each edge $e \in C_1^{\rm int}(\check R_{(2)})$ which is incident to $v \in C_0^{\rm int}(\check R_{(2)})$, there is a unique edge $\hat e \in C_1^{\rm int}(\check R_{(1)})$ which is mapped to e. In particular, one of the endpoints of $\hat e$, denoted by $\hat v$, is mapped to v. The disc corresponding to e in (9.7) is given by the space $\varphi_{(1),\hat v,\hat e}(\mathfrak{x}_{(1),\hat v}, D^2(1))$. Finally if an edge $\hat e \in C_1^{\rm int}(\check R_{(1)})$ is mapped to a vertex $v \in C_0^{\rm int}(\check R_{(2)})$ by π, then the space in (9.8) is identified with the neck region associated to $\hat e$. In particular, the positive number $T_{\hat e}$ is determined by $\sigma_{(1),\hat e}$. The union of the spaces in (9.6) and (9.7) is called the thick part of $\Sigma_{(2),v}(\mathfrak{x}_{(2),v})$, and the spaces in (9.8) form the thin part of $\Sigma_{(2),v}(\mathfrak{x}_{(2),v})$. The above decomposition can be used in an obvious way to define the map $\varphi_{(2),v,e}$ on $\mathcal{V}^{\rm source}_{(2),v} \times {\rm Int}(D^2)$.

We have the following decomposition of $\Sigma_{(2),v}$ as a special case of the above decomposition applied to the point $((\mathfrak{x}_{0,\hat v}), (\sigma_{0,\hat e}))$:

$$\Sigma_{(2),v} = \coprod_{\hat v \in C_0^{\rm int}(\check R_{(1)}),\, \pi(\hat v)=v} \Sigma^-_{(1),v}(\mathfrak{x}_{0,\hat v}) \tag{9.9}$$

$$\cup \coprod_{e \in C_1^{\rm int}(\check R_{(2)}),\, v \in \partial e} D^2 \tag{9.10}$$

$$\cup \coprod_{\hat e \in C_1^{\rm int}(\check R_{(1)}),\, \pi(\hat e)=v} [-5T'_{\hat e}, 5T'_{\hat e}] \times S^1. \tag{9.11}$$

The trivialization $\phi_{(2),v}$ that we intend to define is a family (parametrized by $\mathfrak{x}_{2,v}$) of diffeomorphisms from $\Sigma_{(2),v}$ to $\Sigma_{(2),v}(\mathfrak{x}_{2,v})$. The trivialization $\phi_{(1),v}$ defines a diffeomorphism between the subspaces in (9.6) and (9.9). We then use the coordinate at nodal points, $\varphi_{(1),\hat{v},\hat{e}}$, to extend it to a diffeomorphism from the unions of the subspaces in (9.9) and (9.10) to the union of the subsapces in (9.6) and (9.7). Finally we extend this family of diffeomorphisms in an arbitrary way to the neck region to complete the construction of $\phi_{(2),v}$. This construction of the maps $\varphi_{(2),v,e}$ and $\phi_{(2),v}$ is analogous to [FOOO18, Sublemma 10.15].

Condition 9.12. *We require that the maps $\phi_{(2),v}$ and $\varphi_{(2),v,e}$ of the TSD $\Xi_{(2)}$ are obtained form the TSD $\Xi_{(1)}$ as above.*

Definition 9.13. We say that a TSD $\Xi_{(2)}$ is *induced* from a TSD $\Xi_{(1)}$ if Conditions 9.2 and 9.12 are satisfied.

Definition 9.14. Let $\mathfrak{u}$ be an element of $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ and Ξ be a TSD at $\mathfrak{u}$. An *inconsistent map near* $\mathfrak{u}$ *with respect to* Ξ is an object similar to the ones in Definition 8.37 where we do not require the Cauchy-Riemann equations.

This definition is almost a straightforward generalization of Definition 7.35 with the difference that we also include transversal constraints in (4.31) as one of the requirements for an inconsistent map near $\mathfrak{u}$.

Suppose $\Xi_{(2)}$ is induced from $\Xi_{(1)}$. For any:

$$\begin{aligned}\vec{\sigma}_{(2)} &= (\sigma_{2,e}; e \in C^{\rm int}_1(\check{R}_{(2)})) \in \prod_{e\in C^{\rm int}_1(\check{R}_{(2)})} \mathcal{V}^{\rm deform}_{(2),e} \\ \vec{\mathfrak{x}}_{(2)} &= (\mathfrak{x}_{2,v}; v \in C^{\rm int}_0(\check{R}_{(2)})) \in \prod_{v\in C^{\rm int}_0(\check{R}_{(2)})} \mathcal{V}^{\rm source}_{(2),v},\end{aligned} \tag{9.15}$$

satisfying Definition 8.28 (1) with respect to $\Xi_{(2)}$, there exist:

$$\begin{aligned}\vec{\sigma}_{(1)} &= (\sigma_{1,e}; e \in C^{\rm int}_1(\check{R}_{(1)})) \in \prod_{e\in C^{\rm int}_1(\check{R}_{(1)})} \mathcal{V}^{\rm deform}_{(1),e} \\ \vec{\mathfrak{x}}_{(1)} &= (\mathfrak{x}_{1,v}; v \in C^{\rm int}_0(\check{R}_{(1)})) \in \prod_{v\in C^{\rm int}_0(\check{R}_{(1)})} \mathcal{V}^{\rm source}_{(1),v}.\end{aligned} \tag{9.16}$$

satisfying Definition 8.28 (1) with respect to $\Xi_{(1)}$ such that:

$$\begin{aligned}(\Sigma_{(2)}(\vec{\mathfrak{x}}_{(2)},\vec{\sigma}_{(2)}), \vec{z}_{(2)}(\vec{\mathfrak{x}}_{(2)},\vec{\sigma}_{(2)}) \cup \vec{w}_{(2)}(\vec{\mathfrak{x}}_{(2)},\vec{\sigma}_{(2)})) &\cong \\ (\Sigma_{(1)}(\vec{\mathfrak{x}}_{(1)},\vec{\sigma}_{(1)}), \vec{z}_{(1)}(\vec{\mathfrak{x}}_{(1)},\vec{\sigma}_{(1)}) &\cup \vec{w}_{(1)}(\vec{\mathfrak{x}}_{(1)},\vec{\sigma}_{(1)})).\end{aligned} \tag{9.17}$$

Here $\vec{\mathfrak{x}}_{(1)}, \vec{\sigma}_{(1)}$ depend on $\vec{\mathfrak{x}}_{(2)}, \vec{\sigma}_{(2)}$. (See Figure 15.)

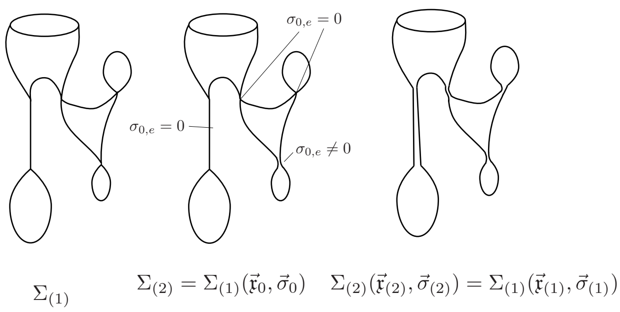


FIGURE 15. $\Sigma_{(1)}$, $\Sigma_{(2)}$ and $\Sigma_{(2)}(\vec{\mathfrak{x}}_{(2)}, \vec{\sigma}_{(2)})$.

Next, let $\mathfrak{y}_{(2)} = (\vec{\mathfrak{x}}_{(2)}, \vec{\sigma}_{(2)}, (u'_{(2),v}), (U'_{(2),v}), (\rho_{(2),e}), (\rho_{(2),i}))$ be an inconsitent map near $\mathfrak{u}_{(2)}$ with respect to $\Xi_{(2)}$. Let $\vec{\mathfrak{x}}_{(1)}$ and $\vec{\sigma}_{(1)}$ be chosen as in the previous paragraph. Let $\hat{v}$ be an interior vertex of $\check{R}_{(1)}$. Identification in (9.17) and Condition 9.12 imply:

$$\Sigma^+_{(1),\hat{v}}(\vec{\mathfrak{x}}_{(1)}, \vec{\sigma}_{(1)}) \subseteq \Sigma^+_{(2),\pi(\hat{v})}(\vec{\mathfrak{x}}_{(2)}, \vec{\sigma}_{(2)}). \tag{9.18}$$

We write $I_{\hat{v}}$ for this inclusion map. Define:

$$U'_{(1),\hat{v}} = \begin{cases} u'_{(2),\pi(\hat{v})} \circ I_{\hat{v}} & \text{if } \lambda_{(2)}(\pi(\hat{v})) = 0 \\ U'_{(2),\pi(\hat{v})} \circ I_{\hat{v}} & \text{if } \lambda_{(2)}(\pi(\hat{v})) > 0 \end{cases} \tag{9.19}$$
$$u'_{(1),\hat{v}} = u'_{(2),\pi(\hat{v})} \circ I_{\hat{v}}.$$

We also define:

$$\rho_{(1),e} = \begin{cases} \rho_{(2),e} & \text{if } e \in C^{\rm int}_{\rm th}(\check{R}_{(2)}) \subset C^{\rm int}_{\rm th}(\check{R}_{(1)}) \\ & \text{and } e \text{ is not a level 0 edge,} \\ 1 & \text{otherwise.} \end{cases} \tag{9.20}$$

We next define:

$$\rho_{(1),i} = \begin{cases} 1 & \text{if } \nu(i-1) = \nu(i) \\ \rho_{(2),i} & \text{otherwise.} \end{cases} \tag{9.21}$$

It is easy to check that $\mathfrak{y}_{(1)} = (\vec{\mathfrak{x}}_{(1)}, \vec{\sigma}_{(1)}, (u'_{(1),\hat{v}}), (U'_{(1),\hat{v}}), (\rho_{(1),e}), (\rho_{(1),i}))$ is an inconsistent map near $\mathfrak{u}_{(1)}$ with respect to $\Xi_{(1)}$. In fact, (8.29)-(8.32) for $\mathfrak{y}_{(1)}$ follow from the corresponding identities for $\mathfrak{y}_{(2)}$ and the definition. This discussion is summarized in the following lemma:

Lemma 9.22. *Suppose $\mathfrak{u}_{(1)}$ and $\mathfrak{u}_{(2)}$ are as above and $\Xi_{(j)}$ is a TSD at $\mathfrak{u}_{(j)}$. We assume that $\Xi_{(2)}$ is induced from $\Xi_{(1)}$. Then an inconsistent map near $\mathfrak{u}_{(2)}$ with respect to $\Xi_{(2)}$ can be regarded as an inconsistent map near $\mathfrak{u}_{(1)}$ with respect to $\Xi_{(1)}$.*

Remark 9.23. The equality (9.20) in the second case shows that the consistency equation is satisfied for such edges.

Using the above argument, we can also verify the following lemma:

Lemma 9.24. *Suppose $\mathfrak{u}_{(1)}$, $\mathfrak{u}_{(2)}$, $\Xi_{(1)}$ and $\Xi_{(2)}$ are given as in Lemma 9.22. In particular, $\mathfrak{u}_{(2)}$ can be regarded as an inconsistent solution*[30]*:*

$$(\vec{\mathfrak{x}}_{(0)}, \vec{\sigma}_{(0)}, (u'_{(0),\hat{v}}), (U'_{(0),\hat{v}}), (\rho_{(0),e}), (\rho_{(0),i}))$$

with respect to $\Xi_{(1)}$. An inconsistent map:

$$\mathfrak{y}_{(1)} = (\vec{\mathfrak{x}}_{(1)}, \vec{\sigma}_{(1)}, (u'_{(1),\hat{v}}), (U'_{(1),\hat{v}}), (\rho_{(1),e}), (\rho_{(1),i}))$$

near $\mathfrak{u}_{(1)}$ with respect to $\Xi_{(1)}$ may be regarded as an inconsistent map near $\mathfrak{u}_{(2)}$ with respect to $\Xi_{(2)}$ if and only if the following conditions hold:

- (i) *For each vertex $\hat{v} \in C^{\rm int}_0(\hat{R}_{(1)})$, the distance between $\mathfrak{x}_{(0),\hat{v}}$ and $\mathfrak{x}_{(1),\hat{v}}$ is less than $\kappa_{(2)}$. (Recall that $\kappa_{(2)}$ is the size of $\Xi_{(2)}$).*
- (ii) *For each edge $e \in C^{\rm int}_1(\hat{R}_{(2)}) \subset C^{\rm int}_1(\hat{R}_{(1)})$, we have $|\sigma_{(1),e}| < \kappa_{(2)}$.*
- (iii) *If $e \in C^{\rm int}_1(\hat{R}_{(2)}) \subset C^{\rm int}_1(\hat{R}_{(1)})$ is (resp. is not) a level 0 edge, then the image of the restriction of $u'_{(1),\hat{v}}$ to $[-5T_e, 5T_e]_{r_e} \times [0,\pi]_{s_e}$ (resp. $[-5T_e, 5T_e]_{r_e} \times S^1_{s_e}$) has a diameter less than $\kappa_{(2)}$.*

[30]More precisely it is an element of $\widetilde{\mathfrak{U}}(\mathfrak{u}, \Xi)$.

(iv) *If* $c(v) = \mathrm{d}$ *or* s*, then the* C^2*-distance between the restrictions of* $u'_{(0),\hat{v}}$ *and* $u'_{(1),\hat{v}}$ *to* $\Sigma_v^-(\vec{\mathfrak{x}},\vec{\sigma})$ *is less than* $\kappa_{(2)}$*. If* $c(v) = \mathrm{D}$*, then we demand that the* C^2*-distance between the restrictions of* $U'_{(0),\hat{v}}$ *and* $U'_{(1),\hat{v}}$ *to* $\Sigma_v^-(\vec{\mathfrak{x}},\vec{\sigma})$ *is less than* $\kappa_{(2)}$*.*

(v) *The consistency equation*

$$\rho_{(1),e} = \rho_{(1),\lambda_{(1),\hat{v}_1}+1} \cdots \rho_{(1),\lambda_{(1),\hat{v}_2}}$$

are satisfied for any edge $e \in C^{\rm int}_{\rm th}(\check{R}_{(1)}) \backslash C^{\rm int}_{\rm th}(\check{R}_{(2)})$ *which is not a level* 0 *edge.*

9.2. **The Choice of Obstruction Spaces.** In order to define an inconsistent *solution*, we need to fix obstruction spaces. For any inconsistent map $\mathfrak{u}$ and for any other inconsistent map $\mathfrak{y}$ which is close to $\mathfrak{u}$, we explained how to make a choice of $\mathcal{E}_{0,\mathfrak{u}}(\mathfrak{y})$ in Section 8. We need to insure that we can arrange for such choices such that they satisfy some nice properties when we move $\mathfrak{u}$. More precisely, we need to pick them so that they are *semi-continuous* with respect to $\mathfrak{u}$. We will prove this property in the next subsection. In this subsection, we explain how we modify our choice of obstruction spaces.

For $j = 1, 2$, let $\mathfrak{u}_{(j)} = ((\Sigma_{(j),v}, \vec{z}_{(j),v}, u_{(j),v}); v \in C^{\rm int}_0(\check{R}_{(j)}))$ be representatives of two elements of $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ in the strata corresponding to very detailed DD-ribbon trees $\check{R}_{(j)}$. We fix a TSD $\Xi_{(j)}$ at $\mathfrak{u}_{(j)}$. We do *not* assume that they are related as in Subsection 9.1. We also fix obstruction bundle data $E = \{E_{\mathfrak{u}_{(1)},v}\}$, where $E_{\mathfrak{u}_{(1)},v} \subset L^2_{m,\delta}(\mathfrak{u}_{(1),v})$ satisfying the conditions in Definition 8.4. We wish to use E to define obstruction spaces for an inconsistent map with respect to $\Xi_{(2)}$, under the assumption that $\mathfrak{u}_{(2)}$ is close to $\mathfrak{u}_{(1)}$. In particular, we assume that $\check{R}_{(2)}$ is obtained from $\check{R}_{(1)}$ by level shrinking, level 0 edge shrinking and fine edge shrinking. As in the previous subsection, we may define a surjective map $\pi : \check{R}_{(1)} \to \check{R}_{(2)}$. Note that in Section 6, we studied the case $\mathfrak{u}_{(2)} = \mathfrak{u}_{(1)}$.

The following lemma is a straightforward consequence of the implicit function theorem: (See [FOOO18, Lemma 9.9].)

Lemma 9.25. *If* $\mathfrak{u}_{(2)}$ *is close enough to* $\mathfrak{u}_{(1)}$ *with respect to the* C^1 *distance, then for any* $\hat{v} \in C^{\rm int}_0(\check{R}_{(1)})$*, there exists a unique choice of* $\vec{w}_{(2),(1),\hat{v}} \subset \Sigma_{(2),v}$ *with* v *being* $\pi(\hat{v})$ *such that:*

(1) $(\Sigma_{(2),v}, \vec{z}_{(2),v} \cup \coprod_{\pi(\hat{v})=v} \vec{w}_{(2),(1),v})$ *is close to:*

$$\coprod_{\pi(\hat{v})=v} (\Sigma_{(1),\hat{v}}, \vec{z}'_{(1),\hat{v}} \cup \vec{w}_{(1),\hat{v}})$$

in the moduli space of stable curves with marked points. Here $\vec{z}'_{(1),\hat{v}}$ *is the subset of* $\vec{z}_{(1),\hat{v}}$ *that is the set of all marked points on* $\Sigma_{(1),v}$ *and nodal points on* $\Sigma_{(1),v}$ *which correspond to the edges* e *incident to* v *with* $\pi(e) \neq v$*.*

(2) $u_{(2),v}(w_{(2),(1),\hat{v},i}) \in \mathcal{N}_{(1),\hat{v},i}$*.*

From now on, we assume that $\mathfrak{u}_{(2)}$ is close enough to $\mathfrak{u}_{(1)}$ such that the claim in Lemma 9.25 holds. Furthermore, let also an inconsistent map:

$$\mathfrak{y} = (\vec{\mathfrak{x}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i))$$

with respect to $\Xi_{(2)} = (\vec{w}_{(2)}, (\mathcal{N}_{(2),v,i}), (\phi_{(2),v}), (\varphi_{(2),v,e}), \kappa_{(2)})$ be fixed. Suppose also $(\Sigma(\vec{\mathfrak{x}},\vec{\sigma}), \vec{z}(\vec{\mathfrak{x}},\vec{\sigma}) \cup \vec{w}(\vec{\mathfrak{x}},\vec{\sigma}))$ denotes the representative of $\vec{\mathfrak{x}}$, as a part of the data of $\mathfrak{y}$.

Let $\hat{v}$ be a vertex of $\check{R}_{(1)}$ and $v = \pi(\hat{v})$. Lemma 9.25 allows us to find $w_{(2),(1),\hat{v},i} \in \Sigma_{(2),v}$. If $\Xi_{(2)}$ is small enough, then we can regard $w_{(2),(1),\hat{v},i}$ as an element of $\Sigma^-_{(2),v}$, and hence an element of $\Sigma^-_{(2),v}(\vec{\mathfrak{x}},\vec{\sigma})$. This implies that if we replace $\vec{w}(\vec{\mathfrak{x}},\vec{\sigma})$ with the points $w_{(2),(1),\hat{v},i}$

to obtain:

$$(\Sigma_v(\vec{\mathfrak{x}},\vec{\sigma}),\vec{z}_v(\vec{\mathfrak{x}},\vec{\sigma})\cup\coprod_{\pi(\hat{v})=v}\vec{w}_{(2),(1),\hat{v}}) \tag{9.26}$$

then (9.26) is close to $\coprod_{\pi(\hat{v})=v}(\Sigma_{(1),\hat{v}},\vec{z}'_{(1),\hat{v}}\cup\vec{w}_{(1),\hat{v}})$.

We use this fact and the target parallel transportation in the same way as in Section 4 to obtain the following map for any $\hat{v}\in C_0^{\rm int}(\check{R}_{(1)})$:

$$\mathcal{P}_{\hat{v}}:E_{\mathfrak{u}_{(1)},\hat{v}}\to L^2_{m,\delta}(\Sigma^-_{(2),v}(\vec{\mathfrak{x}},\vec{\sigma})).$$

We then define for any $v\in\check{R}_{(2)}$:

$$E_{\mathfrak{u}_{(2)},\mathfrak{u}_{(1)},v}(u'_v)=\bigoplus_{\hat{v},\pi(\hat{v})=v}\mathcal{P}_{\hat{v}}(E_{\mathfrak{u}_{(1)},\hat{v}})$$

for $\lambda(v)=0$. We also define $E_{\mathfrak{u}_{(2)},\mathfrak{u}_{(1)},v}(U'_v)$ for $\lambda(v)>0$ by a similar formula.

Now we replace (8.51) by

$$\begin{aligned}\mathcal{E}_{0,\mathfrak{u}_{(2)},\mathfrak{u}_{(1)}}(\mathfrak{y})=&\bigoplus_{v\in C_0^{\rm int}(\check{R}_{(2)}),\,\lambda(v)=0}E_{\mathfrak{u}_{(2)},\mathfrak{u}_{(1)},v}(u'_v)\\&\oplus\bigoplus_{v\in C_0^{\rm int}(\check{R}_{(2)}),\,\lambda(v)>0}E_{\mathfrak{u}_{(2)},\mathfrak{u}_{(1)},v}(U'_v).\end{aligned} \tag{9.27}$$

Lemma 9.28. *Suppose $\Xi_{(1)}$ is small enough such that we can apply the construction of Section 8 to $(\Xi_{(1)},E)$ to obtain a Kuranishi chart at $\mathfrak{u}_{(1)}$. Then for $\Xi_{(2)}$ small enough, applying the construction of Section 8 to $\{E_{\mathfrak{u}_{(2)},\mathfrak{u}_{(1)},v}\}$ (instead of $\{E_{\mathfrak{u}_{(2)},v}\}$) gives rise to a Kuranishi chart at $\mathfrak{u}_{(2)}$.*

This is immediate from the construction of Section 8. In fact, the choice of obstruction bundles we take here satisfies the 'smoothness' condition. See [DF18b, Definition 3.11 (5)] or [FOOO16a, Definition 5.1 (2)]. (Smoothness here means smoothness with respect to $\mathfrak{y}$.)

Remark 9.29. We remark the obstruction spaces we used in Section 8 to construct a Kuranishi neighborhood (in the case of $\mathfrak{u}=\mathfrak{u}_{(2)}$) is slightly different from our choice here. The obstruction space in Section 8 is defined by a target parallel transformation from the obstruction space defined on $\mathfrak{u}_{(2)}$. Here the obstruction space (9.27) is defined by a target parallel transformation from the obstruction space defined on $\mathfrak{u}_{(1)}$.

For each $\mathfrak{p}\in\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$, we consider $\Upsilon_{\mathfrak{p}}=(\Xi_{\mathfrak{p}},E_{\mathfrak{p}})$, where $E_{\mathfrak{p}}:=\{E_{\mathfrak{p},v}\}$ gives obstruction spaces at $\mathfrak{p}$ and $\Xi_{\mathfrak{p}}=(\vec{w}_{\mathfrak{p}},(\mathcal{N}_{\mathfrak{p},v,i}),(\phi_{\mathfrak{p},v}),(\varphi_{\mathfrak{p},v,e}),\kappa_{\mathfrak{p}})$ is a TSD for $\mathfrak{p}$. We assume that $\Xi_{\mathfrak{p}}$ is small enough such that the assumption of Lemma 9.28 holds. Let $\mathfrak{U}(\mathfrak{p}):=\mathfrak{U}(\mathfrak{p},\Xi_{\mathfrak{p}})\cap\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ be a neighborhood of $\mathfrak{p}$ in $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ determined by the TSD $\Xi_{\mathfrak{p}}$. We also fix a compact neighborhood $\mathcal{K}(\mathfrak{p})$ of $\mathfrak{p}$ which is a subset of $\mathfrak{U}(\mathfrak{p})$.

Compactness of $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ implies that we can find a finite subset

$$\mathfrak{J}=\{\mathfrak{p}_j:j=1,\dots,J\}\subset\mathcal{M}^{\rm RGW}_{k+1}(L;\beta) \tag{9.30}$$

such that

$$\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)\subseteq\bigcup_{j=1}^{J}{\rm Int}\,(K(\mathfrak{p}_j)). \tag{9.31}$$

For $\mathfrak{u}\in\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$, we define:

$$\mathfrak{J}(\mathfrak{u})=\{\mathfrak{p}_j\mid\mathfrak{u}\in\mathcal{K}(\mathfrak{p}_j)\}. \tag{9.32}$$

Lemma 9.33. *Let $\check{R}_j$ be the very detailed tree associated to $\mathfrak{p}_j$. We can perturb $\{E_{\mathfrak{p}_j,v} \mid v \in C_0^{\rm int}(\check{R}_j)\}$ by an arbitrary small amount so that the following holds. For any $\mathfrak{u} \in \mathcal{M}_{k+1}^{\rm RGW}(L;\beta)$, the vector spaces $\mathcal{E}_{0,\mathfrak{u},\mathfrak{p}_j}(\mathfrak{u})$ for $\mathfrak{p}_j \in \mathfrak{J}(\mathfrak{u})$ are transversal, i.e., the sum of $\mathcal{E}_{0,\mathfrak{u},\mathfrak{p}_j}(\mathfrak{u})$ for $\mathfrak{p}_j \in \mathfrak{J}(\mathfrak{u})$ is the direct sum:*

$$\bigoplus_{\mathfrak{p}_j\in\mathfrak{J}(\mathfrak{u})} \mathcal{E}_{0,\mathfrak{u},\mathfrak{p}_j}(\mathfrak{u}). \tag{9.34}$$

Proof. The proof is the same as the proof of the analogous statement in the case of the stable map compactification. See the proof of [FOOO18, Lemma 11.7] in [FOOO18, Subsection 11.4]. □

Now we define a Kuranishi chart at each point $\mathfrak{u} \in \mathcal{M}_{k+1}^{\rm RGW}(L;\beta)$ as follows:

Definition 9.35. Let $\Xi_{\mathfrak{u}} = (\vec{w}_{\mathfrak{u}}, (\mathcal{N}_{\mathfrak{u},v,i}, \kappa_{\mathfrak{u}}), (\phi_{\mathfrak{u},v}), (\varphi_{\mathfrak{u},v,e}), \kappa_{\mathfrak{u}})$ be a TSD, which is small enough such that the conclusion of Lemma 9.28 holds for $\mathfrak{u}_{(1)} = \mathfrak{p}_j$, $\mathfrak{u}_{(2)} = \mathfrak{u}$, $\Xi_{(1)} = \Xi_{\mathfrak{p}_j}$ and $\Xi_{(2)} = \Xi_{\mathfrak{u}}$ with $\mathfrak{p}_j$ being an arbitrary element in $\mathfrak{J}(\mathfrak{u})$. We put $\Upsilon_{(j)} = (\Xi_{(j)}, E_{(j)})$, where $E_{(j)}$ is the obstruction space at $\mathfrak{u}_{(1)} = \mathfrak{p}_j$. We define $E_{\mathfrak{u}} = \{E_{(j)} \mid \mathfrak{p}_j \in \mathfrak{J}(\mathfrak{u})\}$ and $\Upsilon_{\mathfrak{u}} = (\Xi_{\mathfrak{u}}, E_{\mathfrak{u}})$. Hereafter we always take this choice of $E_{\mathfrak{u}}$. The Kuranishi neighborhood $\mathcal{U}(\mathfrak{u}, \Upsilon_{\mathfrak{u}})$ is the set of the equivalence classes of inconsistent maps $\mathfrak{y} = (\vec{\mathfrak{r}}, \vec{\sigma}, (u'_v), (U'_v), (\rho_e), (\rho_i))$ near $\mathfrak{u}$ such that

$$\begin{aligned} \overline{\partial}_{j_{\vec{\mathfrak{r}},\vec{\sigma}}} u'_v &\in \bigoplus_{\mathfrak{p}_j\in\mathfrak{J}(\mathfrak{u})} \mathcal{E}_{0,\mathfrak{u},\mathfrak{p}_j}(\mathfrak{y}), \\ \overline{\partial}_{j_{\vec{\mathfrak{r}},\vec{\sigma}}} U'_v &\in \bigoplus_{\mathfrak{p}_j\in\mathfrak{J}(\mathfrak{u})} \mathcal{E}_{0,\mathfrak{u},\mathfrak{p}_j}(\mathfrak{y}). \end{aligned} \tag{9.36}$$

In other words, it is the set of inconsistent solutions (Definition 8.28) where (8.33) and (8.34) are replaced with (9.36).

We also define $\widehat{\mathcal{U}}(\mathfrak{u}, \Upsilon_{\mathfrak{u}})$, $\widetilde{\mathcal{U}}(\mathfrak{u}, \Upsilon_{\mathfrak{u}})$ in the same way as Definition 8.28.

The obstruction bundle $\mathcal{E}_{\mathfrak{u}}$ is defined in the same way as in (8.52) in the following way:

$$\mathcal{E}_{0,\mathfrak{u}}(\mathfrak{y}) = \bigoplus_{\mathfrak{p}_j\in\mathfrak{J}(\mathfrak{u})} \mathcal{E}_{0,\mathfrak{u},\mathfrak{p}_j}(\mathfrak{y}) \tag{9.37}$$

$$\mathcal{E}_{\mathfrak{u},\Xi_{\mathfrak{u}}}(\mathfrak{y}) = \mathcal{E}_{0,\mathfrak{u}}(\mathfrak{y}) \oplus \bigoplus_{e\in C_1^{\rm int}(\check{R}),\, \lambda(e)>0} \mathscr{L}_e. \tag{9.38}$$

where $\check{R}$ is the very detailed tree associated to $\mathfrak{u}$. The Kuranishi map $\mathfrak{s}_{\mathfrak{u}}$ is defined in the same way as in (8.53), (8.54), (8.55), and the parametrization map $\psi_{\mathfrak{u}}$ is defined in the same way as in (8.56).

9.3. **Construction of Coordinate Change I.**

In this and the next subsections, we construct coordinate changes. The next two lemmas state the semi-continuity of our obstruction spaces, a property that we hinted at the beginning of the last subsection.[31]

Lemma 9.39. *For any $\mathfrak{u}_{(1)} \in \mathcal{M}_{k+1}^{\rm RGW}(L;\beta)$, there exists a neighborhood $U(\mathfrak{u}_{(1)})$ of $\mathfrak{u}_{(1)}$ in $\mathcal{M}_{k+1}^{\rm RGW}(L;\beta)$ such that for any $\mathfrak{u}_{(2)} \in U(\mathfrak{u}_{(1)})$:*

$$\mathfrak{J}(\mathfrak{u}_{(2)}) \subseteq \mathfrak{J}(\mathfrak{u}_{(1)}). \tag{9.40}$$

Proof. This is obvious because we pick the subspaces $\mathcal{K}(\mathfrak{p}_j)$ to be closed. □

[31] Compare to [FOOO18, Definition 5.1 (4)] or [DF18b, Definition 3.11 (4)].

Lemma 9.41. *Let $\mathfrak{u}_{(2)} \in U(\mathfrak{u}_{(1)})$. Let $\Xi_{(j)} = (\vec{w}_{(j)}, (\mathcal{N}_{(j),v,i}), (\phi_{(j),v}), (\varphi_{(j),v,e}))$, for $j = 1, 2$, be a TSD. We assume that $\Xi_{(2)}$ is induced from $\Xi_{(1)}$. Let*

$$\mathfrak{y}_{(2)} = (\vec{\mathfrak{x}}_{(2)}, \vec{\sigma}_{(2)}, (u'_{(2),v}), (U'_{(2),v}), (\rho_{(2),e}), (\rho_{(2),i}))$$

be an inconsistent map near $\mathfrak{u}_{(2)}$ with respect to $\Xi_{(2)}$ and

$$\mathfrak{y}_{(1)} = (\vec{\mathfrak{x}}_{(1)}, \vec{\sigma}_{(1)}, (u'_{(1),\hat{v}}), (U'_{(1),\hat{v}}), (\rho_{(1),e}), (\rho_{(1),i}))$$

be the inconsistent map near $\mathfrak{u}_{(1)}$ with respect to $\Xi_{(1)}$, constructed by Lemma 9.22. Let $\mathfrak{p}_j \in \mathfrak{J}(\mathfrak{u}_{(2)})$. Then we have an isomorphism

$$\mathcal{E}_{0,\mathfrak{u}_{(2)};\mathfrak{p}_j}(\mathfrak{y}_{(2)}) \cong \mathcal{E}_{0,\mathfrak{u}_{(1)};\mathfrak{p}_j}(\mathfrak{y}_{(1)}). \tag{9.42}$$

Proof. Let $I_{\hat{v}}$ be the inclusion map (9.18). This is a holomorphic embedding. Then (9.19) induces the required isomorphism. The fact that transversality constraint (8.35) is preserved is a consequence of (9.19) and our choice of transversals $\mathcal{N}_{(j),v,i}$. ☐

Lemma 9.43. *Suppose $\mathfrak{u}_{(1)}$, $\mathfrak{u}_{(2)}$, $\Xi_{(1)}$, $\Xi_{(2)}$, $\mathfrak{y}_{(1)}$ and $\mathfrak{y}_{(2)}$ are given as in Lemma 9.41. We put $\Upsilon_{\mathfrak{u}_{(j)}} = (\Xi_{\mathfrak{u}_{(j)}}, E_{\mathfrak{u}_{(j)}})$, where $E_{\mathfrak{u}_{(j)}}$ is as in Definition 9.35. If $\mathfrak{y}_{(2)}$ is an element of $\widehat{\mathcal{U}}(\mathfrak{u}_{(2)}, \Upsilon_{(2)})$, then $\mathfrak{y}_{(1)}$ is an element of $\widehat{\mathcal{U}}(\mathfrak{u}_{(1)}, \Upsilon_{(1)})$.*

Proof. The isomorphism induced by $I_{\hat{v}}$ send $\overline{\partial} u'_{(2),\hat{v}}$ (resp. $\overline{\partial} U'_{(2),\hat{v}}$) to $\overline{\partial} u'_{(1),\hat{v}}$ (resp. $\overline{\partial} U'_{(1),\hat{v}}$). This is a consequence of (9.19). Therefore, if $\mathfrak{y}_{(2)}$ satisfies (9.36) then $\mathfrak{y}_{(1)}$ satisfies (9.36). ☐

We thus constructed a $\Gamma_{\mathfrak{u}_{(2)}}$-invariant map:

$$\varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}} : \widehat{\mathcal{U}}(\mathfrak{u}_{(2)}, \Upsilon_{(2)}) \to \widehat{\mathcal{U}}(\mathfrak{u}_{(1)}, \Upsilon_{(1)}).$$

It is clear from the construction that the above map can be lifted to a map $\widetilde{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ from $\widetilde{\mathcal{U}}(\mathfrak{u}_{(2)}, \Upsilon_{(2)})$ to $\widetilde{\mathcal{U}}(\mathfrak{u}_{(1)}, \Upsilon_{(1)})$.

Lemma 9.44. *The maps $\varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ and $\widetilde{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ are C^ℓ embeddings.*

Proof. It follows from the definition of $\widetilde{\varphi}_{\mathfrak{u}_{(1)},\mathfrak{u}_{(2)}}$ and the choices of $\Xi_{(j)}$, $\Upsilon_{(j)}$ that the following two diagrams commute:

$$\begin{array}{ccc}
\widetilde{\mathcal{U}}(\mathfrak{u}_{(2)}, \Upsilon_{(2)}) & \xrightarrow{F_1} & \prod_{v \in C_0^{\rm int}(\check{R}_{(2)})} \mathcal{V}^{\rm source}_{(2),v} \times \prod_{e \in C_1^{\rm int}(\check{R}(2))} \mathcal{V}^{\rm deform}_{(2),e} \times (D^2)^{|\lambda_{(2)}|} \\
\Big\downarrow \widetilde{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}} & & \Big\downarrow R_1 \\
\widetilde{\mathcal{U}}(\mathfrak{u}_{(1)}, \Upsilon_{(1)}) & \longrightarrow & \prod_{v \in C_0^{\rm int}(\check{R}_{(1)})} \mathcal{V}^{\rm source}_{(1),v} \times \prod_{e \in C_1^{\rm int}(\check{R}(1))} \mathcal{V}^{\rm deform}_{(1),e} \times (D^2)^{|\lambda_{(1)}|}
\end{array} \tag{9.45}$$

$$\begin{array}{ccc}
\widetilde{\mathcal{U}}(\mathfrak{u}_{(2)}, \Upsilon_{(2)}) & \xrightarrow{F_2} & \begin{array}{l}\prod_{v \in C_0^{\rm int}(\check{R}_{(2)}), \lambda_{(2)}(v)=0} L^2_{m+1}(\Sigma^-_{(2),v}, X \backslash \mathcal{D}) \\ \times \prod_{v \in C_0^{\rm int}(\check{R}_{(2)}), \lambda_{(2)}(v)>0} L^2_{m+1}(\Sigma^-_{(2),v}, \mathcal{N}_{\mathcal{D}} X \backslash \mathcal{D})\end{array} \\
\Big\downarrow \widetilde{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}} & & \Big\downarrow R_2 \\
\widetilde{\mathcal{U}}(\mathfrak{u}_{(1)}, \Upsilon_{(1)}) & \longrightarrow & \begin{array}{l}\prod_{v \in C_0^{\rm int}(\check{R}_{(1)}), \lambda_{(1)}(v)=0} L^2_{m+1}(\Sigma^-_{(1),v}, X \backslash \mathcal{D}) \\ \times \prod_{v \in C_0^{\rm int}(\check{R}_{(1)}), \lambda_{(1)}(v)>0} L^2_{m+1}(\Sigma^-_{(1),v}, \mathcal{N}_{\mathcal{D}} X \backslash \mathcal{D})\end{array}
\end{array} \tag{9.46}$$

Here the horizontal arrows F_1 and F_2 are as in (8.44). The right vertical arrow R_1 of Diagram (9.45) is obtained by requiring (9.17) and is a smooth embedding. Diagram (9.45) commutes since $\widetilde{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ does not change the conformal structure of source (marked) curves. The right vertical arrow R_2 of Diagram (9.46) is obtained by restriction of domain and is a smooth map. Diagram (9.46) commutes because of Condition 9.12. Now the definitions of the C^ℓ structures on $\widetilde{\mathcal{U}}(\mathfrak{u}_{(1)}, \Upsilon_{(1)})$ and $\widetilde{\mathcal{U}}(\mathfrak{u}_{(2)}, \Upsilon_{(2)})$ imply that $\widetilde{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ is a C^ℓ map. Unique continuation implies that the differential of the map $(R_1 \circ F_1, R_2 \circ F_2)$ is injective. In particular, this implies that $\widetilde{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ is an embedding. A similar argument applies to the map $\varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$. □

We next define a bundle map $\overline{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}} : \mathcal{E}_{\mathfrak{u}_{(2)}} \to \mathcal{E}_{\mathfrak{u}_{(1)}}$ which lifts $\varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$. Using (9.40) and (9.42), we obtain a linear embedding:

$$\bigoplus_{\mathfrak{p}_j \in \mathfrak{I}(\mathfrak{u}_{(2)})} \mathcal{E}_{0,\mathfrak{u}_{(2)},\mathfrak{p}_j}(\mathfrak{y}_{(2)}) \to \bigoplus_{\mathfrak{p}_j \in \mathfrak{I}(\mathfrak{u}_{(1)})} \mathcal{E}_{0,\mathfrak{u}_{(1)},\mathfrak{p}_j}(\mathfrak{y}_{(1)}) \tag{9.47}$$

if $\mathfrak{y}_{(1)} = \varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}(\mathfrak{y}_{(2)})$. The map :

$$\bigoplus_{e \in C^{\rm int}_{\rm th}(\check R_{(2)}), \lambda(e)>0} \mathscr{L}_e \to \bigoplus_{e \in C^{\rm int}_{\rm th}(\check R_{(1)}), \lambda(e)>0} \mathscr{L}_e \tag{9.48}$$

is defined as identity on $e \in C^{\rm int}_{\rm th}(\check R_{(2)}) \subset C^{\rm int}_{\rm th}(\check R_{(1)})$ and is zero on the other factors. The bundle map $\overline{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ is defined using (9.47) and (9.48). Analogous to Lemma 9.44, we can prove that $\overline{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ is C^ℓ.

Lemma 9.49.

$$\mathfrak{s}_{(1)} \circ \varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}} = \overline{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}} \circ \mathfrak{s}_{(2)}.$$

Proof. On the factor in (9.47) this is a consequence of the definitions of the map $\varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ and (9.47). Namely, it follows from (9.19) and the fact that $I_{\hat v}$ is bi-holomorphic. For the factor in (9.48), this is a consequence of (9.20). □

Compatibility of the parametrization map with $\varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}$ is also an immediate consequence of the definitions. We thus proved that:

Proposition 9.50. *Let $\mathfrak{u}_{(2)} \in U(\mathfrak{u}_{(1)})$. We assume that $\Xi_{(2)}$ is induced from $\Xi_{(1)}$. Then the pair $(\varphi_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}}, \overline{\varphi}_{\mathfrak{u}_{(1)}\mathfrak{u}_{(2)}})$ is a coordinate change of Kuranishi charts.*

9.4. **Construction of Coordinate Change II.** Let $\mathfrak{u} = ((\Sigma_v, \vec z_v, u_v); v \in C^{\rm int}_0(\check R))$ be an element of $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$. We fix two TSDs

$$\Xi_{(j)} = (\vec w_{(j)}, (\mathcal{N}_{(j),v,i}), (\phi_{(j),v}), (\varphi_{(j),v,e}), \kappa_{(j)}) \qquad j = 1,2$$

at $\mathfrak{u}$ such that we can use Definition 9.35, to form Kuranishi charts

$$\mathcal{U}_{\mathfrak{u},\Upsilon_{(j)}} = (\mathcal{U}(\mathfrak{u}, \Upsilon_{(j)}), \mathcal{E}_{\mathfrak{u},\Upsilon_{(j)}}, \Gamma_{\mathfrak{u}}, \mathfrak{s}_{\mathfrak{u},\Upsilon_{(j)}}, \psi_{\mathfrak{u},\Upsilon_{(j)}}),$$

where $\Upsilon_{(j)} = (\Xi_{(j)}, E)$ and $E = \{E_{\mathfrak{u}}\}$ is as in Definition 9.35.

These Kuranishi charts depend on the choices of the subset $\{\mathfrak{p}_j\}$ of the moduli space $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$, the TSOs $\{\Upsilon_{\mathfrak{p}_j}\}$, the vector spaces $E_{\mathfrak{p}_j,v}$ and the open sets $\mathcal{K}(\mathfrak{p}_j)$. We assume that these choices agree with each other for the above two charts. In this subsection, we will construct a coordinate change from $\mathcal{U}_{\mathfrak{u},\Upsilon_{(2)}}$ to $\mathcal{U}_{\mathfrak{u},\Upsilon_{(1)}}$.[32] The TSD $\Xi_{(j)}$ determines the subspace $\Sigma^-_{(j),v}$ of Σ_v for each interior vertex v of $\check R$. We assume that $\Xi_{(2)}$ is small enough such that $\vec w_{(1)} \cap \Sigma_v$ is a subset of $\Sigma^-_{(2),v}$.

[32] In fact, these two coordinate charts are isomorphic after possibly shrinking $\mathcal{U}(\mathfrak{u}, \Upsilon_{(j)})$ into appropriate open subspaces.

We pick an inconsistent map with respect to $\Upsilon_{(2)}$ denoted by:

$$\mathfrak{y}_{(2)} = (\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}, (u'_{(2),v}), (U'_{(2),v}), (\rho_{(2),e}), (\rho_{(2),i}))$$

Associated to $\mathfrak{y}_{(2)}$, we have $\Sigma^-_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$, which comes with marked points:

$$\vec{z}_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}) \cup \vec{w}_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}).$$

Here the elements of $\vec{z}_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$ are in correspondence with the boundary marked points of Σ_v and $\vec{w}_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$ are in correspondence with the additional marked points $\vec{w}_{(2),v}$ given by $\Xi_{(2)}$. We will write $\vec{z}_{(2)}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$ for the union of all boundary marked points of $\Sigma_{(2)}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$. The following lemma is the analogue of Lemma 9.25:

Lemma 9.51. *There exists $\vec{w}_{(2);(1),v}(\mathfrak{y}_{(2)}) \subset \Sigma^-_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$ such that:*

1. *$w_{(2);(1),v,i}(\mathfrak{y}_{(2)})$ is close to $w_{(1),v,i}$. Here we identify $\Sigma^-_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$ and $\Sigma^-_{(2),v}$ using $\Xi_{(2)}$.*
2. *$u'_{(2),v}(w_{(2);(1),v,i}(\mathfrak{y}_{(2)})) \in \mathcal{N}_{(1),v,i}$.*

We define:

$$\vec{w}_{(2);(1)}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}) = \bigcup_{v \in C_0^{\rm int}(\check{R})} \vec{w}_{(2);(1),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}).$$

Then $(\Sigma_{(2)}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}), \vec{z}_{(2)}(\mathfrak{y}_{(2)}) \cup \vec{w}_{(2);(1)}(\mathfrak{y}_{(2)}))$ is close to $(\Sigma, \vec{z} \cup \vec{w}_{(1)})$ in the moduli space of bordered nodal curves. Therefore, there exists $\vec{\mathfrak{r}}_{(1)}, \vec{\sigma}_{(1)}$ such that:

$$(\Sigma_{(2)}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}), \vec{z}_{(2)}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)}) \cup \vec{w}_{(2);(1)}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})) \cong (\Sigma_{(1)}(\vec{\mathfrak{r}}_{(1)}, \vec{\sigma}_{(1)}), \vec{z}_{(1)}(\vec{\mathfrak{r}}_{(1)}, \vec{\sigma}_{(1)}) \cup \vec{w}_{(1)}(\vec{\mathfrak{r}}_{(1)}, \vec{\sigma}_{(1)})). \tag{9.52}$$

Here we use $\Xi_{(1)}$ to define the right hand side. Let I be an isomorphism from the right hand side of (9.52) to the left hand side. Note that the choices of $\vec{\mathfrak{r}}_{(1)}, \vec{\sigma}_{(1)}$ and I are unique up to an element of $\Gamma_{\mathfrak{u}}$.

We consider decompositions:

$$\Sigma_{(j)}(\vec{\mathfrak{r}}_{(j)}, \vec{\sigma}_{(j)}) = \coprod_{v \in C_0^{\rm int}(\check{R})} \Sigma^-_{(j),v}(\vec{\mathfrak{r}}_{(j)}, \vec{\sigma}_{(j)}) \cup \coprod_{e \in C_1^{\rm int}(\check{R})} [-5T_{(j),e}, 5T_{(j),e}] \times S^1, \tag{9.53}$$

for $j = 1, 2$. In the above identity, we define $T_{(j),e}$ by requiring $e^{-10T_{(j),e}} = |\sigma_{(j),e}|$. Here for simplicity, we assume that $\sigma_{(2),e}$ is non-zero for all interior edges e of $\check{R}$. A similar discussion applies to the case that $\sigma_{(2),e} = 0$ with minor modifications. For example, in (9.53) we need to include two half cylinders for each e that $\sigma_{(2),e} = 0$. We also have:

$$\Sigma_{(j)}(\vec{\mathfrak{r}}_{(j)}, \vec{\sigma}_{(j)}) = \bigcup_{v \in C_0^{\rm int}(\check{R})} \Sigma^+_{(j),v}(\vec{\mathfrak{r}}_{(j)}, \vec{\sigma}_{(j)}). \tag{9.54}$$

Although I in (9.52) is an isomorphism, it does not respect the decompositions in (9.53) or (9.54) for $j = 1, 2$. This is because $\Xi_{(1)} \neq \Xi_{(2)}$.[33] Nevertheless, one can easily prove:

Lemma 9.55. *If $\Xi_{(2)}$ is small enough, then I can be chosen such that the following holds. Let $v \in C_0^{\rm int}(\check{R})$ and $\mathfrak{z} \in \Sigma^+_{(1),v}(\vec{\mathfrak{r}}_{(1)}, \vec{\sigma}_{(1)})$. Then at least one of the following conditions holds:*

(I) $I(\mathfrak{z}) \in \Sigma^+_{(2),v}(\vec{\mathfrak{r}}_{(2)}, \vec{\sigma}_{(2)})$.

[33] Conditions 9.2 and 9.12 are used in Subsection 9.3 to show the compatibility of the similar decompositions. We do not assume them here.

(II) *There exists* $e \in C_1^{\rm int}(\check R)$ *with* $\partial e = \{v, v'\}$ *such that* $I(\mathfrak z) \in \Sigma^+_{(2),v',\mathfrak x_{(2),v'}}$.

This is the consequence of the fact that the decomposition (9.54) is 'mostly preserved' by I. Now we define $u'_{(1),v}$, $U'_{(1),v}$ as follows. If $\lambda(v) = 0$, we have:

$$u'_{(1),v}(\mathfrak z) = \begin{cases} u'_{(2),v}(\mathfrak z) & \text{if (I) holds,} \\ U'_{(2),v'}(\mathfrak z) & \text{if (II) holds, } \lambda(v) < \lambda(v'), \\ u'_{(2),v'}(\mathfrak z) & \text{if (II) holds, } \lambda(v) = \lambda(v'), \end{cases} \tag{9.56}$$

and if $\lambda(v) > 0$, we have:

$$U'_{(1),v}(\mathfrak z) = \begin{cases} U'_{(2),v}(\mathfrak z) & \text{if (I) holds,} \\ U'_{(2),v'}(\mathfrak z) & \text{if (II) holds, } \lambda(v) = \lambda(v'), \\ \mathrm{Dil}_{\rho_{(2),e}} \circ U'_{(2),v'}(\mathfrak z) & \text{if (II) holds, } \lambda(v) < \lambda(v'), \\ \mathrm{Dil}_{1/\rho_{(2),e}} \circ U'_{(2),v'}(\mathfrak z) & \text{if (II) holds, } 0 < \lambda(v') < \lambda(v), \\ \mathrm{Dil}_{1/\rho_{(2),e}} \circ u'_{(2),v'}(\mathfrak z) & \text{if (II) holds, } 0 = \lambda(v') < \lambda(v). \end{cases} \tag{9.57}$$

Using the fact that $\mathfrak y_{(2)}$ satisfies (8.29), (8.30), (8.31), we can easily check that in the case that (I) and (II) are both satisfied the right hand sides coincide.

We also define

$$\rho_{(1),e} = \rho_{(2),e}, \qquad \rho_{(1),i} = \rho_{(2),i}.$$

Lemma 9.58. *The 6-tuple*

$$\mathfrak y_{(1)} = (\vec{\mathfrak x}_{(1)}, \vec\sigma_{(1)}, (u'_{(1),v}), (U'_{(1),v}), (\rho_{(1),e}), (\rho_{(1),i}))$$

is an inconsistent solution near $\mathfrak u$ *with respect to* $\Xi_{(1)}$.

Proof. Definition 8.28 (1), (2), (3) are obvious. (4)-(8), (11) and (12) follow from the definition of $\mathfrak y_{(1)}$ and the corresponding conditions for $\mathfrak y_{(2)}$. (9) and (10) hold by shrinking the size of $\Xi_{(2)}$ if necessary. (13) is a consequence of Lemma 9.51. □

Thus after shrinking the size of $\Xi_{(2)}$ if necessary, we may define:

$$\widetilde\varphi_{(\mathfrak u,\Xi_{(1)})(\mathfrak u,\Upsilon_{(2)})} : \widetilde{\mathcal U}(\mathfrak u, \Upsilon_{(2)}) \to \widetilde{\mathcal U}(\mathfrak u, \Upsilon_{(1)}) \tag{9.59}$$

by:

$$\widetilde\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}(\mathfrak y_{(2)}) = \mathfrak y_{(1)}. \tag{9.60}$$

Similarly, we can define $\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})} : \widehat{\mathcal U}(\mathfrak u, \Upsilon_{(2)}) \to \widehat{\mathcal U}(\mathfrak u, \Upsilon_{(1)})$.

Lemma 9.61. *The maps* $\widetilde\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ *and* $\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ *are* C^ℓ *diffeomorphisms into their images.*

Proof. We cannot apply the same proof as in Lemma 9.44. In fact, Diagram (9.46) does not commute anymore because our TSD $\Xi_{(2)}$ may not be induced from $\Xi_{(1)}$. In order to resolve this issue, we need to modify the definition of right vertical arrow in Diagram (9.46).

Assuming $\Xi_{(2)}$ is small enough, we define a map:

$$\mathfrak I_{v_0} : \prod_{v \in C_0^{\rm int}(\check R)} \mathcal V^{\rm source}_{(2),v} \times \prod_{e \in C_1^{\rm int}(\check R)} \mathcal V^{\rm deform}_{(2),e} \times \Sigma^-_{(1),v_0} \to \Sigma^-_{(2),v_0}$$

for any interior vertex v_0 of $\check R$ as follows. Fix an element $(\vec{\mathfrak x}_{(2)},\vec\sigma_{(2)})$ of $\prod_{v\in C_0^{\rm int}(\check R)}\mathcal V^{\rm source}_{(2),v}\times\prod_{e\in C_1^{\rm int}(\check R)}\mathcal V^{\rm deform}_{(2),e}$, and let $(\vec{\mathfrak x}_{(1)},\vec\sigma_{(1)})\in\prod_{v\in C_0^{\rm int}(\check R)}\mathcal V^{\rm source}_{(1),v}\times\prod_{e\in C_1^{\rm int}(\check R)}\mathcal V^{\rm deform}_{(1),e}$ satisfy (9.52). By taking $\Xi_{(2)}$ small enough, we can form the following composition:

$$\Sigma^-_{(1),v_0}\to\Sigma^-_{(1),v_0}(\vec{\mathfrak x}_{(1)},\vec\sigma_{(1)})\to\Sigma^-_{(2),v_0}(\vec{\mathfrak x}_{(2)},\vec\sigma_{(2)})\to\Sigma^-_{(2),v_0}\tag{9.62}$$

Here the first map is defined using $\Xi_{(1)}$, the second map is induced by the isomorphism (9.52), and the last map is defined using $\Xi_{(2)}$. For $\mathfrak z\in\Sigma^-_{(1),v_0}$, we define $\mathfrak I_{v_0}(\vec{\mathfrak x}_{(2)},\vec\sigma_{(2)},\mathfrak z)$ to be the image of $\mathfrak z$ by the map (9.62). It is clear that $\mathfrak I_{v_0}$ is a smooth map.

For a vertex v_0 with $\lambda(v_0)=0$, define:

$$\begin{aligned}\mathfrak I^*_{v_0}:\prod_{v\in C_0^{\rm int}(\mathcal R)}\mathcal V^{\rm source}_{(2),v}&\times\prod_{e\in C_1^{\rm int}(\mathcal R)}\mathcal V^{\rm deform}_{(2),e}\\&\times L^2_{m+\ell+1}(\Sigma^-_{(2),v_0},X\backslash\mathcal D)\to L^2_{m+1}(\Sigma^-_{(1),v_0},X\backslash\mathcal D)\end{aligned}$$

as follows:

$$\mathfrak I^*_{v_0}(\vec{\mathfrak x}_{(2)},\vec\sigma_{(2)},u')(\mathfrak z)=u'(\mathfrak I_{v_0}(\vec{\mathfrak x}_{(2)},\vec\sigma_{(2)},\mathfrak z)).$$

Note that we pick different Sobolev exponents for the Sobolev spaces on the domain and the target of $\mathfrak I^*_{v_0}$. This allows us to obtain a C^ℓ map $\mathfrak I^*_{v_0}$. Similarly, for a vertex v_0 with $\lambda(v_0)>0$, we can define a C^ℓ map:

$$\begin{aligned}\mathfrak I^*_{v_0}:\prod_{v\in C_0^{\rm int}(\mathcal R)}\mathcal V^{\rm source}_{(2),v}&\times\prod_{e\in C_1^{\rm int}(\mathcal R)}\mathcal V^{\rm deform}_{(2),e}\\&\times L^2_{m+\ell+1}(\Sigma^-_{(2),v_0},\mathcal N_{\mathcal D}X\backslash\mathcal D)\to L^2_{m+1}(\Sigma^-_{(1),v_0},\mathcal N_{\mathcal D}X\backslash\mathcal D)\end{aligned}$$

Now we replace Diagram (9.46) with the following:

$$\begin{array}{ccc}\widetilde{\mathcal U}(\mathfrak u,\Upsilon_{(2)}) & \longrightarrow & \begin{array}{l}\prod_{v\in C_0^{\rm int}(\check R)}\mathcal V^{\rm source}_{(2),v}\times\prod_{e\in C_1^{\rm int}(\check R)}\mathcal V^{\rm deform}_{(2),e}\\ \times\prod_{v\in C_0^{\rm int}(\check R),\ \lambda(v)>0}L^2_{m+\ell+1}(\Sigma^-_{(2),v},\mathcal N_{\mathcal D}X\backslash\mathcal D)\\ \times\prod_{v\in C_0^{\rm int}(\check R),\ \lambda(v)>0}L^2_{m+\ell+1}(\Sigma^-_{(2),v},\mathcal N_{\mathcal D}X\backslash\mathcal D)\end{array}\\ \Big\downarrow{\scriptstyle\widetilde\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}} & & \Big\downarrow{\scriptstyle\mathfrak I^*_v}\\ \widetilde{\mathcal U}(\mathfrak u,\Upsilon_{(1)}) & \longrightarrow & \begin{array}{l}\prod_{v\in C_0^{\rm int}(\check R),\ \lambda(v)=0}L^2_{m+1}(\Sigma^-_{(1),v},X\backslash\mathcal D)\\ \times\prod_{v\in C_0^{\rm int}(\check R),\ \lambda(v)>0}L^2_{m+1}(\Sigma^-_{(1),v},\mathcal N_{\mathcal D}X\backslash\mathcal D)\end{array}\end{array}\tag{9.63}$$

Here horizontal arrows are defined as in (8.44). Commutativity of (9.63) is immediate from the definition. We can also form a diagram similar to Diagram (9.45), which is commutative by the same reason as in Lemma 9.44. Commutativity of these two diagrams and the fact that $\mathfrak I^*_v$ is C^ℓ implies that $\widetilde\varphi_{(\mathfrak u,\Xi_{(1)})(\mathfrak u,\Xi_{(2)})}$ is also C^ℓ. A similar argument applies to $\varphi_{(\mathfrak u,\Xi_{(1)})(\mathfrak u,\Xi_{(2)})}$.

By changing the role of $\Xi_{(1)}$ and $\Xi_{(2)}$, we can similarly obtain C^ℓ maps in different directions. To be more precise we can define maps $\widetilde\varphi_{(\mathfrak u,\Upsilon_{(2)})(\mathfrak u,\Upsilon'_{(1)})}$ and $\varphi_{(\mathfrak u,\Upsilon_{(2)})(\mathfrak u,\Upsilon'_{(1)})}$ where $\Upsilon'_{(1)} = (\Xi'_{(1)}, E_{\mathfrak u_{(1)}})$ and $\Xi'_{(1)}$ is given by a small enough shrinking of $\Xi_{(1)}$. The compositions:

$$\widetilde\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}\circ\widetilde\varphi_{(\mathfrak u,\Upsilon_{(2)})(\mathfrak u,\Upsilon'_{(1)})}\qquad \varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}\circ\varphi_{(\mathfrak u,\Upsilon_{(2)})(\mathfrak u,\Upsilon'_{(1)})}$$

are equal to the identity map. Moreover, the compositions

$$\widetilde\varphi_{(\mathfrak u,\Xi_{(2)})(\mathfrak u,\Upsilon'_{(1)})}\circ\widetilde\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}\qquad \varphi_{(\mathfrak u,\Upsilon_{(2)})(\mathfrak u,\Upsilon'_{(1)})}\circ\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$$

are also equal to the identity map, wherever they are defined. This implies that $\widetilde\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ and $\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ are diffeomorphisms, after possibly shrinking $\Xi_{(2)}$. □

Remark 9.64. By following an argument similar to the case of stable maps, one can show that $\widetilde\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ and $\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ are C^∞ using the above C^ℓ property for all values of ℓ. We omit this argument and refer the reader to [FOOO16a, Section 12] for details of the proof. (See also [DF18b, Remark 3.19].)

We thus constructed a C^ℓ embedding $\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$. One can easily define a lift $\overline\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ of $\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ and obtain embedding of obstruction bundles. The compatibility of the Kuranishi maps and the parametrization maps with the maps $\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ and $\overline\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}$ are immediate from the construction. In summary, we have coordinate change:

$$\Phi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})} = (\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}, \hat\varphi_{(\mathfrak u,\Upsilon_{(1)})(\mathfrak u,\Upsilon_{(2)})}) : \mathcal U_{\mathfrak u,\Upsilon_{(2)}} \to \mathcal U_{\mathfrak u,\Upsilon_{(1)}}.$$

9.5. **Co-cycle Condition for Coordinate Changes.** For $j = 1, 2$, let $\mathfrak u_{(j)}$ be an element of $\mathcal M^{\rm RGW}_{k+1}(L;\beta)$, and $\Xi_{(j)}$ be a TSD at $\mathfrak u_{(j)}$. We assume that $\Xi_{(j)}$ is small enough such that we can form the Kuranish chart $\mathcal U_{\mathfrak u_{(j)},\Upsilon_{(j)}}$ as in Definition 9.35. (Here $\Upsilon_{(j)} = (\Xi_{(j)}, E)$.) We also assume that $\mathfrak u_{(2)}$ is sufficiently close to $\mathfrak u_{(1)}$ in the sense that it belongs to the open subset of $\mathcal M^{\rm RGW}_{k+1}(L;\beta)$ determined by $\mathcal U_{\mathfrak u_{(1)},\Upsilon_{(1)}}$. Therefore, we may use the constructions of Subsection 9.1 to obtain a TSD $\Xi_{(2);(1)}$ at $\mathfrak u_{(2)}$ which is compatible with $\Xi_{(1)}$, namely, it satisfies Conditions 9.2 and 9.12. We put $\Upsilon_{(2);(1)} = (\Xi_{(2);(1)}, E)$. Finally by shrinking $\Xi_{(2)}$, we can assume that we can define the coordinate change $\Phi_{(\mathfrak u_{(2)},\Upsilon_{(2);(1)})(\mathfrak u_{(2)},\Upsilon_{(2)})}$ following the construction of the previous subsection. Now we define:

Definition 9.65. We define the coordinate change:

$$\Phi_{(\mathfrak u_{(1)},\Upsilon_{(1)})(\mathfrak u_{(2)},\Upsilon_{(2)})} : \mathcal U_{\mathfrak u_{(2)},\Upsilon_{(2)}} \to \mathcal U_{\mathfrak u_{(1)},\Upsilon_{(1)}}.$$

as the composition

$$\Phi_{(\mathfrak u_{(1)},\Upsilon_{(1)})(\mathfrak u_{(2)},\Upsilon_{(2)})} = \Phi_{(\mathfrak u_{(1)},\Upsilon_{(1)})(\mathfrak u_{(2)},\Upsilon_{(2);(1)})}\circ\Phi_{(\mathfrak u_{(2)},\Xi_{(2);(1)})(\mathfrak u_{(2)},\Upsilon_{(2)})}. \tag{9.66}$$

Here

$$\Phi_{(\mathfrak u_{(1)},\Upsilon_{(1)})(\mathfrak u_{(2)},\Upsilon_{(2);(1)})} : \mathcal U_{\mathfrak u_{(2)},\Upsilon_{(2);(1)}} \to \mathcal U_{\mathfrak u_{(1)},\Upsilon_{(1)}}$$

is defined in Subsection 9.3 and

$$\Phi_{(\mathfrak u_{(2)},\Upsilon_{(2);(1)})(\mathfrak u_{(2)},\Upsilon_{(2)})} : \mathcal U_{\mathfrak u_{(2)},\Upsilon_{(2)}} \to \mathcal U_{\mathfrak u_{(2)},\Upsilon_{(2);(1)}}$$

is defined in Subsection 9.4.

To complete the construction of the Kuranishi structure on $\mathcal M^{\rm RGW}_{k+1}(L;\beta)$, we need to prove the next lemma.

Lemma 9.67. *For $j = 1, 2, 3$, let $\mathfrak{u}_{(j)} \in \mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$, and $\Xi_{(j)}$ be a TSD at $\mathfrak{u}_{(j)}$ such that we can use Definition 9.65, to define the coordinate changes $\Phi_{(\mathfrak{u}_{(1)},\Upsilon_{(1)})(\mathfrak{u}_{(2)},\Upsilon_{(2)})}$, $\Phi_{(\mathfrak{u}_{(2)},\Upsilon_{(2)})(\mathfrak{u}_{(3)},\Upsilon_{(3)})}$, $\Phi_{(\mathfrak{u}_{(1)},\Upsilon_{(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3)})}$. Then we have:*

$$\Phi_{(\mathfrak{u}_{(1)},\Upsilon_{(1)})(\mathfrak{u}_{(2)},\Upsilon_{(2)})} \circ \Phi_{(\mathfrak{u}_{(2)},\Upsilon_{(2)})(\mathfrak{u}_{(3)},\Upsilon_{(3)})} = \Phi_{(\mathfrak{u}_{(1)},\Upsilon_{(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3)})}. \tag{9.68}$$

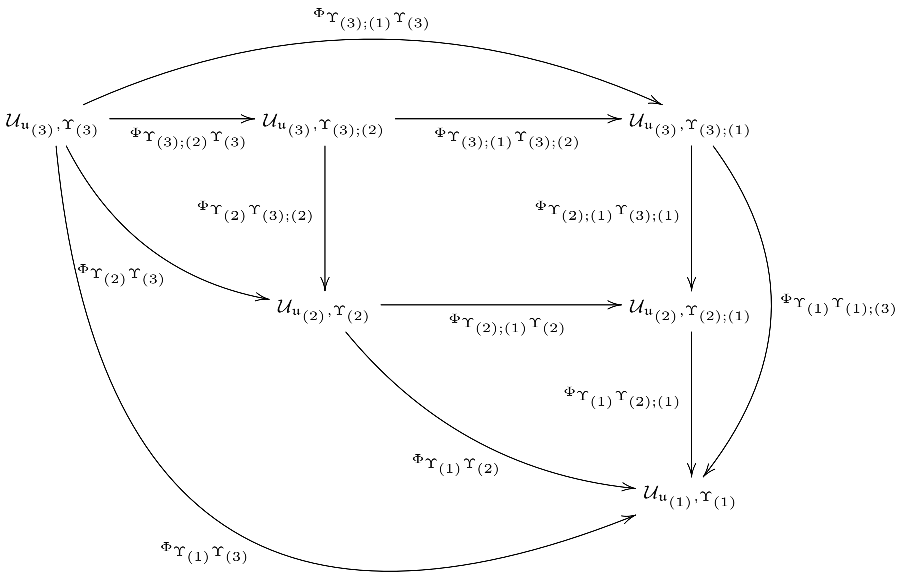
Proof. We use the constructions of Subsection 9.1 to find TSDs $\Xi_{(3);(1)}$, $\Xi_{(3);(2)}$ at $\mathfrak{u}_{(3)}$ such that the pairs $(\Xi_{(1)}, \Xi_{(3);(1)})$ $(\Xi_{(2)}, \Xi_{(3);(2)})$ both satisfy Conditions 9.2 and 9.12. We similarly choose the TSD $\Xi_{(2);(1)}$ at $\mathfrak{u}_{(2)}$. We can easily check the following three formulas:

$$\Phi_{(\mathfrak{u}_{(3)},\Xi_{(3);(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3);(2)})} \circ \Phi_{(\mathfrak{u}_{(3)},\Upsilon_{(3);(2)})(\mathfrak{u}_{(3)},\Upsilon_{(3)})} = \Phi_{(\mathfrak{u}_{(3)},\Upsilon_{(3);(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3)})}$$

$$\Phi_{(\mathfrak{u}_{(1)},\Upsilon_{(1)})(\mathfrak{u}_{(2)},\Upsilon_{(2);(1)})} \circ \Phi_{(\mathfrak{u}_{(2)};\Upsilon_{(2);(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3);(1)})} = \Phi_{(\mathfrak{u}_{(1)},\Upsilon_{(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3);(1)})}$$

$$\begin{aligned}&\Phi_{(\mathfrak{u}_{(2)},\Upsilon_{(2);(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3);(1)})} \circ \Phi_{(\mathfrak{u}_{(3)},\Upsilon_{(3);(1)})(\mathfrak{u}_{(3)},\Upsilon_{(3);(2)})}\\ &= \Phi_{(\mathfrak{u}_{(2)},\Upsilon_{(2);(1)})(\mathfrak{u}_{(2)},\Upsilon_{(2)})} \circ \Phi_{(\mathfrak{u}_{(2)},\Upsilon_{(2)})(\mathfrak{u}_{(3)},\Upsilon_{(3);(2)})}\end{aligned}$$

Then (9.68) is a consequence of these three formulas and Definition 9.65. See the diagram below. In this diagram, the notation $\Phi_{(\mathfrak{u}_{(3)},\Upsilon_{(3)})(\mathfrak{u}_{(3)},\Upsilon_{(3);(2)})}$ is simplified to $\Phi_{\Upsilon_{(3)}\Upsilon_{(3);(2)}}$. Similar notations for other coordinate changes are used.



□

This lemma completes the proof of the following result.

Theorem 9.69. *The space $\mathcal{M}^{\rm RGW}_{k+1}(L;\beta)$ carries a Kuranishi structure.*

We can prove the existence of Kuranishi structure for $\mathcal{M}^{\rm RGW}_{k_1,k_0}(L_1, L_0; p, q; \beta)$ in the same way. The proof of Theorem 1 is now complete. □

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