OBSTRUCTIONS TO RIBBON HOMOLOGY COBORDISMS

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Abstract. We study 4-dimensional homology cobordisms without 3-handles via Heegaard and instanton Floer homologies, character varieties, and Thurston geometries. We provide obstructions to such cobordisms arising from each of these theories, and illustrate some topological applications.

1. Introduction

Recently, there has been a significant amount of attention in the low-dimensional topology community on the particular types of handle decompositions of 4-dimensional cobordisms between 3-manifolds. One particular class of such cobordisms are those that admit a handle decomposition with only 1- and 2-handles, which we call *ribbon cobordisms*. These cobordisms arise in at least two natural ways. The first is as Stein cobordisms between closed contact 3-manifolds. The second is as the exterior of (strongly homotopy-) ribbon surfaces, which are cobordisms between link exteriors. The ribbon surface relation on knots has been studied extensively by Gordon [Gor81] in the case that the surface is a concordance. The key theorem, which is very special to the absence of 3-handles, is the following. Here and below, an *R-homology cobordism* between two compact, oriented 3-manifolds Y_1 and Y_2 is an oriented, smooth cobordism $W: Y_1 \to Y_2$ such that the inclusions $\iota: Y_i \to W$ induce isomorphisms $\iota_*: H_*(Y_i; R) \to H_*(W; R)$ for $i \in \{1, 2\}$.

Theorem 1.1 (Gordon [Gor81, Lemma 3.1])**.** *Let Y_- and Y_+ be compact, connected, oriented 3-manifolds possibly with boundary, and suppose that $W: Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then*

(1) *The map $\iota_*: \pi_1(Y_-) \to \pi_1(W)$ induced by inclusion is injective; and*
(2) *The map $\iota_*: \pi_1(Y_+) \to \pi_1(W)$ induced by inclusion is surjective.*

(While Gordon's original statement is only for exteriors of ribbon concordances, the more general result holds. See the discussion below, as well as Proposition 1.22.)

Gordon uses the theorem above, combined with various properties of knot groups, to study questions related to ribbon concordance; even then, many open questions remain. Very recently, Zemke and his collaborators [Zem19b, MZ19, LZ19] have shown that ribbon concordances induce injections on knot Heegaard Floer homology and Khovanov homology, and this has led to several other interesting results [JMZ19, Sar19], including an exciting relationship between knot Heegaard Floer homology and the bridge index [JMZ19, Corollary 1.9].

Other than in relation to ribbon concordances, the study of ribbon homology cobordisms has been limited so far. In the present article, we study ribbon cobordisms from several complementary

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angles: Heegaard Floer homology, instanton Floer homology, character varieties, and Thurston geometries. Much of the work involving Floer homologies is inspired by the work of Zemke et al.

The employment of these approaches above is motivated by two observations. First, since Gordon's work, there have been many breakthroughs in low-dimensional topology, including the Geometrization Theorem for 3-manifolds, the applications of representation theory and gauge theory, and, relatedly, the advent of Floer theory. Each of these constitutes a new, powerful tool that can be applied in the context of ribbon homology cobordisms, and a major goal of the present article is to systematically carry out these applications. In particular, we will develop obstructions from these theories, which we will then use for topological gain. The interested reader is encouraged to flip forward to Section 1.4 for these applications, including a new result on Dehn surgery. In view of these obstructions, we propose the following conjecture, analogous to [Gor81, Conjecture 1.1].

Conjecture 1.2. *The preorder on the set of homeomorphism classes of closed, connected, oriented 3-manifolds given by ribbon $\mathbb{Q}$-homology cobordisms is in fact a partial order.*

Evidence for Conjecture 1.2 is provided, for example, by Theorem 1.4, Theorem 1.9, Theorem 1.11, Theorem 1.12, Corollary 1.13, Theorem 1.18, Corollary 1.23, Proposition 1.25, and Theorem 1.26 below.

Remark 1.3. Another major open problem regarding ribbon concordance is the Slice–Ribbon Conjecture. In a similar spirit, a natural question to ask is whether a $\mathbb{Z}$-homology sphere bounding a $\mathbb{Z}$-homology 4-ball always bounds a $\mathbb{Z}$-homology 4-ball without 3-handles.

Second, while the approaches reflect very different perspectives, there are interesting theoretical connections between them. To illustrate this point, we discuss Theorem 1.1 further. This theorem follows from the deep property of residual finiteness of 3-manifold groups together with the elegant results of Gerstenhaber and Rothaus [GR62] on the representations of finitely presented groups to a compact, connected Lie group G. (The residual finiteness of closed 3-manifold groups has only been known after the proof of the Geometrization Theorem; this new development is the ingredient that extends Gordon's original statement to closed 3-manifolds in Theorem 1.1.) The statement of Gerstenhaber and Rothaus can be reinterpreted as saying that the G-representations of $\pi_1(Y_-)$ extend to those of $\pi_1(W)$, and Theorem 1.1 (2) implies that any non-trivial representation of $\pi_1(W)$ determines a non-trivial representation of $\pi_1(Y_+)$ by pullback. Thus, Theorem 1.1 naturally leads to the study of the character varieties of $Y_\pm$. Moving further, focusing on $G = \mathrm{SU}(2)$, we observe that the $\mathrm{SU}(2)$-representations of $\pi_1(Y_\pm)$ are related to the instanton Floer homology of $Y_\pm$. Like instanton Floer homology, Heegaard Floer homology is defined by considering certain moduli spaces of solutions; however, while they share many formal properties, the exact relationship between these two theories remains somewhat unclear. Finally, we note that the Geometrization Theorem implies that if $Y_\pm$ is geometric, its geometry can be determined from $\pi_1(Y_\pm)$ in many situations.

In fact, apart from theoretical connections, there is considerable interplay among these perspectives even in their *applications*. We direct the interested reader to Remark 1.24, Theorem 1.27, Theorem 1.34, and Remark 6.5 below for a few examples.

Before we state our results, we set up some conventions for the rest of the article.

Conventions. Unless otherwise specified, all singular homologies have coefficients in $\mathbb{Z}$, Heegaard Floer homologies have coefficients in $\mathbb{Z}/2$, and instanton Floer homologies have coefficients in $\mathbb{Q}$. All 3- and 4-manifolds are assumed to be oriented and smooth, and, except in Section 3.1, they are also assumed to be connected.[1] Accordingly, we also assume that handle decompositions of cobordisms between non-empty 3-manifolds have no 0- or 4-handles. We say that a handle decomposition is

[1] Connectedness is often not essential in our statements, but we impose it for ease of exposition.

ribbon if it has no 3-handles. We always denote the ends of a ribbon homology cobordism by $Y_\pm$; for results that hold for more general cobordisms, we typically denote the cobordism by, for example, $W\colon Y_1 \to Y_2$. All sutured manifolds are assumed to be balanced.

1.1. **Floer homologies and ribbon homology cobordisms.** In the first part of the article, we will prove several results of the following flavor: If $W\colon Y_- \to Y_+$ is a ribbon homology cobordism, then $F(Y_-)$ is a summand of $F(Y_+)$, where F is a version of Floer homology (e.g. sutured instanton Floer homology, involutive Heegaard Floer homology, etc.). In the theorems below, we give the precise statements, which have varying technical hypotheses and conclusions. However, the rough idea is the same throughout and indeed quite simple, which is to show that the double $D(W)$ of W induces an isomorphism on Floer homology. All cobordism maps and isomorphisms can easily be checked to be graded; we leave this task to the reader, although we do use this fact in Theorem 1.27 and Corollary 1.29 below.

We begin with three results for Heegaard Floer homology [OSz04c]. Denote by HF° any of the Heegaard Floer homologies $\widehat{\mathrm{HF}}$, HF^+, HF^-, and HF^∞, and by F_W° the corresponding cobordism map.

Theorem 1.4. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Z}/2$-homology cobordism. Then the cobordism map F_W° includes $\mathrm{HF}^\circ(Y_-)$ into $\mathrm{HF}^\circ(Y_+)$ as a summand. In fact, $\widehat{F}_{D(W)}\colon \widehat{\mathrm{HF}}(Y_-) \to \widehat{\mathrm{HF}}(Y_-)$ is the identity map.*

Remark 1.5. We also provide a Spin^c-refinement of Theorem 1.4; see Theorem 3.9 for the precise statement.

The following is an analogue for the sutured Heegaard Floer homology SFH [Juh06]. We expect that the stated isomorphism below coincides with the cobordism map defined by Juhász [Juh16], although we do not prove it. Here and below, by a cobordism between sutured manifolds, we mean a cobordism obtained by attaching interior handles to a product cobordism; this means that the 3-manifolds have isomorphic sutured boundaries. This definition is narrower than the one used by Juhász [Juh16]. See Definition 3.11 for a precise definition.

Theorem 1.6. *Let (M_-, η_-) and (M_+, η_+) be sutured manifolds, and suppose that there exists a ribbon $\mathbb{Z}/2$-homology cobordism from (M_-, η_-) to (M_+, η_+). Then $\mathrm{SFH}(M_-, \eta_-)$ is isomorphic to a summand of $\mathrm{SFH}(M_+, \eta_+)$.*

Recall that a strongly homotopy-ribbon concordance is a knot concordance in $S^3 \times I$ whose exterior is ribbon.[2] By the isomorphism [Juh06, Proposition 9.2] between the knot Heegaard Floer homology $\widehat{\mathrm{HFK}}$ [OSz04a, Ras03] of a null-homologous knot and SFH of its exterior, Theorem 1.6 immediately implies the following statement for such concordances. This recovers a version of the results in [Zem19b] and [MZ19] when the concordance is in $S^3 \times I$;[3] again, we do not prove that the stated isomorphism coincides with the knot cobordism map.

Corollary 1.7 (cf. [Zem19b, Theorem 1.1] and [MZ19, Theorem 1.2])**.** *Let Y_- and Y_+ be closed 3-manifolds, and let K_- and K_+ be null-homologous knots in Y_- and Y_+ respectively. Suppose that there exists a concordance from K_- to K_+ in a cobordism from Y_- to Y_+, whose exterior is a ribbon $\mathbb{Z}/2$-homology cobordism. Then $\widehat{\mathrm{HFK}}(Y_-, K_-)$ is isomorphic to a summand of $\widehat{\mathrm{HFK}}(Y_+, K_+)$.* ☐

Remark 1.8. One can easily see that the double cover of $S^3 \times I$ branched along the concordance is a ribbon $\mathbb{Z}/2$-homology cobordism, and so Theorem 1.4 applies to show an inclusion of $\mathrm{HF}^\circ(\Sigma_2(K_-))$

[2] All ribbon concordances are strongly homotopy-ribbon.

[3] Note that the exterior of a concordance in $S^3 \times I$ is a $\mathbb{Z}$-homology cobordism.

into $\mathrm{HF}^\circ(\Sigma_2(K_+))$, for knots K_- and K_+ in S^3. A similar statement holds for surgeries along $K_\pm$. We omit these statements for brevity.

We also give an extension for the involutive Heegaard Floer homology $\widehat{\mathrm{HFI}}$ [HM17].

Theorem 1.9. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that there exists a ribbon $\mathbb{Z}$-homology cobordism from Y_- to Y_+. Then $\widehat{\mathrm{HFI}}(Y_-)$ is isomorphic to a summand of $\widehat{\mathrm{HFI}}(Y_+)$.*

We have several results for instanton Floer homology as well. We start with Floer's original homology I for $\mathbb{Z}$-homology spheres [Flo88]. We first make a quick remark.

Remark 1.10. Suppose that Y_+ is a $\mathbb{Z}$-homology sphere. Then for any $\mathbb{Q}$-homology sphere Y_-, a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+ is in fact a ribbon $\mathbb{Z}$-homology cobordism, and the existence of such a cobordism implies that Y_- is also a $\mathbb{Z}$-homology sphere. This is relevant, for example, to Theorem 1.11, Theorem 1.18, Corollary 1.20, and Theorem 1.27.

Theorem 1.11. *Let Y_- and Y_+ be $\mathbb{Z}$-homology spheres, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then the cobordism map $\mathrm{I}(D(W))\colon \mathrm{I}(Y_-) \to \mathrm{I}(Y_-)$ is the identity map, and $\mathrm{I}(W)$ includes $\mathrm{I}(Y_-)$ into $\mathrm{I}(Y_+)$ as a summand.*

Next, we have an analogous statement for the framed instanton Floer homology $\mathrm{I}^\sharp$ [KM11b].

Theorem 1.12. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then the cobordism map $\mathrm{I}^\sharp(D(W))\colon \mathrm{I}^\sharp(Y_-) \to \mathrm{I}^\sharp(Y_-)$ satisfies*

$$\mathrm{I}^\sharp(D(W)) = |H_1(W, Y_-)| \cdot \mathbb{I}_{\mathrm{I}^\sharp(Y_-)},$$

and $\mathrm{I}^\sharp(W)$ includes $\mathrm{I}^\sharp(Y_-)$ into $\mathrm{I}^\sharp(Y_+)$ as a summand.

Theorem 1.12 implies the following corollary, which may also be proved using Theorem 1.4 for ribbon $\mathbb{Z}/2$-homology cobordisms.

Corollary 1.13. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that there exists a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+. Then the unit Thurston norm ball of Y_- includes that of Y_-.*

Proof. This follows from the fact that $\mathrm{I}^\sharp$ detects the Thurston norm [KM10], together with the fact that a ribbon $\mathbb{Q}$-homology cobordism induces a concrete identification between $H_2(Y_-;\mathbb{Q})$ and $H_2(Y_+;\mathbb{Q})$. □

As in Heegaard Floer theory, there is also a version for the sutured instanton Floer homology SHI [KM10].

Theorem 1.14. *Let (M_-, η_-) and (M_+, η_+) be sutured manifolds, and suppose that $N\colon (M_-, \eta_-) \to (M_+, \eta_+)$ is a ribbon $\mathbb{Q}$-homology cobordism. Then the cobordism map $\mathrm{SHI}(D(N))\colon \mathrm{SHI}(M_-, \eta_-) \to \mathrm{SHI}(M_-, \eta_-)$ satisfies*

$$\mathrm{SHI}(D(N)) = |H_1(N, M_-)| \cdot \mathbb{I}_{\mathrm{SHI}(M_-,\eta_-)},$$

and $\mathrm{SHI}(N)$ includes $\mathrm{SHI}(M_-, \eta_-)$ into $\mathrm{SHI}(M_+, \eta_+)$ as a summand.

Recall that for a knot K in a closed 3-manifold Y, the sutured instanton Floer homology of the exterior of K is also denoted by $\mathrm{KHI}(Y, K)$ [KM10]. As in Corollary 1.7, by the isomorphism between KHI and the reduced singular knot instanton Floer homology $\mathrm{I}^\natural$ [KM11a], Theorem 1.14 immediately implies the following result. Note that the isomorphism between KHI and $\mathrm{I}^\natural$ is natural with respect to cobordism maps.

Corollary 1.15. *Let Y_- and Y_+ be closed 3-manifolds, and let K_- and K_+ be knots in Y_- and Y_+ respectively. Suppose that there exists a concordance $C\colon K_- \to K_+$ in a cobordism $W\colon Y_- \to Y_+$, such that the exterior of C is a ribbon $\mathbb{Q}$-homology cobordism. Then the cobordism map $\mathrm{I}^\natural(D(W), D(C))\colon \mathrm{I}^\natural(Y_-, K_-) \to \mathrm{I}^\natural(Y_-, K_-)$ is the identity map, and $\mathrm{I}^\natural(W, C)$ includes $\mathrm{I}^\natural(Y_-, K_-)$ into $\mathrm{I}^\natural(Y_+, K_+)$ as a summand.* □

Remark 1.16. Corollary 1.7 has been used to obtain a genus bound on knots related by ribbon concordance [Zem19b, Theorem 1.3] analogous to Corollary 1.13, and on band connected sums of knots [Zem19b, Theorem 1.6]; Corollary 1.15 provides an alternative proof of these results using knot instanton Floer homology. It also recovers the well-known theorem that, if a ribbon concordance exists in $S^3 \times I$ from K_- to K_+, where K_- and K_+ have the same genus, then the fiberedness of K_+ implies that of K_-.

Remark 1.17. Sherry Gong has informed the authors of a direct proof of a version of Corollary 1.15 with coefficients in $\mathbb{Z}$ for concordances in $Y_- \times I$, without appealing to the isomorphism between KHI and $\mathrm{I}^\natural$. Kang [Kan19] has very recently provided a general proof of Corollary 1.15 for conic strong Khovanov–Floer theories for concordances in $S^3 \times I$, which may be used to recover a version of Corollary 1.15.

As explained in Remark 1.8, one could also use Theorem 1.12 to obtain analogous statements for certain cyclic covers of S^3 branched along $K_\pm$, and for surgeries along $K_\pm$.

We also provide a version for equivariant instanton Floer homologies [Don02, Dae19]. Denote by I° any of the equivariant instanton Floer homologies $\check{\mathrm{I}}$, $\hat{\mathrm{I}}$, and $\bar{\mathrm{I}}$.[4] (We adopt the notation in [Dae19] for these homologies.)

Theorem 1.18. *Let Y_- and Y_+ be $\mathbb{Z}$-homology spheres, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then the cobordism map $\mathrm{I}^\circ(W)$ includes $\mathrm{I}^\circ(Y_-)$ into $\mathrm{I}^\circ(Y_-)$ as a summand.*

Remark 1.19. Equivariant instanton Floer homologies can be extended to $\mathbb{Q}$-homology spheres (with certain auxiliary data) [Mil19, AB96]. We expect (but do not prove) that Theorem 1.18 holds also for these extensions.

Theorem 1.11 also yields an application to the computation of the Furuta–Ohta invariant λ_{FO} for $\mathbb{Z}[\mathbb{Z}]$-homology $S^1 \times S^3$'s [FO93]. Here and below, for a $\mathbb{Z}$-homology sphere Y, denote by $\lambda(Y)$ the Casson invariant of Y.

Corollary 1.20. *Suppose that Y_- and Y_+ are $\mathbb{Z}$-homology spheres, and that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism, and that $\overline{D(W)}$ is the $\mathbb{Z}[\mathbb{Z}]$-homology $S^1 \times S^3$ obtained by gluing the ends of $D(W)$ by the identity. Then $\lambda_{\mathrm{FO}}(\overline{D(W)}) = \lambda(Y_-)$. In particular, $\lambda_{\mathrm{FO}}(\overline{D(W)})$ agrees with the Rokhlin invariant of Y_- mod 2.*

Proof. A standard gluing argument shows that the signed count of elements in the moduli space of index-0 (perturbed) ASD connections on $\overline{D(W)}$ is equal to $2\,\mathrm{Lef}(\mathrm{I}(D(W)\colon \mathrm{I}(Y_-) \to \mathrm{I}(Y_-))$. (See [Don02, Theorem 6.7] for a similar gluing result.) By definition, the former count is equal to $4\lambda_{\mathrm{FO}}(\overline{D(W)})$, and by Theorem 1.11, the Lefschetz number $\mathrm{Lef}(\mathrm{I}(D(W))$ is the Euler characteristic of $\mathrm{I}(Y_-)$, which is precisely twice the Casson invariant of Y_-. □

Remark 1.21. We expect the analogue of Theorem 1.4 to hold also for the monopole Floer homology groups $\widecheck{\mathrm{HM}}$, $\widehat{\mathrm{HM}}$, and $\overline{\mathrm{HM}}$ [KM07]. Note that by the isomorphisms between Heegaard and monopole Floer homologies [KLT, CGH11, Tau10], we already know that $\mathrm{HM}^\circ(Y_-)$ is isomorphic

[4]The homologies $\check{\mathrm{I}}$, $\hat{\mathrm{I}}$, and $\bar{\mathrm{I}}$ may be viewed as analogues of HF^+, HF^-, and HF^∞ respectively.

to a summand of $\mathrm{HM}^\circ(Y_+)$. In order to prove that the isomorphism coincides with the cobordism map, one could, for example, prove a surgery formula analogous to Proposition 3.2 for monopole Floer homology. Although we expect that this surgery formula holds for monopole Floer homology (especially because an analogous result holds for Bauer–Furuta invariants [KLS19, Corollary 1.4]), we do not give a proof of this result for brevity.

We also expect an analogue of Corollary 1.20 to hold for the Mrowka–Ruberman–Saveliev invariant λ_{SW} [MRS11]. Using the splitting theorem [LRS18], we have

$$\lambda_{\mathrm{SW}}(\overline{D(W)}) = -\operatorname{Lef}(\mathrm{HM}^{\mathrm{red}}(D(W))) - h(Y_-),$$

where h is the monopole Frøyshov invariant. Since the Casson invariant of Y_- can alternatively be computed as $\chi(\mathrm{HM}^{\mathrm{red}}(Y_-)) + h(Y_-)$, we would obtain that $\lambda_{\mathrm{SW}}(\overline{D(W)}) = -\lambda(Y_-)$. In particular, we have $\lambda_{\mathrm{SW}}(\overline{D(W)}) = -\lambda_{\mathrm{FO}}(\overline{D(W)})$. This would verify [MRS11, Conjecture B] for the 4-manifolds with the $\mathbb{Z}[\mathbb{Z}]$-homology of $S^1 \times S^3$ which have the form $\overline{D(W)}$.

The rough strategy for proving all of the theorems above is fairly straightforward. First, a topological argument (Proposition 2.1 below) shows that $D(W)$ is given by surgery along a collection of m loops in $(Y_- \times I) \natural\, m(S^1 \times S^3)$. By using surgery formulas, it can be shown that the induced map for $D(W)$ is the same as that for the 4-manifold obtained by surgering along the m cores of the $S^1 \times S^3$ summands, which is just $Y_- \times I$. Of course, this induces the identity map.

We will also outline an alternative proof of Theorem 1.11 in Section 5.5 that passes more directly through the fundamental group and Theorem 1.1.

1.2. **Character varieties and ribbon homology cobordisms.** Given a ribbon $\mathbb{Q}$-homology cobordism $W\colon Y_- \to Y_+$, we will also obtain relations between the character varieties of Y_- and Y_+. Recall that for a group π and compact, connected Lie group G (e.g. $\mathrm{SU}(2)$), we can define the *representation variety* $\mathcal{R}_G(\pi)$, which is the set of G-representations of π; we can also quotient by the conjugation action to obtain the *character variety* $\chi_G(\pi)$. For a path-connected space X, we will write $\mathcal{R}_G(X)$ for $\mathcal{R}_G(\pi_1(X))$, and $\chi_G(X)$ for $\chi_G(\pi_1(X))$. As discussed above, we have the following proposition.

Proposition 1.22. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then any $\rho_- \in \mathcal{R}_G(Y_-)$ can be extended to an element $\rho_W \in \mathcal{R}_G(W)$ that pulls back to an element $\rho_+ \in \mathcal{R}_G(Y_+)$, and distinct elements in $\mathcal{R}_G(Y_-)$ corresponds to distinct elements in $\mathcal{R}_G(Y_+)$. The analogous statement for χ_G also holds.*

See Proposition 5.2 for a restatement and proof. Recall that the Chern–Simons functional gives an $\mathbb{R}/\mathbb{Z}$-valued function on $\mathcal{R}_G(Y)$; the image of this function is a finite subset of $\mathbb{R}/\mathbb{Z}$, which we call the *G–Chern–Simons invariants* of Y. Proposition 1.22 implies a relation between the G–Chern–Simons invariants of Y_- and Y_+.

Corollary 1.23. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that there exists a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+. Then the set of G–Chern–Simons invariants of Y_- is a subset of that of Y_+.*

Proof. Let $W\colon Y_- \to Y_+$ be a ribbon $\mathbb{Q}$-homology cobordism. Let α_- be a flat connection on Y_-, whose holonomy gives an element $\rho_- \in \mathcal{R}_G(Y_-)$. By Proposition 1.22, we may extend ρ_- to an element $\rho_W \in \mathcal{R}_G(W)$, which pulls back to an element $\rho_+ \in \mathcal{R}_G(Y_+)$. We may then choose a corresponding flat connection α_+ on Y_+. By Auckly [Auc94], the Chern–Simons invariants of α_- and α_+ agree. □

Remark 1.24. Stein manifolds provide a large family of 4-manifolds without 3-handles. It is interesting to compare the discussion above with the work of Baldwin and Sivek [BS18], who use instanton Floer homology to prove that if Y is a $\mathbb{Z}$-homology sphere that admits a Stein filling with non-trivial homology, then $\pi_1(Y)$ admits an irreducible SU(2)-representation. In comparison, if $W\colon Y_- \to Y_+$ is a Stein $\mathbb{Q}$-homology cobordism, and $\pi_1(Y_-)$ admits a non-trivial SU(2)-representation, then it extends to an SU(2)-representation of $\pi_1(W)$ that pulls back to a non-trivial SU(2)-representation of $\pi_1(Y_+)$ by Proposition 1.22, which requires no gauge theory.

In fact, with a bit more work, we can compare the local structures of the character varieties. For a path-connected space X and a representation $\rho\colon \pi_1(X) \to G$, recall that the *Zariski tangent space* to $\chi_G(X)$ at the conjugacy class $[\rho]$ is the first group cohomology of $\pi_1(X)$ with coefficients in the adjoint representation associated to ρ, denoted by $H^1(X; \mathrm{Ad}_\rho)$; see Section 5.2 for more details. Below, we also consider the zeroth group cohomology $H^0(X; \mathrm{Ad}_\rho)$.

Proposition 1.25. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Fix $\rho_- \in \mathcal{R}_G(Y_-)$, choose an extension $\rho_W \in \mathcal{R}_G(W)$, and denote by $\rho_+ \in \mathcal{R}_G(Y_+)$ the pullback of ρ_W. Suppose that $\dim_\mathbb{R} H^0(Y_-; \mathrm{Ad}_{\rho_-}) = \dim_\mathbb{R} H^0(Y_+; \mathrm{Ad}_{\rho_+})$. Then*

$$\dim_\mathbb{R} H^1(Y_-; \mathrm{Ad}_{\rho_-}) \leq \dim_\mathbb{R} H^1(W; \mathrm{Ad}_{\rho_W}) \leq \dim_\mathbb{R} H^1(Y_+; \mathrm{Ad}_{\rho_+}).$$

This seemingly technical result will have interesting topological applications in Section 1.4.

1.3. **Thurston geometries and ribbon homology cobordisms.**

Given a ribbon $\mathbb{Q}$-homology cobordism $W\colon Y_- \to Y_+$, we will prove that, roughly speaking, the type of geometry that Y_- admits constrains that which Y_+ admits, and vice versa.

Theorem 1.26. *There is a hierarchy among the Thurston geometries with respect to ribbon $\mathbb{Q}$-homology cobordisms, given by the diagram*

$$S^3 \to (S^2 \times \mathbb{R}) \to \mathbb{R}^3 \to \mathrm{Nil} \to \mathrm{Sol} \to (\mathbb{H}^2 \times \mathbb{R}) \cup \widetilde{\mathrm{SL}(2,\mathbb{R})} \cup \mathbb{H}^3.$$

In other words, suppose that Y_- and Y_+ are compact 3-manifolds with empty or toroidal boundary that admit distinct geometries, and that there exists a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+. Then there is a sequence of arrows from the geometry of Y_- to that of Y_+ in the diagram above.

A more refined version of Theorem 1.26 is stated in Theorem 6.3.

1.4. **Topological applications.**

We now turn to using the results above to obstruct the existence of ribbon homology cobordisms. We begin with an application to Seifert fibered homology spheres that illustrates the use of several different tools described above. We provide the proof here, but postpone a technical proposition within to Section 5.

Theorem 1.27. *Suppose that Y_- and Y_+ are the Seifert fibered homology spheres $\Sigma(a_1, \dots, a_n)$ and $\Sigma(a'_1, \dots, a'_m)$ respectively, and that there exists a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+. Then*

1. *The Casson invariants of Y_- and Y_+ satisfy $|\lambda(Y_-)| \leq |\lambda(Y_+)|$;*
2. *Either Y_- and Y_+ both bound negative-definite plumbings, or both bound positive-definite plumbings; and*
3. *The numbers of fibers satisfy $n \leq m$.*

While the conclusions in Theorem 1.27 seem strong, the authors do not know of any ribbon $\mathbb{Q}$-homology cobordisms between two Seifert fibered homology spheres distinct from S^3, or from a non–Seifert fibered space to a Seifert fibered space. For comparison, given any closed 3-manifold Y_-, one can always construct a ribbon $\mathbb{Q}$-homology cobordism from Y_- to a hyperbolic 3-manifold, and one to a 3-manifold with non-trivial JSJ decomposition.

Proof. (1) This follows from Theorem 1.11, since $|\lambda(Y)| = |\chi(\mathrm{I}(Y))| = \dim \mathrm{I}(Y)$ for a Seifert fibered homology sphere Y [Sav92].

(2) The only Seifert fibered homology sphere with trivial Casson invariant is S^3, which bounds both positive- and negative-definite plumbings. Again by [Sav92], $\mathrm{I}(Y_+)$ is supported in one $\mathbb{Z}/2$-grading; this $\mathbb{Z}/2$-grading determines the sign of $\lambda(Y_+)$ and hence the definiteness of the plumbing Y_+ bounds. Theorem 1.11 implies that $\mathrm{I}(Y_-)$ is supported in the same $\mathbb{Z}/2$-grading.

(3) Fintushel and Stern [FS90] show that the Zariski tangent space to the SU(2)-character variety of $\Sigma(a_1, \ldots, a_n)$ has dimension less than or equal to $2n-6$. By Proposition 5.8, the equality is always realized at some irreducible representation. The result then follows from Proposition 1.25. □

Note that the first two items above can also be proved using Heegaard Floer homology, by [OSz03a, Theorem 1.3] and [OSz03c, Corollary 1.4]. The authors do not know of a Floer-homology proof of (3). Using Heegaard Floer homology, we also have the following two corollaries. Recall that the *reduced Heegaard Floer homology* $\mathrm{HF}^{\mathrm{red}}$ is the U-torsion submodule of HF^-, and a $\mathbb{Q}$-homology sphere Y is an *L-space* if $\mathrm{HF}^{\mathrm{red}}(Y) = 0$.[5]

Corollary 1.28. *Suppose that Y_- and Y_+ are $\mathbb{Q}$-homology spheres, and that Y_+ is an L-space while Y_- is not. Then there does not exist a ribbon $\mathbb{Z}/2$-homology cobordism from Y_- to Y_+.*

Note that this applies whenever Y_- is a toroidal $\mathbb{Z}$-homology sphere, since such a manifold is necessarily not an L-space [Eft18, Theorem 1.1] (see also [HRW17, Corollary 10]).

Proof. Suppose that $W \colon Y_- \to Y_+$ is a ribbon $\mathbb{Z}/2$-homology cobordism; then Theorem 1.4 implies that $F_W^- \colon \mathrm{HF}^-(Y_-) \to \mathrm{HF}^-(Y_+)$ is injective. Under this map, U-torsion elements must be mapped to U-torsion elements; thus, we obtain an injection on $\mathrm{HF}^{\mathrm{red}}$ as well. □

Corollary 1.29. *Suppose that Y_1 and Y_2 are $\mathbb{Q}$-homology spheres that are not L-spaces. Then there does not exist a ribbon $\mathbb{Z}/2$-homology cobordism from $Y_1 \sharp Y_2$ to a Seifert fibered space.*

Proof. If Y_1 and Y_2 both have non-trivial $\mathrm{HF}^{\mathrm{red}}$, then in both $\mathbb{Z}/2$-gradings, $\mathrm{HF}^{\mathrm{red}}(Y_1 \sharp Y_2)$ is not trivial. Indeed, by the Künneth formula [OSz04b, Theorem 1.5], $\mathrm{HF}^{\mathrm{red}}(Y_1 \sharp Y_2)$ contains a summand isomorphic to two copies of $\mathrm{HF}^{\mathrm{red}}(Y_1) \otimes \mathrm{HF}^{\mathrm{red}}(Y_2)$, with one copy shifted in grading by 1. (One comes from the tensor product and one from the Tor term.) Meanwhile, Seifert fibered spaces have $\mathrm{HF}^{\mathrm{red}}$ supported in a single $\mathbb{Z}/2$-grading [OSz03c, Corollary 1.4]. □

Turning to character varieties, we will have the following.

Corollary 1.30. *Let Y and N be compact 3-manifolds, and suppose that $N \ncong S^3$. Then there does not exist a ribbon $\mathbb{Q}$-homology cobordism from $Y \sharp N$ to Y.*

For ease of exposition, we postpone the proof of Corollary 1.30 until Section 5. Note that this result can alternatively be proven if N has non-trivial $\mathrm{HF}^{\mathrm{red}}$, by an application of the Künneth formula, as in the proof of Corollary 1.29. However, such an argument does not work for $N = \Sigma(2,3,5) \sharp (-\Sigma(2,3,5))$, since this is an L-space. (In fact, in this case, $Y \sharp N$ is even $\mathbb{Z}$-homology cobordant to Y.) The same

[5] Technically, Y should be called a $\mathbb{Z}/2$–*Heegaard L-space*. One could also define L-spaces with other coefficients, or with instanton Floer homology. However, we never consider these concepts of L-spaces in the present article.

issue arises for $\mathrm{I}^\sharp$. For I, it is difficult to study the instanton Floer homology of connected sums in general.

Remark 1.31. Eliashberg [Eli90] shows that a Stein filling of a connected sum is a boundary sum of Stein fillings. It is interesting to compare this with the two results above.

Corollary 1.30 can be viewed as obstructing ribbon homology cobordisms from a 3-manifold with an essential sphere to one without; the following is a statement for knots, with a similar flavor.

Corollary 1.32. *There does not exist a strongly homotopy-ribbon concordance from $3_1 \sharp 4_1$ to any knot whose exterior has no closed, non–boundary-parallel, incompressible surfaces.*

Note that the exterior of a torus knot has no closed, non–boundary-parallel, incompressible surfaces.

Proof. The SU(2)-representation variety associated to $K_- = 3_1 \sharp 4_1$ has an open submanifold of dimension 5 on which the conjugation action by SO(3) is free [Kla91, Proposition 13]. Suppose that there is a strongly homotopy-ribbon concordance $C\colon K_- \to K_+$; Proposition 1.22 implies that the SU(2)-representation variety associated to K_+ has an open submanifold of dimension at least 5 on which SO(3) acts freely, and so the associated SU(2)-character variety must have a component of dimension at least 2. (Although the representation variety is not a smooth manifold in general, it is a real algebraic variety and hence can be written as the union of finitely many smooth manifolds [Whi57].) By [Kla91, Proposition 15], this implies that K_+ has a closed, non–boundary-parallel, incompressible surface in its exterior. □

Remark 1.33. The existence results of [KM10] on SU(2)-representations suggest that a statement analogous to the above should hold for an arbitrary composite knot K_-.

Finally, we will discuss some applications of ribbon $\mathbb{Q}$-homology cobordisms to reducible Dehn surgery problems. The following is a sample theorem; see Section 7 for its proof, as well as similar results. There are many other results along these lines that we do not mention, which we leave to the reader.

Theorem 1.34. *Suppose that Y is a Seifert fibered homology sphere, K is a null-homotopic knot in Y, and $Y_0(K) \cong N \sharp (S^1 \times S^2)$. Then $N \cong Y$.*

Remark 1.35. Jennifer Hom and the second author have shown that K must in fact be trivial.

Outline. In Section 2, we give the necessary topological background and give a short application to metrics with positive scalar curvature. In Section 3, we set up the necessary tools for Heegaard Floer homology and prove Theorem 1.4 to Theorem 1.9. In Section 4, after giving an overview of instanton Floer homology, we prove Theorem 1.11 to Theorem 1.14. Then in Section 5, we study the relationships between ribbon homology cobordisms and character varieties, proving Proposition 1.22 and Proposition 1.25, and complete the proofs of Theorem 1.27 and Corollary 1.30; we also outline a proof of Theorem 1.11 via character varieties. In Section 6, we prove Theorem 1.26, pertaining to Thurston geometries. Finally, in Section 7, we provide further applications of ribbon homology cobordisms to Dehn surgery problems, including the proof of Theorem 1.34.

We provide a few routes for the reader. If the sole interest is in Heegaard Floer homology, then refer to Section 2 and Section 3. For instanton Floer homology, see Section 2 and Section 4. The reader solely interested in character varieties, Thurston geometries, or Dehn surgeries can read only Section 5, Section 6, or Section 7, respectively.

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2. Topology of ribbon cobordisms

Recall that the *double* $D(W)$ of a cobordism $W\colon Y_1 \to Y_2$ is formed by gluing W and $-W$ along Y_2. In analogy with the arguments used in ribbon concordance, our strategy to prove Theorem 1.4, Theorem 1.11, and Theorem 1.12 will be to prove the cobordism map on Floer homology induced by $D(W)$ is an isomorphism, when W is ribbon. First, we need a topological description of $D(W)$. In what follows, we will use $\mathbb{F}$ to denote any field. Note that a ribbon $\mathbb{F}$-homology cobordism has the same number of 1- and 2-handles.

Proposition 2.1. *Let Y_- and Y_+ be compact 3-manifolds, and suppose that $W\colon Y_- \to Y_+$ is a ribbon cobordism, where the number of 1-handles is m, and that of 2-handles is ℓ. Then $D(W)$ can be described by surgery on $X \cong (Y_- \times I) \,\sharp\, m(S^1 \times S^3)$ along ℓ disjoint simple closed curves $\gamma_1, \dots, \gamma_\ell$. If, in addition, W is also an $\mathbb{F}$-homology cobordism, then $[\gamma_1] \wedge \cdots \wedge [\gamma_\ell]$ and $\alpha_1 \wedge \cdots \wedge \alpha_\ell$ agree as elements of $(\Lambda^*(H_1(X)/\operatorname{Tors})/\langle H_1(Y_-)/\operatorname{Tors}\rangle) \otimes_{\mathbb{Z}} \mathbb{F}$ up to multiplication by a unit, where $\alpha_i \in H_1(X)$ is homologous to the core of the i^{th} $S^1 \times S^3$ summand, and $\langle H_1(Y_-)/\operatorname{Tors}\rangle$ is the ideal generated by $H_1(Y_-)/\operatorname{Tors}$.*

Of course, working with elements in $\Lambda^*(H_1(X)/\operatorname{Tors})/\langle H_1(Y_-)/\operatorname{Tors}\rangle$ is the same as first projecting $H_1(X)$ to the submodule corresponding to the $S^1 \times S^3$ summands and then working in the exterior algebra there. See Figure 1 for a schematic diagram when $m = 1$.

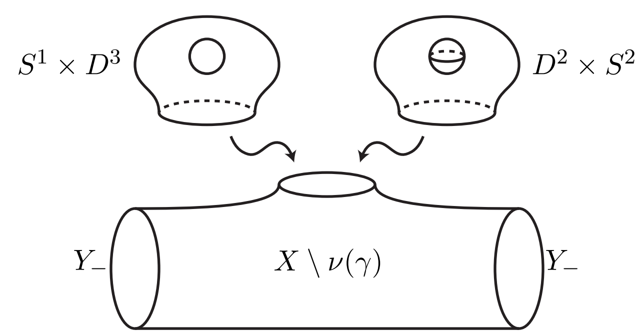


Figure 1. An illustration of Proposition 2.1 in the case of $m = \ell = 1$. Here, $\nu(\gamma)$ denotes a neighborhood of γ. Reattaching the $S^1 \times D^3$ would yield $X \cong (Y_- \times I) \,\sharp\, (S^1 \times S^3)$, while we may obtain $D(W)$ by switching it for the $D^2 \times S^2$.

Before we prove Proposition 2.1, we first establish an elementary fact.

Lemma 2.2. *Let M_1 and M_2 be $(n-1)$-manifolds, and suppose that $N\colon M_1 \to M_2$ is a cobordism associated to attaching an n-dimensional k-handle h. Then the double $D(N)$ can be described by surgery on $M_1 \times I$ along some S^{k-1} that is homologous to the attaching sphere of h.*

Proof. Write $D(N) = (M_1 \times I) \cup h \cup h' \cup (M_1 \times I)$, where h' is the dual handle of h. The cocore of h and the core of h' together form an S^{n-k} with trivial normal bundle, which may be identified with $h \cup h'$. (The case where $n = 4$ and $k = 2$ is described, for example, in [GS99, Example 4.6.3].) Note that h meets the lower $M_1 \times I$, and h' meets the upper $M_1 \times I$, at the same attaching region $S^{k-1} \times D^{n-k} \subset M_1$, with the same framing. Thus, removing $h \cup h'$ from $D(N)$ would result in $(M_1 \times I) \setminus (S^{k-1} \times D^{n-k} \times (-\epsilon, \epsilon))$. In other words, $D(N)$ may be formed by removing $S^{k-1} \times D^{n-k} \times (-\epsilon, \epsilon) \cong S^{k-1} \times D^{n-k+1}$ from $M_1 \times I$ and replacing it with $h \cup h' \cong D^k \times S^{n-k}$, which is the definition of surgery. ☐

Proof of Proposition 2.1. First, decompose W into a cobordism W_1 from Y_- to $\widetilde{Y} \cong Y_- \sharp\, m(S^1 \times S^2)$ and a cobordism W_2 from $\widetilde{Y}$ to Y_+, corresponding to the attachment of 1- and 2-handles respectively. Below, we will compare $D(W) = W_1 \cup W_2 \cup (-W_2) \cup (-W_1)$ with $W_1 \cup (-W_1)$.

Applying Lemma 2.2 to each of the 2-handles in W_2, we see that $W_2 \cup (-W_2)$ can be described by surgery on $\widetilde{Y} \times I$ along some $\gamma_1, \dots, \gamma_\ell$, where the γ_i's are homologous to the attaching circles of the 2-handles. (Perform isotopies and handleslides first, if necessary, to ensure that the attaching regions of the 2-handles lie in $\widetilde{Y}$ and are disjoint.)

Note that $W_1 \cup (-W_1) \cong W_1 \cup (\widetilde{Y} \times I) \cup (-W_1)$ is diffeomorphic to $X \cong (Y_- \times I) \,\sharp\, m(S^1 \times S^3)$. Thus, we see that $D(W) = W_1 \cup W_2 \cup (-W_2) \cup (-W_1)$ can be described by surgery on X along $\gamma_1, \dots, \gamma_\ell$.

Finally, suppose W is a ribbon $\mathbb{F}$-homology cobordism; then $m = \ell$. Present the differential $\partial_2 \colon C_2(Y_-) \to C_1(Y_-)$ by a matrix A; then in the corresponding cellular chain complex of W, the presentation matrix Q of the differential $\partial_2 \colon C_2(W) \to C_1(W)$ is of the form

$$Q = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where C is an $(m \times m)$-matrix representing the attachment of the 2-handles in W_2. As the attaching circles of these 2-handles are homologous to $\gamma_1, \dots, \gamma_m$, we see that C_{ij} is given by the algebraic intersection number of γ_j with $\{p\} \times S^3$ in the i^{th} $S^1 \times S^3$ summand. Since W is an $\mathbb{F}$-homology cobordism, we have $H_1(W, Y_-; \mathbb{F}) = 0$, implying that $C \otimes_{\mathbb{Z}} \mathbb{F} \colon \mathbb{F}^m \to \mathbb{F}^m$ is surjective, and hence invertible. This is equivalent to saying that $[\gamma_1] \wedge \cdots \wedge [\gamma_m]$ and $\alpha_1 \wedge \cdots \wedge \alpha_m$ agree up to multiplication by a unit in $(\Lambda^*(H_1(X)/\operatorname{Tors})/\langle H_1(Y_-)/\operatorname{Tors}\rangle) \otimes_{\mathbb{Z}} \mathbb{F}$. ☐

While it will not be used later in the paper, we conclude this section with the following geometric result, which may be of independent interest.

Proposition 2.3. *Suppose that W is a compact 4-manifold with connected boundary and a ribbon handle decomposition. Then W admits a metric with positive scalar curvature.*

Proof. By Proposition 2.1, $D(W)$ is obtained by surgery on a collection of ℓ loops in $\sharp\, m(S^1 \times S^3)$. First, it is well known that $S^1 \times S^3$ has a p.s.c. metric. By the work of Gromov and Lawson [GL80, Theorem A], $\sharp\, m(S^1 \times S^3)$ admits a p.s.c. metric. Next, surgery on loops is a codimension-3 surgery, and so we may again apply the result of Gromov and Lawson to see that $D(W)$ admits a p.s.c. metric. Since W is a codimension-0 submanifold of $D(W)$, it inherits a p.s.c. metric as well. ☐

3. Heegaard Floer homology and ribbon homology cobordisms

3.1. **Surgery and cobordism maps in Heegaard Floer theory.** In light of Proposition 2.1, our strategy to prove Theorem 1.4 will be to show that the cobordism map for $D(W)$ is actually just determined by that for $X \cong (Y_- \times I) \,\sharp\, m(S^1 \times S^3)$ and the homology classes of the γ_i's, and hence must agree with that of $Y_- \times I$. We will first focus on $\widehat{\mathrm{HF}}$; it will be shown later in the proof of Theorem 1.4 that this is sufficient to recover the result for the other flavors. The necessary tool is Proposition 3.2 below, which shows the behavior of the Heegaard Floer cobordism maps under surgery along circles. This statement is known to experts, and can be derived from the link cobordism TQFT of Zemke; see Remark 3.4 below. It is also already well established in Seiberg–Witten theory; see, for example, [KLS19, Corollary 1.4]. For completeness, we provide a proof in this subsection. Note that, as mentioned in Section 1, we do not assume 3- and 4-manifolds to be connected in this subsection.

Recall that given a connected Spin^c-cobordism $(W,\mathfrak{t})\colon (Y_1,\mathfrak{s}_1)\to(Y_2,\mathfrak{s}_2)$ between closed, connected 3-manifolds, Ozsváth and Szabó [OSz06] define cobordism maps

$$F^\circ_{W,\mathfrak{t}}\colon \mathrm{HF}^\circ(Y_1,\mathfrak{s}_1)\otimes(\Lambda^*(H_1(W)/\operatorname{Tors})\otimes\mathbb{Z}/2)\to\mathrm{HF}^\circ(Y_2,\mathfrak{t}_2).$$

These maps have the property that

$$F^\circ_{W,\mathfrak{t}}(x\otimes\xi)=F^\circ_{W,\mathfrak{t}}(\xi_1\cdot x)+\xi_2\cdot F^\circ_{W,\mathfrak{t}}(x), \tag{3.1}$$

whenever $\xi\in H_1(W)/\operatorname{Tors}$ satisfies $\xi=\iota_1(\xi_1)-\iota_2(\xi_2)$, where $\xi_i\in H_1(Y_i)/\operatorname{Tors}$ and ι_i is induced by inclusion; see [OSz03b, p. 186]. We may also sum over all Spin^c-structures on W, and obtain a total map

$$F^\circ_W\colon \mathrm{HF}^\circ(Y_1)\otimes(\Lambda^*(H_1(W)/\operatorname{Tors})\otimes\mathbb{Z}/2)\to\mathrm{HF}^\circ(Y),$$

satisfying a property analogous to (3.1). We are now ready to state:

Proposition 3.2. *Let Y_1 and Y_2 be closed, connected 3-manifolds, and let $X\colon Y_1\to Y_2$ be a connected cobordism. Suppose that $\gamma_1,\dots,\gamma_\ell\subset\operatorname{Int}(X)$ are loops with disjoint neighborhoods $\nu(\gamma_i)\cong\gamma_i\times D^3$, and denote by Z the result of surgery on X along $\gamma_1,\dots,\gamma_\ell$. Then for $x\in\widehat{\mathrm{HF}}(Y_1)$,*

$$\widehat{F}_X(x\otimes([\gamma_1]\wedge\cdots\wedge[\gamma_\ell]))=\widehat{F}_Z(x). \tag{3.3}$$

Thus, $\widehat{F}_Z$ depends only on X and $[\gamma_1]\wedge\cdots\wedge[\gamma_\ell]\in\Lambda^*(H_1(X)/\operatorname{Tors})\otimes\mathbb{Z}/2$.

Remark 3.4. A surgery formula for link cobordisms and link Floer homology, similar to Proposition 3.2, is provided by Zemke [Zem19a, Proposition 5.4]. One may obtain Proposition 3.2 via an identification, also provided by Zemke [Zem18, Theorem C], of link cobordism maps with maps induced by cobordisms between 3-manifolds. In this paper, we instead provide a direct proof without mentioning any link cobordism theory, in the interest of providing a self-contained discussion.

Before giving the proof, we describe the idea informally. Surgery on γ_i is the result of removing a copy of $S^1\times D^3$ and replacing it with $D^2\times S^2$. The cobordism map for $D^2\times S^2$ agrees with that of $S^1\times D^3$ if one contracts the latter map by the generator of H_1. Composing with the cobordism map for $X\setminus(\coprod\nu(\gamma_i))$, the result follows. However, to prove this carefully, we must cut and re-glue several different codimension-0 submanifolds, and thus need to use the graph TQFT framework by Zemke [Zem15]. Below, we give a brief review of the necessary elements.

Let Y be a possibly disconnected 3-manifold, and let $\mathbf{p}$ be a set of points in Y with at least one point in each component. Let $W\colon Y_1\to Y_2$ be a cobordism, and let Γ be a graph embedded in W with $\partial\Gamma=\mathbf{p}_1\cup\mathbf{p}_2$. Then, Zemke [Zem15] constructs Heegaard Floer homology groups $\widehat{\mathrm{HF}}(Y_i,\mathbf{p}_i)$ and cobordism maps $\widehat{F}_{W,\Gamma}\colon\widehat{\mathrm{HF}}(Y_1,\mathbf{p}_1)\to\widehat{\mathrm{HF}}(Y_2,\mathbf{p}_2)$.

Theorem 3.5 (Zemke [Zem15])**.** *The cobordism maps $\widehat{F}_{W,\Gamma}$ satisfy the following.*

1. *Under disjoint union, we have that $\widehat{\mathrm{HF}}(Y_1\sqcup Y_2,\mathbf{p}_1\sqcup\mathbf{p}_2)=\widehat{\mathrm{HF}}(Y_1,\mathbf{p}_1)\otimes\widehat{\mathrm{HF}}(Y_2,\mathbf{p}_2)$, and $\widehat{F}_{(W_1,\Gamma_1)\sqcup(W_2,\Gamma_2)}=\widehat{F}_{(W_1,\Gamma_1)}\otimes\widehat{F}_{(W_2,\Gamma_2)}$; see* [Zem15, p. 17].
2. *Given $(W,\Gamma)\colon(Y_1,\mathbf{p}_1)\to(Y_2,\mathbf{p}_2)$ and $(W',\Gamma')\colon(Y_2,\mathbf{p}_2)\to(Y_3,\mathbf{p}_3)$, then $\widehat{F}_{W',\Gamma'}\circ\widehat{F}_{W,\Gamma}=\widehat{F}_{W\cup W',\Gamma\cup\Gamma'}$; see* [Zem15, Theorem A (2)].

(3) *$\widehat{F}_{W,\Gamma}$ admits a decomposition by Spin^c-structures in the usual way. In particular, $\widehat{F}_{W,\Gamma} = \sum_{\mathfrak{t}\in\mathrm{Spin}^c(W)} \widehat{F}_{W,\Gamma,\mathfrak{t}}$, and*

$$\widehat{F}_{W',\Gamma',\mathfrak{t}_{W'}} \circ \widehat{F}_{W,\Gamma,\mathfrak{t}_W} = \sum_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(W\cup W')\\ \mathfrak{t}|_W=\mathfrak{t}_W,\, \mathfrak{t}|_{W'}=\mathfrak{t}_{W'}}} \widehat{F}_{W\cup W',\Gamma\cup\Gamma',\mathfrak{t}};$$

see [Zem15, Theorem B (b) and (c)]. *(We take the convention that this equation remains valid when $\mathfrak{t}_W|_{Y_2} \neq \mathfrak{t}_{W'}|_{Y_2}$, in which case both sides of the equation are identically zero.)*

(4) *If (W,Γ) and (W,Γ') are cobordisms from $(Y_1,\mathbf{p}_1)$ to $(Y_2,\mathbf{p}_2)$ such that the exteriors of Γ and Γ' are diffeomorphic, then $\widehat{F}_{W,\Gamma} = \widehat{F}_{W,\Gamma'}$; see* [Zem15, Remark 3.1]. *Also, if λ is an arc from the boundary of some $B^4 \subset W$ to Γ, then $\widehat{F}_{W,\Gamma}(x) = \widehat{F}_{W\setminus B^4,\Gamma\cup\lambda}(x \otimes y)$, where y is the generator of $\widehat{\mathrm{HF}}(\partial B^4)$; see* [Zem15, Lemma 6.1]. *See* [Zem15, Figure 16] *for an illustration of both of these principles.*

(5) *Suppose that Y_1 and Y_2 are connected, $\mathbf{p}_1$ and $\mathbf{p}_2$ each consist of a single point, and Γ is a path. Then $\widehat{F}_W(x) = \widehat{F}_{W,\Gamma}(x)$, where $\widehat{F}_W$ is the original Ozsváth–Szabó cobordism map; see* [Zem15, Theorem A (1)]. *(Implicitly, the Ozsváth–Szabó cobordism map requires a choice of basepoints and a choice of path, but the injectivity statement in Theorem 1.4 is independent of both choices.)*

(6) *Suppose again that Y_1 and Y_2 are connected, $\mathbf{p}_1$ and $\mathbf{p}_2$ each consist of a single point, and Γ is a path. Let γ be a simple closed loop in $\mathrm{Int}(W)$ that intersects Γ at a single point. Then $\widehat{F}_W(x \otimes [\gamma]) = \widehat{F}_{W,\Gamma\cup\gamma}(x)$, where the left-hand side is the Ozsváth–Szabó cobordism map defined above; see* [Zem15, Theorem C].

(7) *Let Y be connected and let $\mathbf{p}$ consist of a single point. Consider $\Gamma = \mathbf{p}\times I \subset Y \times I$. Choose a simple closed loop γ in Y based at $\mathbf{p}$ and let Γ_γ be the graph obtained by appending $\gamma \times \{1/2\}$ to Γ. Denote the cobordism map $\widehat{F}_{Y\times I,\Gamma_\gamma}$ by $\mathcal{F}(\gamma)$. Then, $\mathcal{F}(\gamma)$ depends only on $[\gamma] \in H_1(Y)$* [Zem15, Corollary 7.4]. *Furthermore, $\mathcal{F}(\gamma * \gamma') = \mathcal{F}(\gamma) + \mathcal{F}(\gamma')$* [Zem15, Lemma 7.3] *and $\mathcal{F}(\gamma)\circ\mathcal{F}(\gamma) = 0$* [Zem15, Lemma 7.2]. *Here, $\gamma * \gamma'$ is a simple closed loop in the based homotopy class of the concatenation.*

We now need a slight generalization of Theorem 3.5 (6), i.e. [Zem15, Theorem C], which will allow us to analyze the effect on the cobordism map of appending multiple loops to a path. We begin with the identity cobordism.

Lemma 3.6. *Suppose that Y is connected, and that $\mathbf{p}$ consists of a single point. Suppose that Γ is a graph obtained by taking $\mathbf{p}\times I \subset Y \times I$ and appending to it ℓ disjoint simple closed curves $\gamma_1,\dots,\gamma_\ell$, which each intersect $\mathbf{p}\times I$ only at a single point. Then*

$$\widehat{F}_{Y\times I}(x \otimes ([\gamma_1]\wedge\cdots\wedge[\gamma_\ell])) = \widehat{F}_{Y\times I,\Gamma}(x),$$

where the left-hand side is the Ozsváth–Szabó cobordism map.

Proof. This is implicit in the work of Zemke [Zem15], but we give the proof for completeness. By a homotopy, and hence isotopy, in $Y\times I$, we may arrange that $\gamma_i \subset Y \times \{i/(\ell+1)\}$. Therefore, using Theorem 3.5 (2), we can write $\widehat{F}_{Y\times I,\Gamma}$ as a composition of the maps $\mathcal{F}(\gamma_i)$. Viewing $\mathcal{F}$ as a function from $H_1(Y)$ to $\mathrm{End}_{\mathbb{Z}/2}(\widehat{\mathrm{HF}}(Y))$, Theorem 3.5 (7) implies that this descends to the exterior algebra. □

We move on to more general cobordisms.

Lemma 3.7. *Suppose that Y_1 and Y_2 are connected, and that $\mathbf{p}_1$ and $\mathbf{p}_2$ each consist of a single point. Let $W\colon Y_1 \to Y_2$ be a connected cobordism. Suppose that Γ is a graph obtained by taking* *a path α from $\mathbf{p}_1$ to $\mathbf{p}_2$ and appending to it ℓ disjoint simple closed loops $\gamma_1, \ldots, \gamma_\ell$, which each intersect α only at a single point. Then*

$$\widehat{F}_W(x \otimes ([\gamma_1] \wedge \cdots \wedge [\gamma_\ell])) = \widehat{F}_{W,\Gamma}(x),$$

where the left-hand side is the Ozsváth–Szabó cobordism map.

Proof. This follows the proof of [Zem15, Theorem C]. We may decompose (W, Γ) as a composition of three cobordisms: (W_1, Γ_1), where W_1 consists only of 1-handles and Γ_1 is a path; $(\partial W_1 \times I, \Gamma_*)$, where Γ_* consists of a graph in $\partial W_1 \times I$ as in the statement of Lemma 3.6; and (W_2, Γ_2), where W_2 consists of 2- and 3-handles, and Γ_2 is again a path. The result now follows from Lemma 3.6 together with Theorem 3.5 (2). □

With this generalization, we may now complete the proof of Proposition 3.2.

Proof of Proposition 3.2. Let $\nu(\gamma_i) \cong \gamma_i \times D^3$ be a neighborhood of γ_i, and let $P = X \setminus (\coprod_i \nu(\gamma_i))$. Let $X' = X \setminus (B_1^4 \sqcup \cdots \sqcup B_\ell^4)$, where $B_i^4 \subset \mathrm{Int}(\nu(\gamma_i))$. We construct a properly embedded graph $\Gamma_{X'}$ in X' as follows; see Figure 2.

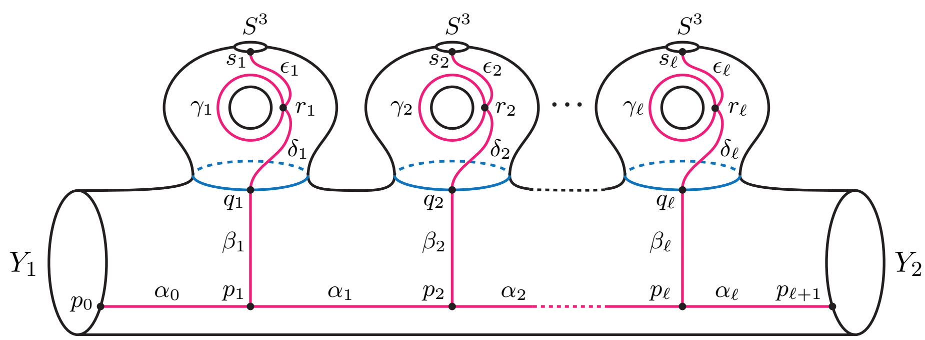


FIGURE 2. The embedded graph $\Gamma_{X'}$ in X'.

We begin with the vertex set. Choose ℓ points $p_1, \ldots, p_\ell$ in the interior of P, and points p_0 and $p_{\ell+1}$ in Y_1 and Y_2 respectively. Choose ℓ points $q_1, \ldots q_\ell$ with $q_i \in \partial\nu(\gamma_i)$, which are copies of $S^1 \times S^2$. Choose ℓ points $r_1, \ldots, r_\ell$ with $r_i \in \gamma_i$. Finally, let s_i be a point in $S_i^3 = \partial B_i^4$ for each i.

Now we define the edge sets. Choose any collection of embedded arcs $\alpha_0, \ldots, \alpha_\ell$ with $\alpha_i \subset P$ connecting p_i and p_{i+1}. Let $\beta_i \subset P$ be an arc from p_i to q_i. Connect q_i and r_i by arcs δ_i, and r_i and s_i by arcs ϵ_i, in $\nu(\gamma_i) \setminus B_i^4$. We may choose the edges above in such a way that their interiors are mutually disjoint, avoid the γ_i, and are contained in the interior of X'. Then, the edge set of $\Gamma_{X'}$ consists of the edges α_i, β_i, γ_i, δ_i, and ϵ_i. In accordance with Theorem 3.5 (1), we view the cobordism map for $(X', \Gamma_{X'})$ as a map

$$\widehat{F}_{X',\Gamma_{X'}} \colon \widehat{\mathrm{HF}}(Y_1) \otimes \left(\bigotimes_{i=1}^{\ell} \widehat{\mathrm{HF}}(S_i^3) \right) \to \widehat{\mathrm{HF}}(Y_2).$$

It follows from Lemma 3.7 as well as Theorem 3.5 (1) and (4) that

$$\widehat{F}_X(x \otimes ([\gamma_1] \wedge \cdots \wedge [\gamma_\ell])) = \widehat{F}_{X',\Gamma_{X'}}(x \otimes y_1 \otimes \cdots \otimes y_\ell),$$

where y_i is the generator of $\widehat{\mathrm{HF}}(S_i^3)$. (We can first contract the homology elements, and then contract the arcs $\beta_i \cup \delta_i \cup \epsilon_i$.) Let Γ_P be the intersection of $\Gamma_{X'}$ with P, which can alternatively be obtained by excising the γ_i, δ_i, and ϵ_i arcs.

Note that $Z = P \cup (\coprod_i (D^2 \times S^2)_i)$. Here, we suppress the choice of gluing from the notation. Similarly, we let $Z' = Z \setminus (B^4_1 \sqcup \cdots \sqcup B^4_\ell)$ where $B^4_i \subset (D^2 \times S^2)_i$; then $Z' = P \cup (\coprod_i R_i)$, where each R_i is a punctured $D^2 \times S^2$. Let ζ_i be an arc in R_i that connects q_i and s_i; then we define Γ_{R_i} in R_i to be ζ_i, and define $\Gamma_{Z'}$ in Z' as the union of the arcs $\alpha_i, \beta_i, \zeta_i$. See Figure 3 for an illustration of $(Z', \Gamma_{Z'})$.

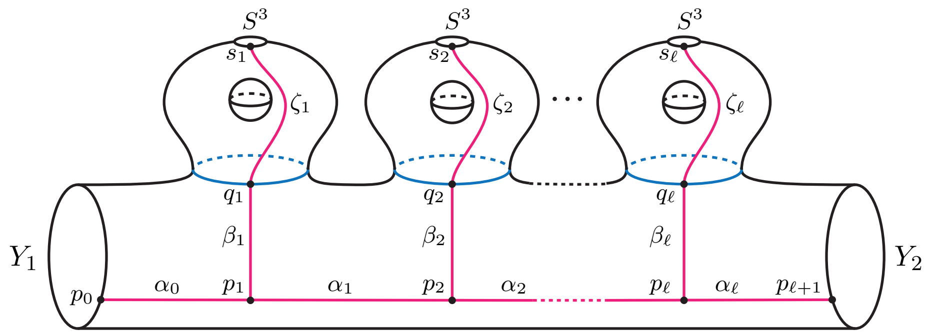


FIGURE 3. The embedded graph $\Gamma_{Z'}$ in Z'.

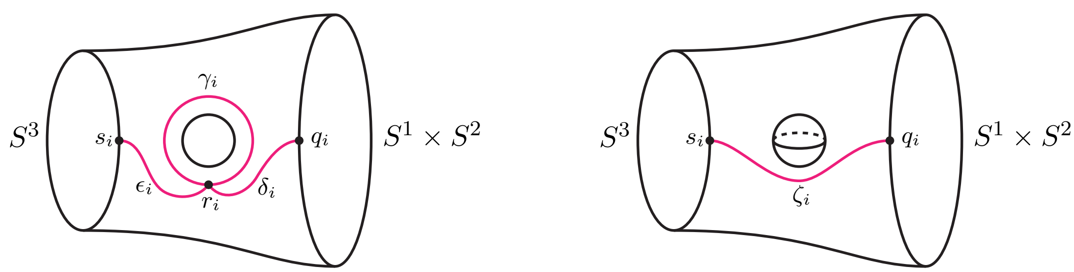
Viewing the cobordism map for $(Z', \Gamma_{Z'})$ as a map $\widehat{F}_{Z',\Gamma_{Z'}} : \widehat{\mathrm{HF}}(Y_1) \otimes (\bigotimes_i \widehat{\mathrm{HF}}(S^3_i)) \to \widehat{\mathrm{HF}}(Y_2)$, we have

$$\widehat{F}_Z(x) = \widehat{F}_{Z',\Gamma_{Z'}}(x \otimes y_1 \otimes \cdots \otimes y_\ell),$$

again by Theorem 3.5 (4). Thus, (3.3) will follow if we can show

$$\widehat{F}_{X',\Gamma_{X'}}(x \otimes y_1 \otimes \cdots \otimes y_\ell) = \widehat{F}_{Z',\Gamma_{Z'}}(x \otimes y_1 \otimes \cdots \otimes y_\ell).$$

To do so, let $Q_i = \nu(\gamma_i) \setminus B^4_i$, and let Γ_{Q_i} be the intersection of $\Gamma_{X'}$ with Q_i. Both (Q_i, Γ_{Q_i}) and (R_i, Γ_{R_i}) are cobordisms from (S^3, s_i) to $(S^1 \times S^2, q_i)$; see Figure 4.



(A) The cobordism (Q_i, Γ_{Q_i}).

(B) The cobordism (R_i, Γ_{R_i}).

FIGURE 4. The cobordisms (Q_i, Γ_{Q_i}) and (R_i, Γ_{R_i}).

Viewing (P, Γ_P) as a cobordism from $(Y_1, p_0) \sqcup (\coprod_i (S^1 \times S^2)_i, q_i) \to (Y_2, p_{\ell+1})$, by Theorem 3.5 (1) and (2), we have that

$$\widehat{F}_{X',\Gamma_{X'}} = \widehat{F}_{P,\Gamma_P} \circ \left(\mathbb{I}_{\widehat{\mathrm{HF}}(Y_1)} \otimes \widehat{F}_{Q_1,\Gamma_{Q_1}} \otimes \cdots \otimes \widehat{F}_{Q_\ell,\Gamma_{Q_\ell}} \right)$$

and

$$\widehat{F}_{Z',\Gamma_{Z'}} = \widehat{F}_{P,\Gamma_P} \circ \left(\mathbb{I}_{\widehat{\mathrm{HF}}(Y_1)} \otimes \widehat{F}_{R_1,\Gamma_{R_1}} \otimes \cdots \otimes \widehat{F}_{R_\ell,\Gamma_{R_\ell}} \right).$$

Thus, we need only to show that $\widehat{F}_{Q_i,\Gamma_{Q_i}} = \widehat{F}_{R_i,\Gamma_{R_i}}$ for each i. On the one hand, Theorem 3.5 (6) together with (3.1) imply that

$$\widehat{F}_{Q_i,\Gamma_{Q_i}}(y_i) = \widehat{F}_{Q_i}(y_i \otimes [\gamma_i]) = [\gamma_i] \cdot \widehat{F}_{Q_i}(y_i).$$

Since Q_i is simply a 1-handle attachment to S^3, its cobordism map, by Ozsváth and Szabó's definition, sends y_i to the topmost generator of $\widehat{\mathrm{HF}}(S^1 \times S^2)$, and the action by $[\gamma_i]$ sends this to the bottommost generator. On the other hand, Theorem 3.5 (5) implies that

$$\widehat{F}_{R_i,\Gamma_{R_i}}(y_i) = \widehat{F}_{R_i}(y_i).$$

Since R_i is simply a 0-framed 2-handle attachment along the unknot in S^3, its cobordism map sends y_i to the bottommost generator of $\widehat{\mathrm{HF}}(S^1 \times S^2)$. Consequently, $\widehat{F}_{Q_i,\Gamma_{Q_i}}(y_i) = \widehat{F}_{R_i,\Gamma_{R_i}}(y_i)$ as desired. □

Proof of Theorem 1.4. We consider the hat flavor first. Consider the double $D(W)$ of W. Then, by Proposition 2.1, $D(W)$ is described by surgery on $X \cong (Y_- \times I) \,\sharp\, m(S^1 \times S^3)$ along m circles $\gamma_1, \dots, \gamma_m$, where $[\gamma_1] \wedge \dots \wedge [\gamma_m] = \alpha_1 \wedge \dots \wedge \alpha_m \in (\Lambda^*(H_1(X)/\operatorname{Tors})/\langle H_1(Y_-)/\operatorname{Tors}\rangle) \otimes_{\mathbb{Z}} \mathbb{Z}/2$, and α_i is homologous to the core of the i^{th} $S^1 \times S^3$ summand. Note that the same description is true of $Y_- \times I$; in this case, the surgery is performed along the core circles γ_i' of the $(S^1 \times S^3)$'s themselves.

Applying Proposition 3.2 with $Z = D(W)$, we have that

$$\widehat{F}_X(x \otimes ([\gamma_1] \wedge \dots \wedge [\gamma_m])) = \widehat{F}_{D(W)}(x) = \widehat{F}_{-W} \circ \widehat{F}_W(x).$$

Now consider $Y_- \times I$ as surgery on X along the cores γ_i'. Applying Proposition 3.2 again, this time with $Z = Y_- \times I$, we have

$$\widehat{F}_X(x \otimes ([\gamma_1'] \wedge \dots \wedge [\gamma_m'])) = \widehat{F}_{Y_- \times I}(x) = \mathbb{I}_{\widehat{\mathrm{HF}}(Y_-)}.$$

Since $[\gamma_1] \wedge \dots \wedge [\gamma_m] = [\gamma_1'] \wedge \dots \wedge [\gamma_m']$ in $\Lambda^*(H_1(X)/\operatorname{Tors})/\langle H_1(Y_-)/\operatorname{Tors}\rangle \otimes_{\mathbb{Z}} \mathbb{Z}/2$, by the linearity of $\widehat{F}$, it suffices to show that $\widehat{F}_X(x \otimes \xi) = 0$ for $x \in \widehat{\mathrm{HF}}(Y_-)$ and $\xi \in \Lambda^m(H_1(X)/\operatorname{Tors}) \cap \langle H_1(Y_-)/\operatorname{Tors}\rangle \otimes_{\mathbb{Z}} \mathbb{Z}/2$. Indeed, this will imply that $\widehat{F}_{-W} \circ \widehat{F}_W = \mathbb{I}_{\widehat{\mathrm{HF}}(Y_-)}$, and we have the desired result for $\widehat{\mathrm{HF}}$.

Note that $\Lambda^m(H_1(X)/\operatorname{Tors}) \cap \langle H_1(Y_-)/\operatorname{Tors}\rangle$ is generated by elements of the form $\omega \wedge (\bigwedge_{i \in I} \alpha_i)$, where ω is a wedge of elements in $H_1(Y_-)/\operatorname{Tors}$ and $I \subsetneq \{1, \dots, m\}$; we would like to show that if $\xi \in \langle H_1(Y_-)/\operatorname{Tors}\rangle \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is of this form, then $\widehat{F}(x \otimes \xi) = 0$ for $x \in \widehat{\mathrm{HF}}(Y_-)$. Therefore, let $\xi = \omega \wedge (\bigwedge_{i \in I} \alpha_i)$ be of this form. The idea is that ξ misses at least one $S^1 \times S^3$ summand, and the cobordism map associated to a twice punctured $S^1 \times S^3$, without an H_1-action, is identically zero. Concretely, choose $j \in \{1, \dots, m\} \setminus I$, and write $X = T_j \cup_{S^3} V$, where T_j is the j^{th} $S^1 \times S^3$ summand punctured once, and $V = ((Y_- \times I) \,\sharp\, (m-1)(S^1 \times S^3)) \setminus B^4$; then ξ determines a graph Γ_ξ in X such that $\widehat{F}_X(x \otimes \xi) = \widehat{F}_{X,\Gamma_\xi}(x)$, and we may assume that $\Gamma_\xi \cap T_j = \emptyset$. Let $X' = X \setminus B^4_j$, where $B^4_j \subset \operatorname{Int}(T_j)$. As in Theorem 3.5 (4), choose an arc λ from ∂B^4_j to Γ_ξ that intersects $\partial T_j = \partial V$ once, and let $\Gamma_\xi' = \Gamma_\xi \cup \lambda$; then we have

$$\widehat{F}_{X,\Gamma_\xi}(x) = \widehat{F}_{X',\Gamma_\xi'}(x \otimes y_j),$$

where y_j is the generator of $\widehat{\mathrm{HF}}(\partial B^4_j)$. Writing $T_j' = T_j \setminus B^4_j$, it is also clear that

$$\widehat{F}_{X',\Gamma_\xi'} = \widehat{F}_{V,\Gamma_\xi' \cap V} \circ \left(\mathbb{I}_{\widehat{\mathrm{HF}}(Y_-)} \otimes \widehat{F}_{T_j',\lambda \cap T_j'}\right).$$

Since $\lambda \cap T'_j$ is simply a path, $\widehat{F}_{T'_j, \lambda\cap T'_j} = \widehat{F}_{T'_j}$. Note that $T'_j \cong (S^3 \times I) \,\sharp\, (S^1 \times S^3)$ is obtained by adding a 1-handle and a 3-handle to $S^3 \times I$, and so a direct computation shows that $\widehat{F}_{T'_j}(y_j) = 0$; thus, $\widehat{F}_X(x \otimes \xi) = 0$, as desired.

To obtain the analogous result for the other flavors of Heegaard Floer homology, we use that the long exact sequences relating the various flavors are natural with respect to cobordism maps. It is straightforward to see that only an isomorphism on HF^+ can induce the identity map on $\widehat{\mathrm{HF}}$, and similarly for HF^-. Finally, only an isomorphism on HF^∞ can induce an isomorphism on both HF^+ and HF^-. ☐

3.2. A Spin^c-refinement of Theorem 1.4.

We now provide a Spin^c-refinement of Theorem 1.4. First, observe that any Spin^c-structure $\mathfrak{t}$ on a cobordism $W \colon Y_1 \to Y_2$ can be extended to a Spin^c-structure $D(\mathfrak{t})$ on $D(W)$, since $\mathfrak{t}$ on W and $\mathfrak{t}$ on $-W$ coincide on the intersection $W \cap -W = Y_2$. We now have the following observation when W is a ribbon $\mathbb{Q}$-homology cobordism.

Lemma 3.8. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W \colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. If a Spin^c-structure $\mathfrak{s}_+$ on Y_+ can be extended to a Spin^c-structure $\mathfrak{t}$ on W, then the extension is unique; moreover, in this case, $D(\mathfrak{t})$ is the unique Spin^c-structure on $D(W)$ that restricts to $\mathfrak{s}_+$ on Y_+.*

Proof. For the first statement, consider

$$H^2(W, Y_+) \to H^2(W) \to H^2(Y_+)$$

from the long exact sequence of the pair (W, Y_+). By the Poincaré Duality, $H^2(W, Y_+) \cong H_2(W, Y_-)$. Take a ribbon handle decomposition of W; since W is a $\mathbb{Q}$-homology cobordism, the numbers m of 1- and 2- handles are the same, and the differential $\partial_2 \colon C_2(W, Y_-) \to C_1(W, Y_-)$ in the cellular chain complex is given by a homomorphism $R \colon \mathbb{Z}^m \to \mathbb{Z}^m$ such that $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism. This means that R, and hence ∂_2, are injective, and so $H_2(W, Y_-) = 0$. Thus, the map $H^2(W) \to H^2(Y_+)$ induced by inclusion is injective, proving that any extension $\mathfrak{t}$ of $\mathfrak{s}_+$ is unique.

For the second statement, consider

$$H^1(W) \oplus H^1(-W) \to H^1(Y_+) \to H^2(D(W)) \to H^2(W) \oplus H^2(-W)$$

from the Mayer–Vietoris exact sequence; we wish to prove the first map is surjective. In fact, we will prove that the map $H^1(W) \to H^1(Y_+)$ is an isomorphism. To do so, consider the map $H_1(Y_+) \to H_1(W)$. Since W is a $\mathbb{Q}$-homology cobordism, we have $\mathrm{rk}_{\mathbb{Z}} H_1(Y_+) = \mathrm{rk}_{\mathbb{Z}} H_1(W)$, and we denote this number by k; then the map in question is given by some homomorphism $R' \colon \mathbb{Z}^k \oplus T_1 \to \mathbb{Z}^k \oplus T_2$, where T_1 and T_2 are torsion, with matrix

$$R' = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

Viewing W upside down, it is built from Y_+ by adding 2- and 3-handles, which implies that R' is surjective; in particular, $A \colon \mathbb{Z}^k \to \mathbb{Z}^k$ is also surjective, and thus an isomorphism. By the Universal Coefficient Theorem, the map $H^1(W) \to H^1(Y_+)$ is exactly given by the transpose $A^{\mathrm{T}} \colon \mathbb{Z}^k \to \mathbb{Z}^k$, which is also an isomorphism. Returning to the exact sequence, we see that the third map is injective, showing that the extension from $\mathfrak{t}$ to $D(\mathfrak{t})$ is unique. ☐

We are now ready to state the following refinement of Theorem 1.4.

Theorem 3.9. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Z}/2$-homology cobordism. Fix a Spin^c-structure $\mathfrak{s}$ on Y_-. Then the sum of cobordism maps*

$$\left(\sum_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(W)\\ \mathfrak{t}|_{Y_-}=\mathfrak{s}}} F^\circ_{W,\mathfrak{t}}\right)\colon \mathrm{HF}^\circ(Y_-,\mathfrak{s}) \to \bigoplus_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(W)\\ \mathfrak{t}|_{Y_-}=\mathfrak{s}}} \mathrm{HF}^\circ(Y_+,\mathfrak{t}|_{Y_+})$$

includes $\mathrm{HF}^\circ(Y_-,\mathfrak{s})$ into the codomain as a summand. In fact,

$$\left(\sum_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(W)\\ \mathfrak{t}|_{Y_-}=\mathfrak{s}}} \widehat{F}_{D(W),D(\mathfrak{t})}\right)\colon \widehat{\mathrm{HF}}(Y_-,\mathfrak{s}) \to \widehat{\mathrm{HF}}(Y_-,\mathfrak{s})$$

is the identity map.

Proof. The assertion that the first map is injective is a direct consequence of Theorem 1.4, since it is simply the restriction of F°_W to the summand $\mathrm{HF}^\circ(Y_-,\mathfrak{s})$ of $\mathrm{HF}^\circ(Y_-)$. (However, in writing the codomain as the direct sum above, we have implicitly used the fact that for distinct $\mathfrak{t}_1,\mathfrak{t}_2 \in \mathrm{Spin}^c(W)$, their restrictions $\mathfrak{t}_1|_{Y_+},\mathfrak{t}_2|_{Y_+} \in \mathrm{Spin}^c(Y_+)$ are distinct, which is a consequence of Lemma 3.8.) The second assertion is obtained by restricting the identity map $\widehat{F}_{D(W)}$ in Theorem 1.4 to the summand $\widehat{\mathrm{HF}}(Y_-,\mathfrak{s})$, and observing that all Spin^c-structures on $D(W)$ are of the form $D(\mathfrak{t})$, which follows from Lemma 3.8. □

With the additional condition that W is a $\mathbb{Z}$-homology cobordism, a Spin^c structure $\mathfrak{s}_-$ on Y_- determines a unique $\mathfrak{t}$ on W, and hence a unique $\mathfrak{s}_+$ on Y_+. We have:

Corollary 3.10. *Let Y_- and Y_+ be 3-manifolds, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Z}$-homology cobordism. Fix a Spin^c-structure $\mathfrak{s}_-$ on Y_-, and let $\mathfrak{t}$ and $\mathfrak{s}_+$ be the corresponding Spin^c structures on W and Y_+ respectively. Then the cobordism map $F^\circ_{W,\mathfrak{t}}\colon \mathrm{HF}^\circ(Y_-,\mathfrak{s}_-) \to \mathrm{HF}^\circ(Y_+,\mathfrak{s}_+)$ includes $\mathrm{HF}^\circ(Y_-,\mathfrak{s}_-)$ into $\mathrm{HF}^\circ(Y_+,\mathfrak{s}_+)$ as a summand.* □

3.3. **Sutured Heegaard Floer theory.**

We first define what we mean by a cobordism of sutured manifolds. Note that this definition is narrower than the one used by Juhász [Juh16].

Definition 3.11. Let (M_1,η_1) and (M_2,η_2) be sutured manifolds. A *cobordism* $W\colon (M_1,\eta_1) \to (M_2,\eta_2)$ is a 4-manifold W obtained by a sequence of interior handle attachments on $M_1 \times I$. In particular, there is a natural diffeomorphism of ∂M_1 and ∂M_2 that identifies η_1 with η_2.

We now use Theorem 3.9 to prove the sutured analogue.

Proof of Theorem 1.6. Recall from Lekili's work [Lek13, Theorem 24] that the sutured Floer homology of a sutured manifold (M,η) can be described in terms of the Heegaard Floer homology of the sutured closure $\widehat{M} = M \cup (F_{g,d} \times [-1,1])$ and a closed surface R in $\widehat{M}$ obtained from $F_{g,d}$, where $F_{g,d}$ is a surface of genus $g \geq 2$ and d boundary components. For more details on the construction of $\widehat{M}$ and R, see Section 4.5. Then we have the isomorphism

$$\mathrm{SFH}(M,\eta) \cong \bigoplus_{\langle c_1(\mathfrak{s}),R\rangle = 2g-2} \mathrm{HF}^+(\widehat{M},\mathfrak{s}).$$

Now, given a ribbon $\mathbb{Z}/2$-homology cobordism $N\colon (M_-,\eta_-) \to (M_+,\eta_+)$ between sutured manifolds, we can attach $F_{g,d} \times [-1,1] \times I$ to obtain a ribbon $\mathbb{Z}/2$-homology cobordism $\widehat{N}$ between the sutured closures. Furthermore, for any Spin^c-structure $\mathfrak{t}$ on $\widehat{N}$,

$$\left\langle c_1\left(\mathfrak{t}|_{\widehat{M}_-}\right), \left[R_{\widehat{M}_-}\right]\right\rangle = \left\langle c_1\left(\mathfrak{t}|_{\widehat{M}_+}\right), \left[R_{\widehat{M}_+}\right]\right\rangle.$$

Consequently, the desired result follows from Theorem 3.9. ☐

3.4. **Involutive Heegaard Floer theory.** We now extend our work in Section 3.1 to prove Theorem 1.9. Recall that $\widehat{\mathrm{HFI}}(Y)$ is defined as the homology of the mapping cone of $1+\iota$, where ι is a chain homotopy equivalence on $\widehat{\mathrm{CF}}(Y)$ coming from Spin^c-conjugation. Since we are working over $\mathbb{Z}/2$, we have that $\widehat{\mathrm{HFI}}(Y)$ is in fact isomorphic to the homology of the mapping cone of $1+\iota_*\colon \widehat{\mathrm{HF}}(Y) \to \widehat{\mathrm{HF}}(Y)$. Unfortunately, the theory of cobordism maps is not fully developed in the theory, but we can still compare the involutive Heegaard Floer homologies under ribbon homology cobordisms.

Proof of Theorem 1.9. Fix a self-conjugate Spin^c-structure $\mathfrak{s}_-$ on Y_-, which determines a unique Spin^c-structure $\mathfrak{t}$ on W and a unique $\mathfrak{s}_+$ on Y_+. Then we have the commutative diagram

$$\begin{array}{ccccc}
\widehat{\mathrm{HF}}(Y_-,\mathfrak{s}_-) & \xrightarrow{\widehat{F}_{W,\mathfrak{t}}} & \widehat{\mathrm{HF}}(Y_+,\mathfrak{s}_+) & \xrightarrow{\widehat{F}_{-W,\mathfrak{t}}} & \widehat{\mathrm{HF}}(Y_-,\mathfrak{s}_-) \\
\downarrow{\scriptstyle 1+\iota_*} & & \downarrow{\scriptstyle 1+\iota_*} & & \downarrow{\scriptstyle 1+\iota_*} \\
\widehat{\mathrm{HF}}(Y_-,\mathfrak{s}_-) & \xrightarrow{\widehat{F}_{W,\mathfrak{t}}} & \widehat{\mathrm{HF}}(Y_+,\mathfrak{s}_+) & \xrightarrow{\widehat{F}_{-W,\mathfrak{t}}} & \widehat{\mathrm{HF}}(Y_-,\mathfrak{s}_-).
\end{array}$$

The result now follows from Theorem 3.9. ☐

4. Instanton Floer homology and ribbon homology cobordisms

4.1. **The Chern–Simons functional.** Let G be a compact, connected, simply connected, simple Lie group, and let P be a principal G-bundle on Y. Any such bundle can be trivialized, and we fix one such trivialization. Denote by $\mathrm{ad}(P)$ the adjoint bundle associated to P; this vector bundle is induced by the adjoint action of G on its Lie algebra $\mathfrak{g}$. The space of connections $\mathcal{A}(P)$ on P is an affine space modeled on $\Omega^1(Y;\mathrm{ad}(P))$, with a distinguished element Θ, which is the trivial connection (associated to the trivialization we chose). Given a connection $B \in \mathcal{A}(P)$, let A be the connection on the bundle $P\times\mathbb{R}$ over $Y\times\mathbb{R}$ that is equal to the pull-back of B on $P\times(-\infty,-1]$ and the pull-back of Θ on $P\times[1,\infty)$. The *Chern–Simons functional of* B is defined by the Chern–Weil integral

$$\widetilde{\mathrm{CS}}(B) = -\frac{1}{32\pi^2\check{h}}\int_{Y\times\mathbb{R}} \mathrm{tr}(\mathrm{ad}(F(A))\wedge\mathrm{ad}(F(A))), \tag{4.1}$$

where $F(A)$ is the $\mathrm{ad}(P)$-valued curvature 2-form, and $\mathrm{ad}(F(A))$ is the corresponding induced element of $\mathrm{End}(\mathrm{ad}(P))$. The constant $\check{h}$ is the *dual Coxeter number*, which depends on G; it is equal to N when $G=\mathrm{SU}(N)$.

Let $\mathcal{G}_G$ be the space of smooth maps from Y to G. This space can be identified with the group of automorphisms of P, known as the *gauge group*; in particular, any $g\in\mathcal{G}_G$ acts on $\mathcal{A}(P)$ by mapping a connection A to its pull-back $g^*(A)$. The integral in (4.1) is not necessarily invariant with respect to this $\mathcal{G}_G$-action; however, it always changes by multiples of a fixed constant, and the normalization in (4.1) is chosen such that the change in $\widetilde{\mathrm{CS}}$ is always an integer. In particular, if we denote by $\mathcal{B}(P)$ the quotient of $\mathcal{A}(P)$ by this $\mathcal{G}_G$-action, then (4.1) induces a map $\mathrm{CS}:\mathcal{B}(P)\to\mathbb{R}/\mathbb{Z}$. An important feature of CS is that it is a topological function, in that its definition does not require a metric on Y.

It is not hard to see from the definition that a connection B is a critical point of $\widetilde{\mathrm{CS}}$ if and only if B has vanishing curvature, i.e. if B is flat. Given a flat connection, one may take its holonomy along closed loops in Y to obtain a homomorphism $\rho\colon\pi_1(Y)\to G$, i.e. an element of the representation variety $\mathcal{R}_G(Y)$. This is not necessarily a one-to-one correspondence, but if we quotient the space of flat connections by the gauge group action and quotient $\mathcal{R}_G(Y)$ by conjugation, we do get an identification of the isomorphism classes of flat connections with the character variety $\chi_G(Y)$. In other words, $\chi_G(Y)$ is the set of critical points of CS. Further, the set of critical values of the Chern–Simons functional CS is a finite set, which is a topological invariant of Y.

In the definition of the Chern–Simons functional, the assumptions on the Lie group G are not essential. Indeed, we may take G to be a compact, connected, simple Lie group that is possibly not simply connected, with universal cover $\widetilde{G}$. An important example to keep in mind is when $G = \mathrm{SO}(3)$ and $\widetilde{G} = \mathrm{SU}(2)$. Instead of a trivial principal bundle, we consider a possibly non-trivial principal G-bundle on Y. We may still form the space of connections $\mathcal{A}(P)$ as before, and we may form the configuration space $\mathcal{B}(P)$ by quotienting $\mathcal{A}(P)$ by the $\mathcal{G}_{\widetilde{G}}$-action (rather than the $\mathcal{G}_G$-action). There is no longer a distinguished element $\Theta \in \mathcal{A}(P)$. Instead, we arbitrarily choose a connection $B_0 \in \mathcal{A}(P)$, which plays the role of Θ in the definitions of $\widetilde{\mathrm{CS}}$; this determines an $\mathbb{R}/\mathbb{Z}$-valued functional CS on $\mathcal{B}(P)$ that is well defined up to addition by a constant (representing the indeterminacy of the choice of B_0). The critical points of CS are isomorphism classes of flat connections on P. Moreover, the set of (relative) values of the Chern–Simons functional at this set of critical points is a topological invariant of the pair (Y, P).

4.2. **An overview of instanton Floer theory.** In this section, we review the two main versions of instanton Floer homology and develop some properties of the associated cobordism maps. (Other versions will be discussed later in this section.) Throughout, we work only with coefficients in $\mathbb{Q}$. We begin with Floer's original version of instanton Floer homology [Flo88], which associates to any $\mathbb{Z}$-homology sphere Y a $\mathbb{Z}/8$-graded vector space $\mathrm{I}(Y)$. To a $\mathbb{Q}$-homology cobordism $W\colon Y_1 \to Y_2$ of $\mathbb{Z}$-homology spheres, the theory associates a homomorphism $\mathrm{I}(W)\colon \mathrm{I}(Y_1) \to \mathrm{I}(Y_2)$ of vector spaces [Don02].[6]

The vector space $\mathrm{I}(Y)$ is the homology of a chain complex $(\mathrm{C}(Y), d)$. The chain complex $\mathrm{C}(Y)$ is defined roughly as the Morse homology of the Chern–Simons functional CS with the Lie group $G = \mathrm{SU}(2)$ and the trivial bundle on Y. Recall from Section 4.1 that the critical set of CS is exactly the space of isomorphism classes of flat connections; in this setup, all non-trivial flat connections are irreducible. Here, a connection is *irreducible* if its isotropy group is $\{\pm 1\}$; when the connection is flat, this is equivalent to the condition that the associated representation is irreducible.

In order to achieve Morse–Smale transversality, one perturbs the Chern–Simons functional. The critical set of the perturbed Chern–Simons functional still contains the trivial connection; the other critical points are no longer necessarily flat, but the perturbation can be chosen to be small, which guarantees that the non-trivial critical points are still (isomorphism classes of) irreducible connections.[7] We denote the set of all non-trivial critical points by $\mathfrak{S}(Y)$.[8] Then $\mathrm{C}(Y)$ is the $\mathbb{Q}$-vector space generated by the elements of $\mathfrak{S}(Y)$, equipped with the differential d, where the coefficients $\langle d(\alpha), \beta\rangle$ are given by the signed count of index-1 gradient flow lines of the perturbation of CS that are asymptotic to α and β. A useful observation, which is also essential in the development of the analytical aspects of the theory, is that the gradient flow lines of (a perturbation of) CS may be viewed as the solutions of (a corresponding perturbation of) the ASD (anti–self-dual) equation for the trivial SU(2)-bundle on $Y \times \mathbb{R}$.

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[6]The homomorphism $\mathrm{I}(W)$ is also defined for more general cobordisms W; see [Don02] for details. We focus on $\mathbb{Q}$-homology cobordisms here for ease of exposition, as this specialization suffices for our purposes.

[7]For simplicity, it is customary to blur the line between connections and isomorphism classes of connections (i.e. connections up to the gauge group action). From now on, we will often follow this custom; for example, by an irreducible element of $\mathfrak{S}(Y)$, we will mean an isomorphism class of irreducible connections.

[8]Although it is not reflected in the notation, the set $\mathfrak{S}(Y)$ depends on the choice of perturbation of the Chern–Simons functional.

The cobordism map $\mathrm{I}(W)\colon \mathrm{I}(Y_1) \to \mathrm{I}(Y_2)$ is also defined with the aid of the ASD equation. We first attach cylindrical ends to W and fix a Riemannian metric on this new manifold, which we also denote by W by abuse of notation. For any pair $(\alpha_1, \alpha_2) \in \mathfrak{S}(Y_1) \times \mathfrak{S}(Y_2)$, we may form a moduli space $\mathcal{M}(W; \alpha_1, \alpha_2)$ of connections that satisfy a perturbed ASD equation for the trivial SU(2)-bundle on W and that are asymptotic to α_1 and α_2 on the ends. Here, the perturbation of the ASD equation is chosen such that it is compatible with the perturbations of the Chern–Simons functionals of Y_1 and Y_2, and guarantees that each connected component of $\mathcal{M}(W; \alpha_1, \alpha_2)$ is a smooth manifold, of possibly different dimensions. We write $\mathcal{M}(W; \alpha_1, \alpha_2)_d$ for the union of the d-dimensional connected components of $\mathcal{M}(W; \alpha_1, \alpha_2)$. The value of d mod 8 is determined by α_1 and α_2. We then define a chain map $\mathrm{C}(W)\colon \mathrm{C}(Y_1) \to \mathrm{C}(Y_2)$ by

$$\mathrm{C}(W)(\alpha_1) = \sum_{\alpha_2 \in \mathfrak{S}(Y_2)} \#\mathcal{M}(W; \alpha_1, \alpha_2)_0 \cdot \alpha_2 \in \mathrm{C}(Y_2). \tag{4.2}$$

Here, $\#\mathcal{M}(W; \alpha_1, \alpha_2)_0$ is the signed count of the elements of $\mathcal{M}(W; \alpha_1, \alpha_2)_0$. The homomorphism $\mathrm{I}(W)\colon \mathrm{I}(Y_1) \to \mathrm{I}(Y_2)$ is the map induced by $\mathrm{C}(W)$ at the level of homology. It turns out that this map depends only on W and is independent of the choice of Riemannian metric on W and perturbation of the ASD equation.

A variation of instanton Floer homology is obtained by replacing the trivial SU(2)-bundles with non-trivial SO(3)-bundles. Fix a closed 3-manifold Y. The isomorphism class of an SO(3)-bundle P on Y is determined by its second Stiefel–Whitney class $w = w_2(P) \in H^2(Y; \mathbb{Z}/2)$. As described in Section 4.1, we may define a Chern–Simons functional CS_w on the configuration space $\mathcal{B}(P)$ of connections on P up to gauge group action. We say that (Y, w) is an *admissible pair* if the pairing of w with $H_2(Y)$ is not trivial. This condition guarantees that the set of critical points of CS_w, or equivalently, the set of flat connections on P, consists only of irreducible elements of $\mathcal{B}(P)$. This assumption considerably simplifies the analytical aspects of gauge theory and allows us to define an instanton Floer homology $\mathrm{I}(Y, w)$ for an admissible pair, analogous to instanton Floer homology of a $\mathbb{Z}$-homology sphere. As in the previous case, we apply a small perturbation to CS_w to obtain a Morse–Smale functional with the critical set $\mathfrak{S}(Y, w)$. Again, the critical points of the perturbed functional are no longer necessarily flat, but they remain irreducible. We define $\mathrm{C}(Y, w)$ to be the $\mathbb{Q}$-vector space generated by $\mathfrak{S}(Y, w)$, equipped with a differential d defined using gradient flow lines of the perturbed Chern–Simons functional.

Instanton Floer homology of admissible pairs is also functorial with respect to cobordisms. Let (Y_1, w_1) and (Y_2, w_2) be admissible pairs, let $W\colon Y_1 \to Y_2$ be an arbitrary cobordism (i.e. not necessarily a $\mathbb{Q}$-homology cobordism), and let $c \in H^2(W; \mathbb{Z}/2)$ be a cohomology class whose restriction to Y_i is equal to w_i. The cohomology class c determines an SO(3)-bundle on W, and solutions to a perturbed ASD equation for connections on this bundle that are asymptotic to $\alpha_1 \in \mathfrak{S}(Y_1, w_1)$ and $\alpha_2 \in \mathfrak{S}(Y_2, w_2)$ give rise to the moduli space $\mathcal{M}(W, c; \alpha_1, \alpha_2)$. As in the previous case, the perturbation of the ASD equation is chosen such that it is compatible with the perturbations of the Chern–Simons functionals of (Y_1, w_1) and (Y_2, w_2) and that each component of $\mathcal{M}(W, c; \alpha_1, \alpha_2)$ a smooth manifold. As in (4.2), these moduli spaces can be used to define a homomorphism $\mathrm{I}(W, c)\colon \mathrm{I}(Y_1, w_1) \to \mathrm{I}(Y_2, w_2)$. In general, this map is defined only up to a sign; this sign can be determined if we fix a homology orientation on W, which is an orientation of $\Lambda^{\mathrm{top}} H^1(W; \mathbb{Q}) \otimes \Lambda^{\mathrm{top}} H^+(W; \mathbb{Q}) \otimes \Lambda^{\mathrm{top}} H^1(Y_2; \mathbb{Q})$. Here $H^+(W; \mathbb{Q})$ is the subspace of $H^2(W; \mathbb{Q})$ represented by L^2 self-dual harmonic 2-forms on W. (See, for example, [KM11b] for more details on how to use homology orientations to remove the sign ambiguity of $\mathrm{I}(W, c)$.) In particular, for a $\mathbb{Q}$-homology cobordism W, there is a canonical choice of homology orientation.

There are more general cobordism maps defined for instanton Floer homology of admissible pairs. Let $\mathbb{A}(W)$ be the $\mathbb{Z}$-graded algebra $\mathrm{Sym}^*(H_2(W;\mathbb{Q})\oplus H_0(W;\mathbb{Q}))\otimes\Lambda^*(H_1(W;\mathbb{Q}))$, where the elements in $H_i(W;\mathbb{Q})$ have degree $4-i$. For any $z\in\mathbb{A}(W)$ with degree i, a standard construction gives rise to a cohomology class $\mu(z)$ of degree i in $\mathcal{M}(W,c;\alpha_1,\alpha_2)_d$, represented by a linear combination of submanifolds $V(W,c;\alpha_1,\alpha_2;z)_{d-i}$ of codimension i; see, for example, [DK90, Chapter 5]. Then the homomorphism $\mathrm{C}(W,c;z)\colon \mathrm{C}(Y_1,w_1)\to\mathrm{C}(Y_2,w_2)$ defined by

$$\mathrm{C}(W,c;z)(\alpha_1)=\sum_{\alpha_2\in\mathfrak{S}(Y_2,w_2)}\#V(W,c;\alpha_1,\alpha_2;z)_0\cdot\alpha_2\in C(Y_2,w_2) \tag{4.3}$$

is a chain map, and the induced homomorphism $\mathrm{I}(W,c;z)$ at the level of homology is independent of the choice of metric, perturbation, and the representative submanifold $V(W,c;\alpha_1,\alpha_2;z)_0$. The homomorphism $\mathrm{I}(W,c;z)$ depends linearly on z, and is again defined up to a sign that can be fixed using a homology orientation on W. It is also functorial: Let (Y_1,w_1), (Y_2,w_2), and (Y_3,w_3) be admissible pairs, $W\colon Y_1\to Y_2$ and $W'\colon Y_2\to Y_3$ be cobordisms equipped with homology orientations, and $c_\circ$ be an element of $H^2(W'\circ W;\mathbb{Z}/2)$ whose restrictions to W and W' are denoted by c and c' respectively, and fix $z\in\mathbb{A}(W)$ and $z'\in\mathbb{A}(W')$; then $\mathrm{I}(W'\circ W,c_\circ;z\cdot z')$, defined using the composed homology orientation, is equal to $\mathrm{I}(W',c';z')\circ\mathrm{I}(W,c;z)$.

It is natural to ask whether for a cobordism W between $\mathbb{Z}$-homology spheres, the definition of the cobordism map $\mathrm{I}(W)$ can also be extended to a homomorphism $\mathrm{I}(W;z)$ for $z\in\mathbb{A}(W)$. In this context, it would also be useful to define $\mathrm{I}(W;z)$ when W is not a $\mathbb{Q}$-homology cobordism, e.g. when $b_1(W)>0$; to do so, we would also need to make use of homology orientations to remove the sign ambiguity. In general, the main obstruction to defining this extension is the existence of reducible ASD connections on W: One can still define a subspace $V(W;\alpha_1,\alpha_2;z)_0$ of $\mathcal{M}(W;\alpha_1,\alpha_2)_d$ in the case that $\deg(z)=d$, but $V(W;\alpha_1,\alpha_2;z)_0$ might not be compact because of the existence of reducible connections. Thus one cannot proceed easily, as in (4.3), to define $\mathrm{I}(W;z)$. In the case that $b^+(W)>1$, the cobordism map $\mathrm{I}(W;z)\colon\ \mathrm{I}(Y_1)\to\mathrm{I}(Y_2)$ is defined for any $z\in\mathbb{A}(W)$; see [Don02, Chapter 6]. For our purposes, we need to consider the case where $b^+(W)=0$ and the degree of z is sufficiently small. The following compactness result provides the essential analytical input to define $\mathrm{I}(W;z)$ in this context.

Lemma 4.4. *Let Y_1 and Y_2 be $\mathbb{Z}$-homology spheres, and let $\alpha_1\in\mathfrak{S}(Y_1)$ and $\alpha_2\in\mathfrak{S}(Y_2)$. Suppose that $W\colon Y_1\to Y_2$ is a cobordism with $b_1(W)=m$ and $b^+(W)=0$, and that $\{A_i\}_{i=1}^\infty$ is a sequence of connections on W each representing an element of $\mathcal{M}(W;\alpha_1,\alpha_2)_d$, where $d\le 3m+4$. Then there are $\alpha_1'\in\mathfrak{S}(Y_1)\cup\{\Theta\}$, $\alpha_2'\in\mathfrak{S}(Y_2)\cup\{\Theta\}$, a finite set of points $\{p_1,\dots,p_\ell\}\subset W$, and an irreducible connection A_0 on W representing an element of $\mathcal{M}(W;\alpha_1',\alpha_2')_{d'}$, such that*

1. *$0\le d'\le d-8\ell$; and*
2. *after possibly passing to a subsequence and changing each connection A_i by an action of the gauge group, the sequence of connections $\{A_i\}$ converges in $\mathcal{C}^\infty$-norm to A_0 on any compact subspace of the complement of $\{p_1,\dots,p_\ell\}$.*

Proof. This is a consequence of the standard compactness theorem for the solutions of the ASD equation on manifolds with cylindrical ends (see, for example, [Don02, Chapter 5]), together with the following observation. If the chosen perturbations of the Chern–Simons functionals of Y_1 and Y_2 and of the ASD equation on W are small enough, then any reducible ASD connection on W is a (singular) element of a moduli space of the form $\mathcal{M}(W;\Theta,\Theta)_e$, where Θ is the trivial connection, and $e\ge 3m-3$. A straightforward index computation shows that such reducible connections do not appear as limits of a sequence in $\mathcal{M}(W;\alpha_1,\alpha_2)_d$ when $d\le 3m+4$. ∎

Suppose that W is a cobordism as in the statement of Lemma 4.4. We equip W with a homology orientation by fixing an orientation for the vector space $H^1(W;\mathbb{Q})$. Suppose also that $z \in \mathbb{A}(W)$ has degree at most $3m+3$. Lemma 4.4 together with a standard counting argument shows that the moduli space $V(W;\alpha_1,\alpha_2;z)_0$ is compact. Thus we may use a formula similar to (4.3) to define the cobordism map $\mathrm{I}(W;z)\colon \mathrm{I}(Y_1) \to \mathrm{I}(Y_2)$. A standard argument shows that this map is independent of the choice of metric, perturbation, and representative submanifold for the cohomology class associated to z.

4.3. **Surgery and cobordism maps in instanton Floer theory.** We first start with two basic propositions, which are the counterparts of Proposition 3.2 for instanton Floer homology. In both propositions, we will relate certain cobordism maps associated to two cobordisms X and Z, where Z is the result of surgery on X along a loop γ. Note that $H^+(X;\mathbb{Q}) \cong H^+(Z;\mathbb{Q})$, and $H_1(X;\mathbb{Q}) \cong H_1(Z;\mathbb{Q})$ or $H_1(X;\mathbb{Q}) \cong \langle[\gamma]\rangle \oplus H_1(Z;\mathbb{Q})$ depending on whether γ is rationally null-homologous or not. Fix a homology orientation $\mathfrak{o}_Z$ on Z; we may obtain a homology orientation $\mathfrak{o}_X$ on X as follows. In the former case, we set $\mathfrak{o}_X = \mathfrak{o}_Z$; in the latter case, we set $\mathfrak{o}_X = \omega \wedge \mathfrak{o}$, where $\omega \in H^1(X;\mathbb{Q})$ is determined by $[\gamma]$. In the statements of the propositions, the cobordism maps for X and Z are computed using $\mathfrak{o}_X$ and $\mathfrak{o}_Z$ respectively, for any fixed $\mathfrak{o}_Z$.

First, we have a surgery formula for instanton Floer homology of admissible pairs.

Proposition 4.5. *Let (Y_1,w_1) and (Y_2,w_2) be admissible pairs, and let $X\colon Y_1 \to Y_2$ be a cobordism. Suppose that $\gamma \subset \mathrm{Int}(X)$ is a loop with neighborhood $\nu(\gamma) \cong \gamma \times D^3$, and denote by Z the result of surgery on X along γ. Fix a properly embedded surface $S \subset \mathrm{Int}(X)$ supported away from $\nu(\gamma)$, such that the cohomology class $c_X \in H^2(X;\mathbb{Z}/2)$ dual to $[S]$ restricts to w_1 and w_2 on Y_1 and Y_2 respectively, and denote by c_Z the class in $H^2(Z;\mathbb{Z}/2)$ determined by $[S]$. Suppose that $z_X \in \mathbb{A}(X)$ admits representatives for its homology classes that are supported away from $\nu(\gamma)$, and denote by z_Z the class in $\mathbb{A}(Z)$ determined by these representatives. Then*

$$\mathrm{I}(X, c_X; [\gamma]\cdot z_X) = \mathrm{I}(Z, c_Z; z_Z).$$

Proof. This is essentially [Don02, Theorem 7.16], and the same proof works in this set up. Note that both sides of the equation vanish when $[\gamma] = 0$. ☐

Similarly, we have a surgery formula for instanton Floer homology of $\mathbb{Z}$-homology spheres.

Proposition 4.6. *Let Y_1 and Y_2 be $\mathbb{Z}$-homology spheres, and suppose that $X\colon Y_1 \to Y_2$ is a cobordism with $b_1(X) = m$ and $b^+(X) = 0$. Suppose that $\gamma \in \mathrm{Int}(X)$ is a loop with neighborhood $\nu(\gamma) \cong \gamma \times D^3$, and denote by Z the result of surgery on X along γ. Suppose that $z_X \in \mathbb{A}(X)$ has degree at most $3m-3$ and admits representatives for its homology classes that are supported away from $\nu(\gamma)$, and denote by z_Z the class in $\mathbb{A}(Z)$ determined by these representatives. Then*

$$\mathrm{I}(X; [\gamma]\cdot z_X) = \mathrm{I}(Z; z_Z).$$

Proof. This is again essentially [Don02, Theorem 7.16]. ☐

We now use the propositions above to study ribbon homology cobordisms. First, we verify an analogue of Theorem 1.11 for admissible pairs, which we will use in the following subsections.

Theorem 4.7. *Let (Y_-,w_-) and (Y_+,w_+) be admissible pairs, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Fix a properly embedded surface $S \subset \mathrm{Int}(W)$ supported away from the cocores in a ribbon handle decomposition of W, such that the cohomology class $c_W \in H^2(W;\mathbb{Z}/2)$ dual to $[S]$ restricts to w_- and w_+ on Y_- and Y_+ respectively, and denote by $c_{D(W)} \in H^2(D(W);\mathbb{Z}/2)$*

and $c_{Y_-\times I} \in H^2(Y_- \times I; \mathbb{Z}/2)$ *the cohomology classes determined by* $D(S)$*. Then the cobordism map* $\mathrm{I}(D(W), c_{D(W)})\colon \mathrm{I}(Y_-, w_-) \to \mathrm{I}(Y_-, w_-)$ *satisfies*

$$\mathrm{I}(D(W), c_{D(W)}) = |H_1(W, Y_-)| \cdot \mathrm{I}(Y_- \times I, c_{Y_-\times I}).$$

In particular, if $c_{Y_-\times I}$ *is the pull-back of* w_-*, then*

$$\mathrm{I}(D(W), c_{D(W)}) = |H_1(W, Y_-)| \cdot \mathbb{I}_{\mathrm{I}(Y_-, w_-)},$$

and $\mathrm{I}(W, c_W)$ *includes* $\mathrm{I}(Y_-, w_-)$ *into* $\mathrm{I}(Y_+, w_+)$ *as a summand.*

Proof. By Proposition 2.1, $D(W)$ is described by surgery on $X \cong (Y_- \times I) \,\sharp\, m(S^1 \times S^3)$ along m disjoint loops $\gamma_1, \dots, \gamma_m$. Moreover, if $\alpha_i \in H_1(X)$ is homologous to the core of the i^{th} $S^1 \times S^3$ summand, then we may write

$$[\gamma_i] = \sigma_i + \sum_{j=1}^{m} a_i^j \alpha_j,$$

where $\sigma_i \in H_1(Y_-)$ and $|\det(a_i^j)| = |H_1(W, Y_-)|$. Applying Proposition 4.5 with $Z = D(W)$ and some fixed homology orientation $\mathfrak{o}_{D(W)}$ on $D(W)$, we have that

$$\mathrm{I}(X, c_X; [\gamma_1] \wedge \cdots \wedge [\gamma_m]) = \mathrm{I}(D(W), c_{D(W)}),$$

where the left-hand side is computed using the homology orientation $\mathfrak{o}_{X,\gamma} = [\gamma_1] \wedge \cdots \wedge [\gamma_m] \wedge \mathfrak{o}_{D(W)}$. We claim that

$$\mathrm{I}(X, c_X; [\gamma_1] \wedge \cdots \wedge [\gamma_m]) = \det(a_i^j) \cdot \mathrm{I}(X, c_X; \alpha_1 \wedge \cdots \wedge \alpha_m), \tag{4.8}$$

where we are using $\mathfrak{o}_{X,\gamma}$ on both sides of the equation; indeed, by the linearity of I, it suffices to show that $\mathrm{I}(X, c_X; \xi) = 0$ for $\xi \in \Lambda^m(H_1(X; \mathbb{Q})) \cap \langle H_1(Y_-; \mathbb{Q}) \rangle$. To see this, we may apply Proposition 4.6 in the opposite direction to see that $\mathrm{I}(X, c_X; \xi) = \mathrm{I}(Z', c_{Z'})$ for some cobordism Z' with at least one $S^1 \times S^3$ connected summand; the general vanishing theorem for connected sums implies that this map is zero. (The interested reader may compare this argument with the penultimate paragraph of the proof of Theorem 1.4 in Section 3.1.) If we instead compute $\mathrm{I}(X, c_X; \alpha_1 \wedge \cdots \wedge \alpha_m)$ using the homology orientation $\mathfrak{o}_{X,\alpha} = \alpha_1 \wedge \cdots \wedge \alpha_m \wedge \mathfrak{o}_{D(W)}$, then (4.8) needs to be corrected by the sign of $\det(a_i^j)$:

$$\begin{aligned}\mathrm{I}(X, c_X; [\gamma_1] \wedge \cdots \wedge [\gamma_m]) &= |\det(a_i^j)| \cdot \mathrm{I}(X, c_X; \alpha_1 \wedge \cdots \wedge \alpha_m)\\ &= |H_1(W, Y_-)| \cdot \mathrm{I}(X, c_X; \alpha_1 \wedge \cdots \wedge \alpha_m).\end{aligned}$$

Applying Proposition 4.5 again with $Z = Y_- \times I$ and the canonical homology orientation on $Y_- \times I$, we see that

$$\mathrm{I}(X, c_X; \alpha_1 \wedge \cdots \wedge \alpha_m) = \mathrm{I}(Y_- \times I, c_{Y_-\times I}),$$

where the left-hand side is computed using $\mathfrak{o}_{X,\alpha}$. This completes our proof. □

Similarly, we prove Theorem 1.11.

Proof of Theorem 1.11. The proof is completely analogous to that of Theorem 1.11, without the need to keep track of the cohomology classes or consider elements of $H_1(Y_-)$. □

4.4. **Framed instanton Floer theory.** Instanton Floer homology of admissible pairs can be used to define a 3-manifold invariant called framed instanton Floer homology [KM11b]. First, by a *framed manifold*, we mean a closed 3-manifold with a framed basepoint. Fix (T^3, u) to be the admissible pair of the 3-dimensional torus and the element of $H^2(T^3; \mathbb{Z}/2)$ given by the dual of $S^1 \times \{q\} \subset T^3$ for some point $q \in T^2$. Let Y be a framed manifold with a framed basepoint $p \in Y$. Then define $Y^\sharp$ to be $Y \sharp T^3$, where the connected sum takes place in a neighborhood of p, and let $w^\sharp \in H^2(Y^\sharp; \mathbb{Z}/2)$ be the class induced by the trivial class in Y and u in T^3. Let $x \in \mathbb{A}(Y^\sharp \times I)$ be the class of degree 4 determined by the homology class of a point in $Y^\sharp \times I$. The operator $\mu(x) = \mathrm{I}(Y^\sharp \times I, \pi_1^*(w^\sharp); x)$ acts on the $\mathbb{Z}/8$-graded vector space $\mathrm{I}(Y^\sharp, w^\sharp)$, and satisfies $\mu(x)^2 = 4 \cdot \mathbb{I}_{\mathrm{I}(Y^\sharp, w^\sharp)}$ [KM10, Corollary 7.2]. The *framed instanton Floer homology* of Y, denoted by $\mathrm{I}^\sharp(Y)$, is defined to be the kernel of $\mu(x) - 2$; it inherits a $\mathbb{Z}/4$-grading from $\mathrm{I}(Y^\sharp, w^\sharp)$. This flavor of instanton Floer homology is conjectured to agree with the hat flavor of Heegaard Floer homology, when both are computed over $\mathbb{Q}$.

Framed instanton Floer homology is functorial with respect to cobordisms of framed manifolds. Given framed 3-manifolds Y_1 and Y_2 with framed basepoints p_1 and p_2 respectively, a *framed cobordism* $W\colon Y_1 \to Y_2$ is a cobordism together with a choice of an embedded framed path in W between p_1 and p_2. A framed cobordism $W\colon Y_1 \to Y_2$ can be used to define a cobordism $W^\sharp\colon Y_1^\sharp \to Y_2^\sharp$ by taking the sum with $T^3 \times I$ along a regular neighborhood of the framed path in W. A homology orientation on W induces a homology orientation on $W^\sharp$ in an obvious way. Moreover, the dual of $S^1 \times \{q\} \times I \subset T^3 \times I$ defines a cohomology class $c \in H^2(W^\sharp; \mathbb{Z}/2)$ that restricts to $w_1^\sharp$ and $w_2^\sharp$ on $Y_1^\sharp$ and $Y_2^\sharp$ respectively. The functoriality of instanton Floer homology of admissible pairs implies that

$$\mathrm{I}(W^\sharp, c) \circ \mathrm{I}(Y_1^\sharp \times I, \pi_1^*(w_1^\sharp); x_1) = \mathrm{I}(Y_2^\sharp \times I, \pi_1^*(w_2^\sharp); x_2) \circ \mathrm{I}(W^\sharp, c).$$

In particular, $\mathrm{I}(W^\sharp, c)$ gives rise to a homomorphism $\mathrm{I}^\sharp(W)\colon\ \mathrm{I}^\sharp(Y_1) \to \mathrm{I}^\sharp(Y_2)$.

Proof of Theorem 1.12. Let $W\colon Y_- \to Y_+$ be a ribbon $\mathbb{Q}$-homology cobordism of framed 3-manifolds. We also denote by $w_-^\sharp$ and $w_+^\sharp$ the cohomology classes in $Y_-^\sharp$ and $Y_+^\sharp$ induced by u respectively. Then $(Y_-^\sharp, w_-^\sharp)$, $(Y_+^\sharp, w_+^\sharp)$, $W^\sharp$, and $S^1 \times \{q\} \times I \subset W^\sharp$ satisfy the conditions of Theorem 4.7, and we can thus apply it to conclude that $\mathrm{I}(D(W)^\sharp, c)$ is equal to multiplication by $|H_1(W, Y_-)|$. Since this map clearly respects the eigenspace decomposition of $\mu(x)$, we obtain the analogous statement for $\mathrm{I}^\sharp(D(W))$. □

4.5. **Sutured instanton Floer theory.** First, we mention that the definition of a cobordism of sutured manifolds is given in Definition 3.11.

If Y is a framed 3-manifold, then we can define a sutured manifold (M, η), where M is the complement of a regular neighborhood of the basepoint diffeomorphic to the 3-ball, and α is the equator in ∂M. A framed cobordism $W\colon Y_1 \to Y_2$ of framed 3-manifolds then induces a cobordism of the sutured manifolds associated to Y_1 and Y_2.

More generally, the theory of instanton Floer homology of admissible pairs can be also used to define a functorial invariant of sutured manifolds, generalizing the framed instanton Floer construction. Instanton homology of sutured manifolds is defined using *closures* of sutured manifolds, which we now recall.

Let (M, η) be a sutured manifold whose set of sutures η has d elements. Denote by $F_{g,d}$ the genus-g surface with d boundary components. Fix an arbitrary $g \geq 1$; we glue (M, η) to the product sutured manifold $F_{g,d} \times [-1, 1]$ by identifying $A(\eta)$ with $(\partial F_{g,d}) \times [-1, 1]$. The resulting space has two boundary components $\widehat{R}_\pm \cong R_\pm(\eta) \cup (F_{g,d} \times \{\pm 1\})$, which are closed surfaces of the same genus; we choose a diffeomorphism ϕ of these two boundary components that fixes some point $p \in F_{g,d}$, and glue $\widehat{R}_\pm$ together by ϕ to obtain a closed 3-manifold $\widehat{M}$. Then $\{p\} \times [-1,1] \subset F_{g,d} \times [-1,1]$ determines a closed curve in $\widehat{M}$, and we write $w \in H^2(\widehat{M};\mathbb{Z}/2)$ for its Poincaré dual. The image of $\widehat{R}_\pm$ gives rise to an embedded oriented surface R of a certain genus g' in $\widehat{M}$ with $g' \geq g$, and $(\widehat{M}, w)$ is an admissible pair because the pairing of w with R is not trivial. Moreover, R induces an endomorphism

$$\mu(R) = \mathrm{I}(\widehat{M} \times I, \pi_1^*(w); R)\colon\ \mathrm{I}(\widehat{M}, w) \to \mathrm{I}(\widehat{M}, w).$$

If $g' > 1$, then the instanton homology of (M, η) is defined by

$$\mathrm{SHI}(M,\eta) = \ker(\mu(R) - (2g'-2)) \quad \text{if } g' > 1.$$

In the case that $g' = 1$, the operator $\mu(R)$ acts trivially and the definition of $\mathrm{I}(M,\eta)$ should be modified using the operator $\mu(x) = \mathrm{I}(\widehat{M} \times I, \pi_1^*(w); x)$, where $x \in \mathbb{A}(\widehat{M} \times I)$ is the class given by a point. Then we may define:

$$\mathrm{SHI}(M,\eta) = \ker(\mu(x) - 2) \quad \text{if } g' = 1.$$

In any case, the key fact is that this construction of $\mathrm{I}(M,\eta)$ above is independent of all choices made in the process. (The interested reader may compare the above with the proof of Theorem 1.6 in Section 3.3, in the context of sutured Heegaard Floer theory.)

We also have an analogous construction for a cobordism of sutured manifolds $N\colon (M_1,\eta_1) \to (M_2,\eta_2)$. First, fix $g \geq 1$, and glue the product of an interval and the product sutured manifold $F_{g,d} \times [-1,1]$ to N to obtain a cobordism of manifolds with boundary, where the induced cobordism of the boundary components is the trivial cobordism $(\widehat{R}_+ \times I) \sqcup (\widehat{R}_- \times I)$ to itself. Using the diffeomorphism ϕ of $\widehat{R}_+$ and $\widehat{R}_-$, we identify $\widehat{R}_+ \times I$ with $\widehat{R}_- \times I$ to obtain a cobordism $\widehat{N}$ from a closure $\widehat{M}_1$ of (M_1,η_1) to a closure $\widehat{M}_2$ of (M_2,η_2). In this case, the map induced by $\widehat{N}$ on instanton Floer homology respects the eigenspace decompositions of $\mathrm{I}(\widehat{M}_1, w_1)$ and $\mathrm{I}(\widehat{M}_2, w_2)$, and we obtain a homomorphism $\mathrm{SHI}(N)\colon\ \mathrm{SHI}(M_1,\eta_1) \to \mathrm{SHI}(M_2,\eta_2)$.

Proof of Theorem 1.14. This follows directly from Theorem 4.7 together with the description of sutured instanton Floer homology as the eigenspace of the instanton Floer homology for an admissible pair. □

The sutured instanton homology of the sutured manifold associated to a framed 3-manifold Y is isomorphic to $\mathrm{I}^\sharp(Y)$. In fact, the manifold $Y^\sharp$ can be obtained as a closure of the sutured manifold associated to Y, where we use the product sutured manifold $F_{1,1} \times I$ in the construction of the closure.

4.6. **Equivariant instanton Floer theory.** For a $\mathbb{Z}$-homology sphere Y, one can define a stronger invariant that contains the information of $\mathrm{I}(Y)$ and $\mathrm{I}^\sharp(Y)$. Let $(\mathrm{C}(Y), d)$ be the instanton Floer chain complex whose homology is equal to $\mathrm{I}(Y)$. We consider a larger chain complex $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ defined by $\widetilde{\mathrm{C}}(Y) = \mathrm{C}(Y) \oplus \mathbb{Q} \oplus \mathrm{C}(Y)[3]$, where $\mathrm{C}(Y)[3]$ denotes the complex $\mathrm{C}(Y)$ with the $\mathbb{Z}/8$-grading shifted up by 3. The complex $\widetilde{\mathrm{C}}(Y)$ is equipped with a $\mathbb{Z}/8$-grading on $\widetilde{C}(Y)$ by assigning degree 0 to the summand $\mathbb{Q}$. With respect to the direct sum decomposition of $\widetilde{\mathrm{C}}(Y)$ above, the differential $\widetilde{d}$, which has degree -1, has the matrix form

$$\widetilde{d} = \begin{pmatrix} d & 0 & 0 \\ D_1 & 0 & 0 \\ U & D_2 & -d \end{pmatrix}, \tag{4.9}$$

where $U\colon\ \mathrm{C}(Y) \to \mathrm{C}(Y)[4]$ is a degree-preserving map, D_1 is a functional on $\mathrm{C}(Y)$ that is not zero only on elements of degree 1, and $D_2(1)$ is a degree-4 element in $\mathrm{C}(Y)$. We refer the reader to

[Don02, Frø02] for more details on the definition of U, D_1, and D_2. Here we use the same conventions as in [Dae19], where an exposition of the definition of $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ is given. The characterizing feature of the special form of $\widetilde{d}$ in (4.9) is that it anti-commutes with the endomorphism of $\widetilde{\mathrm{C}}(Y)$ given by

$$\chi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We call a chain complex $(\widetilde{C}, \widetilde{d})$ over $\mathbb{Q}$ whose differential has the form in (4.9) an *$\mathcal{SO}$-complex*.[9]

The chain complex $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ depends on some auxiliary choices, namely the Riemannian metric on Y and the perturbation of the Chern–Simons functional of Y. However, the chain homotopy type of $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ is an invariant of Y in an appropriate sense. Suppose $(\widetilde{\mathrm{C}}'(Y), \widetilde{d}')$ is the chain complex that is obtained from another set of auxiliary choices. Then there is a degree-0 chain map $\widetilde{\lambda}\colon \widetilde{\mathrm{C}}(Y) \to \widetilde{\mathrm{C}}'(Y)$, such that

$$\widetilde{\lambda} = \begin{pmatrix} \lambda & 0 & 0 \\ \Delta_1 & 1 & 0 \\ \mu & \Delta_2 & \lambda \end{pmatrix}. \tag{4.10}$$

Notice that the map $\widetilde{\lambda}$ commutes with χ. Similarly, there is a degree-0 chain map $\widetilde{\lambda}'\colon \widetilde{\mathrm{C}}'(Y) \to \widetilde{\mathrm{C}}(Y)$ with the same form as (4.10), with λ' playing the role of λ. Moreover, there are degree-1 maps $\widetilde{K}\colon \widetilde{\mathrm{C}}(Y) \to \widetilde{\mathrm{C}}(Y)$ and $\widetilde{K}'\colon \widetilde{\mathrm{C}}'(Y) \to \widetilde{\mathrm{C}}'(Y)$ that anti-commute with χ, such that

$$\widetilde{K} \circ \widetilde{d} + \widetilde{d} \circ \widetilde{K} = \widetilde{\lambda}' \circ \widetilde{\lambda} - \mathbb{I}_{\widetilde{\mathrm{C}}(Y)}, \qquad \widetilde{K}' \circ \widetilde{d}' + \widetilde{d}' \circ \widetilde{K}' = \widetilde{\lambda} \circ \widetilde{\lambda}' - \mathbb{I}_{\widetilde{\mathrm{C}}'(Y)}.$$

As is customary in Floer theories, the existence of the maps $\widetilde{\lambda}$ and $\widetilde{\lambda}'$ is a consequence of a more general functoriality of the theory. In fact, for any $\mathbb{Z}$-homology cobordism $W\colon Y_1 \to Y_2$, there is a chain map $\widetilde{\lambda}(W)\colon \widetilde{\mathrm{C}}(Y_1) \to \widetilde{\mathrm{C}}(Y_2)$ of the form in (4.10). In particular, this morphism contains in its data a chain map $\lambda(W)\colon\ \mathrm{C}(Y_1) \to \mathrm{C}(Y_2)$, which induces the cobordism map $\mathrm{I}(W)\colon\ \mathrm{I}(Y_1) \to \mathrm{I}(Y_2)$ on the level of homology.

For general $\mathcal{SO}$-complexes, a chain map of the form (4.10), with the number 1 possibly replaced by a non-zero rational number, is called an *$\mathcal{SO}$-morphism*. An *$\mathcal{SO}$-homotopy* from an $\mathcal{SO}$-morphism $\widetilde{\lambda}_1$ to another $\mathcal{SO}$-morphism $\widetilde{\lambda}_2$ is given by a map $\widetilde{K}$ of degree 1 that anti-commutes with χ, such that

$$\widetilde{K} \circ \widetilde{d} + \widetilde{d} \circ \widetilde{K} = \widetilde{\lambda}_2 - \widetilde{\lambda}_1,$$

and we say that two $\mathcal{SO}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ are *$\mathcal{SO}$-homotopy equivalent* if there are $\mathcal{SO}$-morphisms $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ and $\widetilde{\lambda}'\colon \widetilde{C}' \to \widetilde{C}$ such that $\widetilde{\lambda}' \circ \widetilde{\lambda}$ and $\widetilde{\lambda} \circ \widetilde{\lambda}'$ are $\mathcal{SO}$-homotopic to identity maps. In other words, the discussion above shows that the $\mathcal{SO}$-homotopy type of $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ is an invariant of Y.

The $\mathcal{SO}$-homotopy type of the complex $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ contains the information of the instanton homology groups $\mathrm{I}(Y)$ and $\mathrm{I}^\sharp(Y)$. It is clear from the definition that $\mathrm{I}(Y)$ is the homology of the quotient complex $(\mathrm{C}(Y), d)$, whose chain homotopy type can be recovered from the $\mathcal{SO}$-homotopy type of $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$. The homology of the chain complex $(\widetilde{\mathrm{C}}(Y), \widetilde{d} + 4\chi)$ is also isomorphic to $\mathrm{I}^\sharp(Y)$ [Sca15].

One could extract from $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ several other homologies, which are analogous to HF^-, HF^+, and HF^∞ in Heegaard Floer theory respectively. Following [Don02, Dae19], consider the $\mathbb{Z}/8$-graded

[9] For a topological space with an SO(3)-action that has a unique fixed point, one can form an $\mathcal{SO}$-complex whose homology is the homology of the space. This justifies the terminology $\mathcal{SO}$-complex.

chain complexes $(\widehat{\mathrm{C}}(Y), \widehat{d})$ and $(\check{\mathrm{C}}(Y), \check{d})$ defined by

$$\widehat{\mathrm{C}}(Y) = \mathrm{C}(Y)[3] \oplus \mathbb{Q}[x], \qquad \widehat{d}\left(\alpha, \sum_{i=0}^{N} a_i x^i\right) = \left(d\alpha - \sum_{i=0}^{N} U^i D_2(a_i), 0\right),$$

$$\check{\mathrm{C}}(Y) = \mathrm{C}(Y) \oplus (\mathbb{Q}[\![x^{-1}, x]/\mathbb{Q}[x]), \qquad \check{d}\left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i\right) = \left(d\alpha, \sum_{i=-\infty}^{-1} D_1 U^{-i-1}(\alpha) x^i\right).$$

Here, the degree of x is defined to be -4. The homology of $(\widehat{\mathrm{C}}(Y), \widehat{d})$ and $(\check{\mathrm{C}}(Y), \check{d})$ are denoted by $\widehat{\mathrm{I}}(Y)$ and $\check{\mathrm{I}}(Y)$ respectively. They are modules over the polynomial ring $\mathbb{Q}[x]$, with the action of x given by the endomorphisms

$$x\colon \widehat{\mathrm{C}}(Y) \to \widehat{\mathrm{C}}(Y), \qquad x \cdot \left(\alpha, \sum_{i=0}^{N} a_i x^i\right) = \left(U\alpha, D_1(\alpha) + \sum_{i=0}^{N} a_i x^{i+1}\right),$$

$$x\colon \check{\mathrm{C}}(Y) \to \check{\mathrm{C}}(Y), \qquad x \cdot \left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i\right) = \left(U\alpha + D_2(a_{-1}), \sum_{i=-\infty}^{-2} a_i x^{i+1}\right).$$

We define $(\overline{\mathrm{C}}(Y), \overline{d})$ to be the $\mathbb{Q}[x]$-module $\mathbb{Q}[\![x^{-1}, x]$ with the trivial differential; although it is independent of Y, it is convenient to consider it and its homology $\overline{\mathrm{I}}(Y) \cong \mathbb{Q}[\![x^{-1}, x]$. Together, $\widehat{\mathrm{I}}(Y)$, $\check{\mathrm{I}}(Y)$, and $\overline{\mathrm{I}}(Y)$ are called the *equivariant instanton Floer homologies* of Y.

The modules $\widehat{\mathrm{I}}(Y)$, $\check{\mathrm{I}}(Y)$ and $\overline{\mathrm{I}}(Y)$ fit into an exact triangle

$$\begin{array}{ccc} \check{\mathrm{I}}(Y) & \xrightarrow{\;j_*\;} & \widehat{\mathrm{I}}(Y) \\ {\scriptstyle p_*}\nwarrow & & \swarrow{\scriptstyle i_*} \\ & \overline{\mathrm{I}}(Y) & \end{array} \tag{4.11}$$

where the module homomorphisms are induced by the maps

$$i\colon \widehat{\mathrm{C}}(Y) \to \overline{\mathrm{C}}(Y), \qquad i\left(\alpha, \sum_{i=0}^{N} a_i x^i\right) = \sum_{i=-\infty}^{-1} D_1 U^{-i-1}(\alpha) x^i + \sum_{i=0}^{N} a_i x^i,$$

$$j\colon \check{\mathrm{C}}(Y) \to \widehat{\mathrm{C}}(Y), \qquad j\left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i\right) = (-\alpha, 0),$$

$$p\colon \overline{\mathrm{C}}(Y) \to \check{\mathrm{C}}(Y), \qquad p\left(\sum_{i=-\infty}^{N} a_i x^i\right) = \left(\sum_{i=0}^{N} U^i D_2(a_i), \sum_{i=-\infty}^{-1} a_i x^i\right).$$

As is apparent from the definitions, the construction of the equivariant instanton homologies and the exact triangle (4.11) from $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$ is completely algebraic and does not require any additional geometric input. In particular, for any $\mathcal{SO}$-complex $(\widetilde{C}, \widetilde{d})$, one can define the chain complexes $(\widehat{C}, \widehat{d})$, $(\check{C}, \check{d})$, their homologies $\widehat{I}$, $\check{I}$, and the analogue of the exact triangle (4.11). These constructions are functorial; given an $\mathcal{SO}$-morphism $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$, there are corresponding chain maps $\widehat{\lambda}\colon \widehat{C} \to \widehat{C}'$ and $\check{\lambda}\colon \check{C} \to \check{C}$, which induce module homomorphisms $\widehat{\lambda}_*\colon \widehat{I} \to \widehat{I}'$ and $\check{\lambda}_*\colon \check{I} \to \check{I}'$ that commute with the exact triangles associated to $\widetilde{C}$ and $\widetilde{C}'$. An $\mathcal{SO}$-homotopy between two $\mathcal{SO}$-morphisms $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ induces a homotopy between $\check{\lambda}_1$ and $\check{\lambda}_2$ and a homotopy between $\check{\lambda}_1$ and $\check{\lambda}_2$. Moreover, the maps corresponding to the composition $\widetilde{\lambda}' \circ \widetilde{\lambda}$ of two $\mathcal{SO}$-morphisms are equal to the compositions of the maps corresponding to $\widetilde{\lambda}'$ and $\widetilde{\lambda}$. As a consequence of this functoriality, the equivariant instanton homologies $\widehat{\mathrm{I}}(Y)$, $\check{\mathrm{I}}(Y)$, $\bar{\mathrm{I}}(Y)$ and the exact triangle (4.11) are invariants of Y, and do not depend on the auxiliary choices in the definition of $(\widetilde{\mathrm{C}}(Y), \widetilde{d})$.

We now turn our attention to the behavior of equivariant instanton Floer homologies under ribbon $\mathbb{Q}$-homology cobordisms. (Recall from Remark 1.10 that $\mathbb{Q}$-homology cobordisms between $\mathbb{Z}$-homology spheres are in fact $\mathbb{Z}$-homology cobordisms.) The key statement is the following proposition about the associated $\mathcal{SO}$-complexes.

Proposition 4.12. *Let Y_- and Y_+ be $\mathbb{Z}$-homology spheres, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then the $\mathcal{SO}$-morphism $\widetilde{\lambda}(D(W))\colon \widetilde{\mathrm{C}}(Y_-) \to \widetilde{\mathrm{C}}(Y_-)$ is $\mathcal{SO}$-homotopic to an $\mathcal{SO}$-isomorphism.*

Proof. Write the differential $\widetilde{d}$ of the $\mathcal{SO}$-complex $(\widetilde{\mathrm{C}}(Y_-), \widetilde{d})$ as in (4.9), with the maps d, U, D_1, and D_2, and write the $\mathcal{SO}$-morphism $\widetilde{\lambda}(D(W))$ as in (4.10), with the maps $\lambda(D(W))$, Δ_1, Δ_2, and μ. Since we are working with chain complexes over a field, our argument in Section 4.3 shows that there is a chain homotopy $K\colon\ \mathrm{C}(Y_-) \to \mathrm{C}(Y_-)$ such that $K \circ d + d \circ K = \mathbb{I}_{\mathrm{C}(Y_-)} - \lambda(D(W))$. Defining the map $\widetilde{K}\colon \widetilde{\mathrm{C}}(Y_-) \to \widetilde{\mathrm{C}}(Y_-)$ by

$$\widetilde{K} = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -K \end{pmatrix},$$

we immediately see that $\widetilde{K}$ anti-commutes with χ; moreover, we can compute that

$$\widetilde{K} \circ \widetilde{d} + \widetilde{d} \circ \widetilde{K} + \widetilde{\lambda}(D(W)) = \begin{pmatrix} \mathbb{I}_{\mathrm{C}(Y_-)} & 0 & 0 \\ * & 1 & 0 \\ * & * & \mathbb{I}_{\mathrm{C}(Y_-)} \end{pmatrix},$$

and so $\widetilde{K}$ is an $\mathcal{SO}$-homotopy between $\widetilde{\lambda}(D(W))$ and $\widetilde{q} = \widetilde{K} \circ \widetilde{d} + \widetilde{d} \circ \widetilde{K} + \widetilde{\lambda}(D(W))$, which is clearly invertible over $\mathbb{Q}$. □

Proof of Theorem 1.18. Proposition 4.12 and the discussion above it together imply that $\widehat{\mathrm{I}}(D(W)) = \widehat{q}_*$ and $\check{\mathrm{I}}(D(W)) = \check{q}_*$, where $\widehat{q}\colon \widehat{\mathrm{C}}(Y_-) \to \widehat{\mathrm{C}}(Y_-)$ and $\check{q}\colon \check{\mathrm{C}}(Y_-) \to \check{\mathrm{C}}(Y_-)$ are the chain maps corresponding to some $\mathcal{SO}$-isomorphism $\widetilde{q}\colon \widetilde{\mathrm{C}}(Y_-) \to \widetilde{\mathrm{C}}(Y_-)$. It is clear that $\widehat{q}_*$ and $\check{q}_*$ are $\mathbb{Q}[x]$-module isomorphisms. □

5. π_1, CHARACTER VARIETIES, AND RIBBON HOMOLOGY COBORDISMS

In this section, we study the behavior of character varieties under ribbon cobordisms. One goal is to prove the following, which is a restatement of Theorem 1.27 (3).

Theorem 5.1. *Suppose that there exists a ribbon $\mathbb{Z}$-homology cobordism from the Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ to $\Sigma(a'_1, \dots, a'_m)$. Then the numbers of fibers satisfy $n \le m$.*

Note that Theorem 5.1 immediately implies that a ribbon concordance between determinant-1 Montesinos knots cannot decrease the number of strands.

5.1. **Background.**

Throughout, we let G denote a compact, connected Lie group. For a group π, let $\mathcal{R}_G(\pi)$ denote the space of G-representations. If X is a connected manifold, we write $\mathcal{R}_G(X)$ for $\mathcal{R}_G(\pi_1(X))$. We write $\chi_G(\pi)$ for the set of conjugacy classes of G-representations. We will omit G from the notation when $G = \mathrm{SU}(2)$.

We first prove the following proposition, which is a restatement of Theorem 1.1 and Proposition 1.22. The argument, using work of Gerstenhaber and Rothaus [GR62], repeats that of Gordon [Gor81] and also that of Cornwell, Ng, and Sivek [CNS16].

Proposition 5.2. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then the inclusion $\iota_+\colon Y_+ \to W$ induces a surjection $(\iota_+)_*\colon \pi_1(Y_+) \to \pi_1(W)$ and an injection $\iota_+^*\colon \mathcal{R}_G(W) \to \mathcal{R}_G(Y_+)$, and the inclusion $\iota_-\colon Y_- \to W$ induces an injection $(\iota_-)_*\colon \pi_1(Y_-) \to \pi_1(W)$ and a surjection $\iota_-^*\colon \mathcal{R}_G(W) \to \mathcal{R}_G(Y_-)$.*

Proof. Since W consists entirely of 1- and 2-handles, we may flip W upside down and view it as a cobordism from $-Y_+$ to $-Y_-$. From this perspective, W is obtained by attaching 2- and 3-handles to $-Y_+$. It follows that the inclusion from $-Y_+$ into W induces a surjection from $\pi_1(-Y_+)$ to $\pi_1(W)$.

For $\iota_-\colon Y_- \to W$, we will prove the second claim first. Choose a representation $\rho\colon \pi_1(Y_-) \to G$. Since W is a $\mathbb{Q}$-homology cobordism, it admits a handle decomposition with an equal number m of 1- and 2-handles. This allows us to write $\pi_1(W) \cong (\pi_1(Y_-) * \langle b_1, \dots, b_m\rangle)/\langle\!\langle v_1, \dots, v_m\rangle\!\rangle$, where the generators b_i are induced by the 1-handles and the relators v_i are induced by the 2-handles. The words v_i induce a map $K\colon G^m \to G^m$, and the existence of an extension of ρ to $\pi_1(W)$ is equivalent to solving the equation $K = \vec{e}$. (To handle the elements in $\pi_1(Y_-)$ that appear in v_i, we apply ρ to the element to view it in G.) By quotienting out by $\pi_1(Y_-)$, each element v_i induces a word v_i' in the free group $\langle b_1, \dots, b_n\rangle$. Consider the matrix B whose $(ij)^{\text{th}}$ coordinate is the signed number of times that b_j appears in v_i'. Since $H_1(W, Y_-; \mathbb{Q}) = 0$, we see that $\det(B) \neq 0$. It now follows from [GR62, Theorem 1] that there exists a solution to the equation $K = \vec{e}$.

Now we show that the inclusion map $(\iota_-)_*$ from $\pi_1(Y_-)$ to $\pi_1(W)$ is injective. The residual finiteness property of 3-manifold groups implies that for any non-trivial $x \in \pi_1(Y_-)$, there exists a finite quotient H of $\pi_1(Y_-)$ by a normal subgroup N such that $x \notin N$. According to [GR62, Theorem 2], the induced map $\overline{(\iota_-)_*}\colon H \to \pi_1(W)/\langle\!\langle(\iota_-)_*(N)\rangle\!\rangle$ is injective. This implies that $(\iota_-)_*(x)$ is a non-trivial element of $\pi_1(W)$. ☐

Corollary 5.3. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. If $\chi_G(Y_+)$ is finite, then $\chi_G(Y_-)$ is finite.* ☐

In the next subsection, we give a more structured comparison of the character varieties with a bit more work.

5.2. **Group cohomology computations and Zariski tangent spaces.** We briefly review some definitions and constructions in group cohomology; see [Bro94] for more details. Let π be a group and let M be a $\mathbb{Z}[\pi]$-module. The group cohomology $H^*(\pi; M)$ with coefficients in M is defined by taking a projective $\mathbb{Z}[\pi]$-resolution $\cdots \to P_1 \to P_0 \to \mathbb{Z}$ of $\mathbb{Z}$, where $\mathbb{Z}$ has the $\mathbb{Z}[\pi]$-module structure where π acts by the identity. Then $C^*(\pi; M)$ is defined by applying $\operatorname{Hom}_{\mathbb{Z}[\pi]}(-, M)$, and omitting $\mathbb{Z}$, as in

$$0 \to \operatorname{Hom}_{\mathbb{Z}[\pi]}(P_0, M) \to \operatorname{Hom}_{\mathbb{Z}[\pi]}(P_1, M) \to \cdots,$$

and $H^*(\pi; M)$ is the cohomology of this cochain complex. A natural way of constructing a free resolution of $\mathbb{Z}$ is as follows. Consider an aspherical CW complex X with $\pi_1(X) = \pi$, and lift this to a CW structure on the universal cover $\widetilde{X}$. Then, the (augmented/reduced) CW chain complex for $\widetilde{X}$ naturally inherits a $\mathbb{Z}[\pi]$-module structure, where π acts by the deck transformation group action, and this is a free $\mathbb{Z}[\pi]$-resolution of $\mathbb{Z}$. (The lift of an individual cell in X yields a π's worth of cells upstairs, and these constitute a single copy of $\mathbb{Z}[\pi]$ in the cellular chain complex for the universal cover.) Recall that a presentation $\pi = \langle a_\alpha \mid w_\beta\rangle$ determines a CW structure on X with one 0-cell, one 1-cell e^1_α for each generator a_α, and one 2-cell e^2_β for each relator w_β; then $H^*(\pi; M)$ can be computed from a cochain complex with Abelian groups

$$C^0(\pi; M) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi], M) \cong M, \quad C^1(\pi; M) \cong \prod_\alpha M, \quad C^2(\pi; M) \cong \prod_\beta M,$$

and possibly non-trivial higher cochain groups $C^i(\pi; M)$ for $i > 2$ that we will not be concerned with. The $(\alpha\beta)^{\text{th}}$ component of the differential from $C^1(\pi; M)$ to $C^2(\pi; M)$ is non-zero only if a_α appears in w_β. Indeed, in X, if $\overline{e^1_\alpha} \cap \overline{e^2_\beta} = \emptyset$, then the same is true in the universal cover.

Now, given a representation $\rho \in \mathcal{R}_G(\pi)$, we can consider the $\mathbb{Z}[\pi]$-module Ad_ρ, which is the Lie algebra $\mathfrak{g}$ of G with the $\mathbb{Z}[\pi]$-action where π acts by the composition of ρ and the adjoint representation. Note that Ad_ρ is in fact an $\mathbb{R}[\pi]$-module, and so $H^*(\pi; \mathrm{Ad}_\rho)$ is an $\mathbb{R}$–vector space. Recall also that $H^1(\pi; \mathrm{Ad}_\rho)$ is the *Zariski tangent space* of $\chi_G(\pi)$ at $[\rho]$. We are now ready to show that ribbon homology cobordisms induce relations between the Zariski tangent spaces.

Proof of Proposition 1.25. We begin by comparing $\dim_\mathbb{R} H^1(W; \mathrm{Ad}_{\rho_W})$ and $\dim_\mathbb{R} H^1(Y_+; \mathrm{Ad}_{\rho_+})$. First, we recall the inflation–restriction exact sequence in group cohomology (see, for example, [Wei94, 6.8.3]), which says that, given a normal subgroup N of π and a $\mathbb{Z}[\pi]$-module M, there exists an injection of $H^1(\pi/N; M^N)$ into $H^1(\pi; M)$, where M^N is the subgroup of elements of M fixed by the action of π restricted to N. It is clear that M^N naturally inherits a $\mathbb{Z}[\pi/N]$-module structure. (Further, if M actually has an $\mathbb{R}[\pi]$-module structure, then everything respects the $\mathbb{R}$–vector space structures.)

In our case, we take $\pi = \pi_1(Y_+)$, take N to be the kernel of the quotient map from $\pi_1(Y_+)$ to $\pi_1(W)$, and take $M = \mathrm{Ad}_{\rho_+}$; then $(\mathrm{Ad}_{\rho_+})^N$ is a $\mathbb{Z}[\pi_1(W)]$-module. By construction, $N \subset \ker(\rho_+) \subset \ker(\mathrm{Ad} \circ \rho_+)$; thus, N acts by the identity on Ad_{ρ_+}, and so $(\mathrm{Ad}_{\rho_+})^N$ is in fact Ad_{ρ_W}. Therefore, we conclude that $\dim_\mathbb{R} H^1(W; \mathrm{Ad}_{\rho_W}) \le \dim_\mathbb{R} H^1(Y_+; \mathrm{Ad}_{\rho_+})$.

Next, we consider the restriction of ρ_W to $\pi_1(Y_-)$. Suppose that $\pi_1(Y_-)$ has a presentation of the form $\pi_1(Y_-) = \langle a_1, \ldots, a_g \,|\, w_1, \ldots, w_r \rangle$. (We do not require $Y_\pm$ to be closed, and so there may not exist a balanced presentation.) Then $\pi_1(W)$ admits a presentation of the form

$$\pi_1(W) = \langle a_1, \ldots, a_g, b_1, \ldots, b_m \,|\, w_1, \ldots, w_r, v_1, \ldots, v_m \rangle .$$

As discussed above, $H^*(Y_-; \mathrm{Ad}_{\rho_-})$ is the cohomology of a cochain complex of the form

$$0 \to \mathfrak{g} \xrightarrow{\psi} \bigoplus_{i=1}^{g} \mathfrak{g} \xrightarrow{\phi} \bigoplus_{j=1}^{r} \mathfrak{g} \to \cdots . \tag{5.4}$$

Thus, $H^0(Y_-; \mathrm{Ad}_{\rho_-}) = \ker(\psi)$, and $\dim_\mathbb{R} H^1(Y_-; \mathrm{Ad}_{\rho_-}) = \dim_\mathbb{R} \ker(\phi) - \dim_\mathbb{R} \mathrm{im}(\psi)$. We consider a similar setup for $\pi_1(W)$, where $C^1(W; \mathrm{Ad}_{\rho_W})$ (resp. $C^2(W; \mathrm{Ad}_{\rho_W})$) has $g+m$ (resp. $r+m$) copies of $\mathfrak{g}$, and write ψ' and ϕ' for the associated differentials. It is obvious now that the condition $\dim_\mathbb{R} H^0(Y_-; \mathrm{Ad}_{\rho_-}) = \dim_\mathbb{R} H^0(W; \mathrm{Ad}_{\rho_W})$ implies that $\dim_\mathbb{R} \mathrm{im}(\psi) = \dim_\mathbb{R} \mathrm{im}(\psi')$.

We now aim to compare $H^1(Y_-; \mathrm{Ad}_{\rho_-})$ and $H^1(W; \mathrm{Ad}_{\rho_W})$. Note that we have an $\mathbb{R}$–vector space decomposition $C^i(W; \mathrm{Ad}_{\rho_W}) = C^i(Y_-; \mathrm{Ad}_{\rho_-}) \oplus \mathfrak{g}^m$ for $i \in \{1, 2\}$. Since the relators $w_1, \ldots, w_r$ do not interact with the m additional generators in $\pi_1(W)$, we have a block decomposition

$$\phi' = \begin{pmatrix} \phi & 0 \\ \eta & \gamma \end{pmatrix}.$$

Writing $\dim_\mathbb{R} \mathfrak{g} = d$, we note that η is a $(dm \times dg)$-matrix and γ is a $(dm \times dm)$-matrix. We deduce

$$\begin{aligned} \dim_\mathbb{R} H^1(Y_-; \mathrm{Ad}_{\rho_-}) &= \dim_\mathbb{R} \ker(\phi) - \dim_\mathbb{R} \mathrm{im}(\psi) \\ &= dg - \dim_\mathbb{R} \mathrm{im}(\phi) - \dim_\mathbb{R} \mathrm{im}(\psi) \end{aligned}$$

$$\begin{aligned} &= dg + dm - (\dim_{\mathbb{R}} \operatorname{im}(\phi) + dm) - \dim_{\mathbb{R}} \operatorname{im}(\psi') \\ &\leq d(g+m) - \dim_{\mathbb{R}} \operatorname{im}(\phi') - \dim_{\mathbb{R}} \operatorname{im}(\psi') \\ &= \dim_{\mathbb{R}} \ker(\phi') - \dim_{\mathbb{R}} \operatorname{im}(\psi') \\ &= \dim_{\mathbb{R}} H^1(W; \operatorname{Ad}_{\rho_W}), \end{aligned}$$

which completes the proof. □

5.3. **Ribbon cobordisms between Seifert fibered homology spheres.** In this subsection, we prove Theorem 5.1; in particular, we focus on the case $G = \mathrm{SU}(2)$. We begin by mentioning a basic fact about $\mathrm{SU}(2)$-representations. Every representation $\rho\colon \pi \to \mathrm{SU}(2)$ is either trivial, Abelian, or irreducible, and $\dim_{\mathbb{R}} H^0(\pi; \operatorname{Ad}_\rho)$ is respectively 3, 1, or 0, according to this trichotomy.

We now review some useful facts from the work of Fintushel and Stern on $\mathrm{SU}(2)$-representations for Seifert fibered homology spheres [FS90] (see also the work of Boyer [Boy88]). To fix our notation, the Seifert fibered homology sphere $\Sigma(a_1, \ldots, a_n)$ has base orbifold S^2 and presentation $(b; (a_1, b_1), \ldots, (a_n, b_n))$, where we do not require that $0 < b_i < a_i$, but do require that a_i and b_i are relatively prime, and that

$$-b + \sum_{i=1}^{n} \frac{b_i}{a_i} = \frac{1}{a_1 \cdots a_n}.$$

Then the fundamental group of $\Sigma(a_1, \ldots, a_n)$ is given by

$$\pi_1(\Sigma(a_1, \ldots, a_n)) \cong \left\langle x_1, \ldots, x_n, h \,\middle|\, h \text{ central},\ x_i^{a_i} = h^{-b_i},\ x_1 \cdots x_n = h^{-b} \right\rangle.$$

Theorem 5.5 (Fintushel and Stern [FS90])**.** *Suppose that $\rho \in \mathcal{R}(\Sigma(a_1, \ldots, a_n))$ is irreducible. Then*

(1) [FS90, Lemma 2.1] $\rho(h) = \pm 1$*;*
(2) [FS90, Lemma 2.2] $\rho(x_i) \neq \pm 1$ *for at least three values of $i \in \{1, \ldots, n\}$; and*
(3) [FS90, Proposition 2.5] $\dim_{\mathbb{R}} H^1(\Sigma(a_1, \ldots, a_n); \operatorname{Ad}_\rho) = 2t - 6$*, where t is the number of x_i's such that $\rho(x_i) \neq \pm 1$.*

We also recall their recipe for constructing conjugacy classes of irreducible $\mathrm{SU}(2)$-representations. For this, it will be useful to think of elements of $\mathrm{SU}(2)$ as unit quaternions. After choosing the sign of $\rho(h)$, we may choose an integer ℓ_1 and define $\rho(x_1) = e^{i\pi\ell_1/a_1}$, as long as $0 \leq \ell_1 \leq a_1$ and $(-1)^{\ell_1} = (\rho(h))^{-b_1}$. Next, for each $q \in \{2, \ldots, n\}$, we choose an integer ℓ_q with analogous constraints and consider $e^{i\pi\ell_q/a_q}$; we will eventually define $\rho(x_q)$ to be some element conjugate to this. (The choice of the integers ℓ_q is also subject to Theorem 5.5 (2).) Note that once we choose $\rho(x_2), \ldots, \rho(x_{n-1})$, they will determine $\rho(x_n)$ by the equation

$$\rho(x_1) \cdots \rho(x_n) = (\rho(h))^{-b}; \tag{5.6}$$

the difficulty, however, lies in ensuring that $\rho(x_n)$ is also conjugate to $e^{i\pi\ell_n/a_n}$ for some integer ℓ_n with analogous constraints. Plugging this last condition into (5.6), we see that $\rho(x_2), \ldots, \rho(x_{n-1})$ must satisfy

$$(\rho(x_1) \cdots \rho(x_{n-1}))^{a_n} = (\rho(h))^{-ba_n + b_n} = \pm 1. \tag{5.7}$$

To fulfill this condition, let S_q denote the set of elements in $\mathrm{SU}(2)$ conjugate to $e^{i\pi\ell_q/a_q}$, for each $q \in \{2, \ldots, n-1\}$. If $e^{i\pi\ell_q/a_q} \neq \pm 1$, then S_q is a copy of S^2. In any case, consider the map $\phi\colon S_2 \times \cdots \times S_{n-1} \to [0, \pi]$ given by

$$\phi(s_2, \ldots, s_{n-1}) = \operatorname{Arg}(\rho(x_1) s_2 \cdots s_{n-1}),$$

where $\operatorname{Arg}(z)$ is defined to be the value $\theta \in [0, \pi]$ such that z is conjugate to $e^{i\theta}$. If $\pi\ell'_n/a_n$ is in the image of ϕ, then, for each integer ℓ'_n such that $0 \le \ell'_n \le a_n$ and $(-1)^{\ell'_n} = (\rho(h))^{-ba_n+b_n}$, there exists some choice of $\rho(x_2), \dots, \rho(x_{n-1})$ such that $\rho(x_1)\cdots\rho(x_{n-1})$ is conjugate to $e^{i\pi\ell'_n/a_n}$ (and hence (5.7) holds). This determines a well-defined representation ρ. Finally, note that since the Abelianization of $\pi_1(\Sigma(a_1, \dots, a_n))$ is trivial, there are no non-trivial Abelian SU(2)-representations, and every non-trivial representation is irreducible.

We now proceed towards the proof of Theorem 5.1. The main technical proposition we will prove is the following. While this is well-known, we include a direct proof for completeness.

Proposition 5.8. *Suppose that Y is the Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$. Then there exists an irreducible $\rho \in \mathcal{R}(Y)$ such that $H^1(Y; \mathrm{Ad}_\rho)$ has the maximal dimension possible, i.e. $2n-6$.*

We now briefly describe our strategy to prove this proposition. By Theorem 5.5 (3), we would like to show that $\pi_1(\Sigma(a_1, \dots, a_n))$ admits an irreducible SU(2)-representation that does not send any x_i to ± 1. The idea is to reduce to the case where there are exactly three singular fibers; in other words, we will construct such a representation from an irreducible SU(2)-representation of $\pi_1(\Sigma(a_1, a_2, a_3 \cdots a_n))$, by a pinching argument. A subtlety here is that for this pinching argument to work, we will require the representation of $\pi_1(\Sigma(a_1, a_2, a_3 \cdots a_n))$ to be of a certain form; to show that this exists, we will assume the primality of the a_i's. Thus, we begin by reducing to the case that all the a_i's are prime.

Lemma 5.9. *Let $r \in \mathbb{Z}_{\ge 0}$. Suppose that $\pi_1(Y)$ admits an irreducible representation ρ such that $\dim_{\mathbb{R}} H^1(Y; \mathrm{Ad}_\rho) = r$ for $Y \cong \Sigma(a_1, \dots, a_n)$. Then the same holds for $Y \cong \Sigma(a_1, \dots, a_{n-1}, ka_n)$, where k is relatively prime to $a_1, \dots, a_{n-1}$.*

Proof. Fix a presentation $(b; (a_1, b_1), \dots, (a_{n-1}, b_{n-1}), (ka_n, b_n))$ for $\Sigma(a_1, \dots, a_{n-1}, ka_n)$; then

$$\Sigma(a_1, \dots, a_n) = S^2(kb; (a_1, kb_1), \dots, (a_{n-1}, kb_{n-1}), (a_n, b_n)).$$

We denote by x_i and y_i the respective generators of $\pi_1(\Sigma(a_1, \dots, a_{n-1}, ka_n))$ and $\pi_1(\Sigma(a_1, \dots, a_n))$ associated to the singular fibers, but we abusively write h for the central generator in both groups. Consider the homomorphism $\phi\colon \pi_1(\Sigma(a_1, \dots, a_{n-1}, ka_n)) \to \pi_1(\Sigma(a_1, \dots, a_n))$ defined by

$$\phi(h) = h^k, \quad \phi(x_i) = y_i \text{ for all } i \in \{1, \dots, n\},$$

which can be easily checked to be well-defined. (For completeness, we note that ϕ is induced by the k-fold cover of $\Sigma(a_1, \dots, a_n)$ branched over the singular fiber of order a_n.)

Since $\rho \in \mathcal{R}(\Sigma(a_1, \dots, a_n))$ is irreducible, so is $\phi^*\rho \in \mathcal{R}(\Sigma(a_1, \dots, a_{n-1}, ka_n))$, and the number of x_i's such that $\phi^*\rho(x_i) = \pm 1$ is exactly the same as the number of y_i's such that $\rho(y_i) = \pm 1$. Therefore, by Theorem 5.5 (3), we conclude that $\dim_{\mathbb{R}} H^1(\Sigma(a_1, \dots, a_{n-1}, ka_n); \mathrm{Ad}_{\phi^*\rho}) = \dim_{\mathbb{R}} H^1(\Sigma(a_1, \dots, a_n); \mathrm{Ad}_\rho) = r$. ☐

Next, we reduce to the case where there are exactly three singular fibers. Given $\Sigma(a_1, \dots, a_n)$, let $p = a_3 \cdots a_n$. Denote the generators for $\pi_1(\Sigma(a_1, \dots, a_n))$ corresponding to the singular fibers by $x_1, \dots, x_n$, and those for $\pi_1(\Sigma(a_1, a_2, p))$ by y_1, y_2, and z respectively; we continue to write h for the central generator. Note that the a_i's are not assumed to be prime in the following lemma; their primality will instead be used later.

Lemma 5.10. *Suppose that $n \ge 4$. Then there exists a surjective homomorphism*

$$f\colon \pi_1(\Sigma(a_1, \dots, a_n)) \to \pi_1(\Sigma(a_1, a_2, p))$$

such that, for every irreducible $\rho \in \mathcal{R}(\Sigma(a_1, a_2, p))$ where

- *$\rho(y_1) \ne \pm 1$ and $\rho(y_2) \ne \pm 1$; and*

- *$\rho(z)$ is conjugate to $e^{i\pi\ell_z/p}$, where ℓ_z is relatively prime to p,*

we have that $f^*\rho \in \mathcal{R}(\Sigma(a_1,\dots,a_n))$ is irreducible and $f^*\rho(x_i) \neq \pm 1$ for all i; in other words, $\dim_{\mathbb{R}} H^1(\Sigma(a_1,\dots,a_n);\mathrm{Ad}_{f^*\rho})$ is maximal.

Recall that there exists a degree-1 map from $\Sigma(a_1,\dots,a_n)$ to $\Sigma(a_1,a_2,a_3\cdots a_n)$ given by pinching along a suitable vertical torus in the Seifert fibration. The homomorphism f above is induced by this map.

Proof. Fix a presentation $(b;(a_1,b_1),\dots,(a_n,b_n))$ for $\Sigma(a_1,\dots,a_n)$, and let $q=\sum_{i=3}^n pb_i/a_i$. Then

$$\Sigma(a_1,a_2,p)=S^2(b;(a_1,b_1),(a_2,b_2),(p,q)).$$

(Since the a_i's are pairwise relatively prime, p and q are relatively prime.) The two fundamental groups are

$$\pi_1(\Sigma(a_1,\dots,a_n)) \cong \left\langle x_1,\dots,x_n,h \,\middle|\, h \text{ central},\ x_i^{a_i}=h^{-b_i},\ x_1\cdots x_n=h^{-b} \right\rangle,$$
$$\pi_1(\Sigma(a_1,a_2,p)) \cong \left\langle y_1,y_2,z,h \,\middle|\, h \text{ central},\ y_i^{a_i}=h^{-b_i},\ z^p=h^{-q},\ y_1y_2z=h^{-b} \right\rangle.$$

With these presentations, we now define f. Since the a_i's are pairwise relatively prime, for each $i\geq 3$, we may choose an integer η_i such that $\eta_i p/a_i \equiv 1 \bmod a_i$. Clearly, η_i and a_i are relatively prime for each i. We define

$$f(x_1)=y_1,\quad f(x_2)=y_2,\quad f(x_i)=z^{\alpha_i}h^{\beta_i} \text{ for } i\geq 3,\quad f(h)=h,$$

where $\alpha_i=\eta_i p/a_i$ and $\beta_i=(\eta_i q-b_i)/a_i$. (Note that f does not depend on the choice of η_i.) Observe that $\sum_{j=3}^n \alpha_j \equiv 1 \bmod a_i$ for each $i\geq 3$, which implies that $\sum_{j=3}^n \alpha_j \equiv 1 \bmod p$; using this fact, it is straightforward to check that f is a well-defined group homomorphism.

We now claim that, for an irreducible $\rho\in\mathcal{R}(\Sigma(a_1,a_2,p))$ satisfying the conditions in the lemma, we have $f^*\rho(x_i)\neq\pm1$ for $i=1,\dots,n$. This is clear for $i=1$ and $i=2$. For $i\geq 3$, suppose that $f^*\rho(x_i)=\pm1$ for some $i\geq3$; then $\rho(z)^{\eta_i p/a_i}=\pm1$, implying that $e^{i\pi\ell_z\eta_i/a_i}=\pm1$, which is a contradiction since η_i is relatively prime to a_i. ∎

We now demonstrate the existence of an irreducible $\rho\in\mathcal{R}(\Sigma(a_1,a_2,p))$ satisfying the conditions of Lemma 5.10, in the case that the a_i's are pairwise prime.

Proposition 5.11. *Suppose that $n\geq 4$, and that $a_1<\cdots<a_n$ are positive prime numbers. There exists an irreducible $\rho\in\mathcal{R}(\Sigma(a_1,a_2,p))$ such that*

- *$\rho(y_1)\neq\pm1$ and $\rho(y_2)\neq\pm1$; and*
- *$\rho(z)$ is conjugate to $e^{i\pi\ell_z/p}$, where ℓ_z is relatively prime to p.*

Proof. We continue to write $\Sigma(a_1,a_2,p)=S^2(b;(a_1,b_1),(a_2,b_2),(p,q))$, and use the same presentation for $\pi_1(\Sigma(a_1,a_2,p))$ as before. First, we make some general observations. Recall the construction of irreducible SU(2)-representations in the paragraph after Theorem 5.5. Observe that $\rho(z)$ is conjugate to $e^{i\pi\ell_z/p}$ for some ℓ_z relatively prime to p if and only if $\rho(y_1)\rho(y_2)$ is conjugate to $e^{i\pi\ell_z'/p}$ for some ℓ_z' relatively prime to p; thus, the goal is to show that after picking appropriate values for $\rho(h)$, ℓ_1, and ℓ_2 satisfying $(-1)^{\ell_j}=(\rho(h))^{-b_j}$ (for us to define $\rho(y_1)=e^{i\pi\ell_1/a_1}$ and decree $\rho(y_2)$ to be conjugate to $e^{i\pi\ell_2/a_2}$), there exists an integer ℓ_z' such that

(1) ℓ_z' is relatively prime to p;
(2) $(-1)^{\ell_z'}=(\rho(h))^{-bp+q}$; and
(3) $\pi\ell_z'/p$ is in the image of $\phi\colon S_2\to[0,\pi]$.

Since ϕ is continuous, to satisfy (3), we simply need to exhibit choices $s_2, s_2' \in S_2$ (i.e. elements s_2 and s_2' that are conjugate to $e^{i\pi\ell_2/a_2}$) such that

$$\mathrm{Arg}(\rho(y_1)s_2) \leq \frac{\pi\ell_z'}{p} \leq \mathrm{Arg}(\rho(y_1)s_2'). \tag{5.12}$$

Our strategy will be to find s_2, s_2', and two values of ℓ_z' of opposite parities satisfying (1) and (3); then exactly one of them will satisfy (2). Finally, by construction, ρ is not trivial, and thus is irreducible.

First, we consider the special case that $a_1 = 2$. In this case, we may choose a presentation where $b_1 = 1$ and b_2 is odd (at the expense of changing b). We take $\rho(h) = -1$, $\ell_1 = 1$, and $\ell_2 = 1$. We claim that the image of ϕ contains $\pi r_\pm/p$, where $\ell_z' = r_\pm = (p \pm 1)/2$. Note that these two numbers are *both* integers relatively prime to p, and have opposite parities; thus, if $\pi r_\pm/p$ are both in the image of ϕ, the proof will be complete in this case. To prove our claim, note that if we choose $s_2 = e^{-i\pi/a_2}$ and $s_2' = e^{i\pi/a_2}$, which are both conjugate to $e^{i\pi/a_2}$, then since $a_2 < p$, we have

$$\mathrm{Arg}(e^{i\pi/2}e^{-i\pi/a_2}) = \frac{\pi(a_2-2)}{2a_2} \leq \frac{\pi(p\pm 1)}{2p} \leq \frac{\pi(a_2+2)}{2a_2} = \mathrm{Arg}(e^{i\pi/2}e^{i\pi/a_2});$$

in other words, (5.12) is satisfied.

We may now assume that all the a_i's are odd. Next, we consider the special case that $a_1 = 3$ and $a_2 = 5$. Again, we may choose a presentation where b_1 and b_2 are both odd, and take $\rho(h) = -1$ and $\ell_1 = \ell_2 = 1$. We again claim that the image of ϕ contains $\pi r_\pm/p$, where $\ell_z' = r_\pm = (p \pm 1)/2$. Indeed, if we choose $s_2 = e^{-i\pi/5}$ and $s_2' = e^{i\pi/5}$, both of which are conjugate to $e^{i\pi/5}$, then since $p > 15$, we have

$$\mathrm{Arg}(e^{i\pi/3}e^{-i\pi/5}) = \frac{2\pi}{15} < \frac{\pi(p\pm 1)}{2p} < \frac{8\pi}{15} = \mathrm{Arg}(e^{i\pi/3}e^{i\pi/5}),$$

and (5.12) is satisfied.

By dispensing with the two cases above, we may assume that all the a_i's are odd, and further that $1/a_1 + 1/a_2 < 1/2$. In this case, there are several choices we could take for ℓ_j and $\rho(h)$; for concreteness, we choose a presentation where b_1 and b_2 are both even, and take $\ell_1 = \ell_2 = 2$ and $\rho(h) = 1$. We choose $s_2 = e^{-i\pi 2/a_2}$ and $s_2' = e^{i\pi 2/a_2}$, and compute the arguments to be

$$0 < \mathrm{Arg}(e^{i\pi 2/a_1}e^{\pm i\pi 2/a_2}) = 2\pi\left(\frac{1}{a_1} \pm \frac{1}{a_2}\right) < \pi.$$

As before, we now wish to find two values r and r' for ℓ_z', of opposite parities, each relatively prime to p, such that (5.12) is satisfied, i.e.

$$2\left(\frac{1}{a_1} - \frac{1}{a_2}\right) \leq \frac{r}{p}, \frac{r'}{p} \leq 2\left(\frac{1}{a_1} + \frac{1}{a_2}\right).$$

Let I denote the interval governed by the above inequality. Note that the length of this interval is $4/a_2$, and $4/a_3 < 4/a_2 < 1$. Therefore, we may choose an integer k with $0 < k < k+2 < a_3$, such that $[k/a_3, (k+2)/a_3] \subset I$; we can rewrite this as

$$\left[\frac{k\frac{p}{a_3}}{p}, \frac{(k+2)\frac{p}{a_3}}{p}\right] \subset I.$$

In fact, since $p/a_3 > a_3$, we have

$$\frac{k\frac{p}{a_3} + a_3}{p}, \frac{k\frac{p}{a_3} + 2a_3}{p} \in I.$$

Let $r = kp/a_3 + a_3$ and $r' = kp/a_3 + 2a_3$. Note that r and r' are between 0 and p, and have opposite parities since a_3 is odd. It remains to see that r and r' are relatively prime to p. First, since $0 < k < a_3$ and a_3 is prime, we see that k is relatively prime to a_3; thus, $r, r' \equiv kp/a_3 \not\equiv 0 \bmod a_3$. At the same time, for $i > 3$, we observe that $r \equiv a_3 \bmod a_i$ and $r' \equiv 2a_3 \bmod a_i$; since the a_i's are odd, prime, and greater than a_3, we have that a_3 and $2a_3$ are also relatively prime to a_i. Putting this together, we conclude that r, r' are relatively prime to p, which completes the proof. □

Proof of Proposition 5.8. Since the Casson invariant of any Seifert fibered homology sphere is never zero, we have that the result trivially holds for $n = 3$. For $n \geq 4$, the result follows from combining Lemma 5.9, Lemma 5.10, and Proposition 5.11. □

Proof of Theorem 5.1. This follows directly from Proposition 5.8, by an application of Proposition 1.25. □

5.4. **Ribbon homology cobordisms from connected sums.** In this subsection, we prove Corollary 1.30, which will follow quickly from Proposition 5.2.

Proof of Corollary 1.30. First, we fix some notation. Let π be a group, and let G be a compact, connected Lie group. Fix a presentation $\langle a_1, \dots, a_g \,|\, w_1, \dots, w_r \rangle$ of π. For each $\rho \in \mathcal{R}_G(\pi)$, the words w_i give a smooth map $\Phi\colon G^g \to G^r$; denote by ϕ_ρ the derivative of Φ at $(\rho(a_1), \dots, \rho(a_g))$, and we define $\omega_G\colon \mathcal{R}_G(\pi) \to \mathbb{Z}_{\geq 0}$ by

$$\omega_G(\rho) = \dim_{\mathbb{R}} \ker(\phi_\rho).$$

(The reader may wish to compare ϕ_ρ here with the map ϕ in (5.4).) It is not difficult to check that ω_G is independent of the presentation of π. We also define

$$\omega_G(\pi) = \max_{\rho \in \mathcal{R}_G(\pi)} \omega_G(\rho),$$

and write $\omega_G(X)$ for $\omega_G(\pi_1(X))$, for a path-connected space X.

Suppose that Y_-, Y_+, W, ρ_-, ρ_+, and ρ_W are as in Proposition 1.22. By Proposition 5.2, we have

$$\omega_G(\rho_-) \leq \omega_G(\rho_W) \leq \omega_G(\rho_+).$$

Indeed, since $\pi_1(W)$ is obtained from $\pi_1(Y_+)$ by adding relations, the matrix for $\omega_G(\rho_W)$ contains that for $\omega_G(\rho_+)$ as a block, with additional rows; similarly, the matrix for ϕ_{ρ_W} contains that for ϕ_{ρ_-} as a block, with m additional rows and columns. Consequently, we see that

$$\omega_G(Y_-) \leq \omega_G(W) \leq \omega_G(Y_+). \tag{5.13}$$

Now suppose that there exists a ribbon $\mathbb{Q}$-homology cobordism $W\colon Y \,\sharp\, N \to Y$. Homology considerations show that N must be a $\mathbb{Z}$-homology sphere (cf. Remark 1.10 and Lemma 6.1). Therefore, there exists a non-trivial, finite, perfect quotient H of $\pi_1(N)$, with an irreducible representation from H to $\mathrm{U}(n)$, for some $n \geq 2$. (Of course, the possible choices for n depend on H.) Let $\eta\colon \pi_1(N) \to \mathrm{U}(n)$ be the associated representation, and choose $\rho \in \mathcal{R}_{\mathrm{U}(n)}(Y)$ that maximizes $\omega_{\mathrm{U}(n)}$. Consider the free product representation $\rho * \eta\colon \pi_1(Y \,\sharp\, N) \to \mathrm{U}(n)$. It follows from the definitions that

$$\omega_{\mathrm{U}(n)}(\rho * \eta) = \omega_{\mathrm{U}(n)}(\rho) + \omega_{\mathrm{U}(n)}(\eta).$$

It is easy to see that since η is irreducible, we have $\omega_{\mathrm{U}(n)}(\eta) > 0$. We see that $\omega_{\mathrm{U}(n)}(Y \sharp N) > \omega_{\mathrm{U}(n)}(Y)$, which contradicts (5.13). □

5.5. **A character variety approach to Theorem 1.11.** In this subsection, we sketch a different approach to prove Theorem 1.11 and Theorem 1.12. For simplicity, we focus on the proof of Theorem 1.11. In particular, let Y_- and Y_+ be $\mathbb{Z}$-homology spheres, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. (See Remark 1.10.) Our approach in this section is based on the relationship between the character varieties of Y_- and Y_+. A key component of our proof is an energy argument that also appears in [BD95b, Fuk96] and the forthcoming work [DFL].

Fix a Riemannian metric on Y_- and a cylindrical metric on $D(W)$ that is compatible with the metric on Y_-. For simplicity, we first assume that these metrics allow us to define the instanton Floer homology $\mathrm{I}(Y_-)$ and the cobordism map $\mathrm{I}(D(W))$ without perturbing the Chern–Simons functional of Y_- or the ASD equation on $D(W)$. In particular, $\mathrm{I}(Y_-)$ is the homology of a chain complex $(\mathrm{C}(Y_-), d)$, where $\mathrm{C}(Y_-)$ is generated by gauge equivalence classes of non-trivial flat connections, or equivalently, non-trivial elements of the character variety of Y_-. The cobordism map $\mathrm{I}(D(W))$ is defined using the moduli spaces $\mathcal{M}(D(W); \alpha_1, \alpha_2)$, i.e. gauge equivalence classes of solutions of the (unperturbed) ASD equation

$$F(A)^+ = 0, \tag{5.14}$$

where A is an $\mathrm{SU}(2)$-connection on $D(W)$ asymptotic to the flat $\mathrm{SU}(2)$-connections α_1 and α_2 on the ends of $D(W)$.

Let $\mathcal{B}(D(W); \alpha_1, \alpha_2)$ be the space of gauge equivalence classes of connections on $D(W)$ that are asymptotic to α_1 and α_2 on the ends (that may or may not satisfy (5.14)). For a connection A representing an element of $\mathcal{B}(D(W); \alpha_1, \alpha_2)$, the topological energy of A, given by the Chern–Weil integral

$$\mathcal{E}(A) = \frac{1}{8\pi^2} \int_{D(W)} \mathrm{tr}(F(A) \wedge F(A)),$$

can easily be verified to be invariant under the action of the gauge group, and also under continuous deformation of A. Moreover, (5.14) implies that, for connections A that represent an element in $\mathcal{M}(D(W); \alpha_1, \alpha_2)$, we always have $\mathcal{E}(A) \geq 0$, and $\mathcal{E}(A) = 0$ if and only if A is a flat connection. We will also need the following fact, which says that the topological energy of A determines the dimension of the component of the moduli space $\mathcal{M}(D(W); \alpha_1, \alpha_2)$ that contains A.

Lemma 5.15. *There exists a function ϵ that associates to each flat connection α on Y_- a real number $\epsilon(\alpha)$, such that the equality*

$$d = 8\mathcal{E}(A) + \epsilon(\alpha_1) - \epsilon(\alpha_2)$$

holds whenever $[A] \in \mathcal{M}(D(W); \alpha_1, \alpha_2)_d$.

Proof. To each connection A representing an element of $\mathcal{M}(D(W); \alpha_1, \alpha_2)$, we may associate the ASD operator $\mathcal{D}_A$, which is an elliptic operator; if A represents an element of $\mathcal{M}(D(W); \alpha_1, \alpha_2)_d$, then d is equal to the index of $\mathcal{D}_A$. Therefore, it suffices to show that, for some choice of ϵ,

$$\mathrm{ind}(\mathcal{D}_A) = 8\mathcal{E}(A) + \epsilon(\alpha_1) - \epsilon(\alpha_2). \tag{5.16}$$

We first verify this formula when $\alpha_1 = \alpha_2$. Since index and topological energy are invariant under continuous deformation, we may assume without loss of generality that the connection A is the pull-back of a fixed flat connection α_1 on the cylindrical ends of $D(W)$. In particular, A induces a connection $\overline{A}$ on the closed 4-manifold $\overline{D(W)}$ obtained by gluing the incoming and the outgoing ends of $D(W)$ by the identity. Clearly, the topological energy of A and $\overline{A}$ are equal to each other. Moreover, the additive property of indices with respect to gluing (see [Don02, Chapter 3]) implies that $\mathrm{ind}(\mathcal{D}_A) = \mathrm{ind}(\mathcal{D}_{\overline{A}})$. Since $\overline{D(W)}$ has the same $\mathbb{Z}$-homology as $S^1 \times S^3$, the standard index theorems for the ASD operator on closed 4-manifolds imply that

$$\mathrm{ind}(\mathcal{D}_{\overline{A}}) = 8\mathcal{E}(\overline{A}).$$

This shows that (5.16) holds in the case that $\alpha_1 = \alpha_2$.

In the more general case, we fix an arbitrary flat connection α_1 on Y_-, and set $\epsilon(\alpha_1) = 0$. For any other flat connection α, we take an arbitrary connection B on $Y_- \times \mathbb{R}$ that is equal to the pull-backs of representatives of α_1 and α on $(-\infty, -1] \times Y_-$ and $[1, \infty) \times Y_-$ respectively, and define

$$\epsilon(\alpha) = \mathrm{ind}(\mathcal{D}_B) - \frac{1}{\pi^2} \int_{Y_- \times \mathbb{R}} \mathrm{tr}(F(B) \wedge F(B)).$$

Another application of the additive property of the index of ASD operators with respect to gluing completes the proof of the lemma. □

Lemma 5.17. *If the moduli space $\mathcal{M}(D(W); \alpha_1, \alpha_2)_0$ is not empty, then either $\epsilon(\alpha_2) > \epsilon(\alpha_1)$, or $\alpha_1 = \alpha_2$. Moreover, the moduli space $\mathcal{M}(D(W); \alpha_1, \alpha_1)_0$ consists of an odd number of flat connections.*

Proof. Lemma 5.15 implies that, if there is an element $[A]$ in $\mathcal{M}(D(W); \alpha_1, \alpha_2)_0$, then $\epsilon(\alpha_2) \geq \epsilon(\alpha_1)$. Moreover, if $\epsilon(\alpha_1) = \epsilon(\alpha_2)$, then the connection A has to be flat, which is to say that A represents an element of the character variety $\chi_{\mathrm{SU}(2)}(D(W))$. In particular, Proposition 5.2 implies that $\alpha_1 = \alpha_2$. By assumption, any element of $\mathcal{M}(D(W); \alpha_1, \alpha_1)_0$ is cut out regularly, and we do not need to perform any perturbation. Regularity of a flat connection A on $D(W)$ is equivalent to the property that $H^1(D(W); \mathrm{Ad}_A)$ is trivial. The proof of Proposition 5.2 implies that the SU(2)-representations of $\pi_1(D(W))$ that extend a given representation of $\pi_1(Y_-)$ is the set of solutions of $K(g_1, \ldots, g_m) = 1$, where $K\colon \mathrm{SU}(2)^m \to \mathrm{SU}(2)^m$ is a map of degree ± 1. Since the solutions of these equations are cut out transversely, the number of solutions of this extension problem is an odd integer. □

Lemma 5.17 implies that if we sort flat connections on Y_- based on their ϵ-values, then the chain map $\mathrm{C}(D(W))$ is upper triangular with non-zero diagonal entries. In particular, $\mathrm{I}(D(W))$ is an isomorphism.

In general, we need to consider perturbations of the Chern–Simons functional of Y_- and the ASD equation on $D(W)$. There are standard functions on the space of connections on Y_- that give rise to perturbed Chern–Simons functionals of Y_- (see [Don02, Chapter 5]) that are sufficient to define the instanton Floer homology $\mathrm{I}(Y_-)$. Any such perturbation can be extended to a perturbation of the ASD equation on $D(W)$ that is *time independent* in the sense defined by Braam and Donaldson [BD95a]. The main point of considering such perturbations is that, even after we slightly modify the definition of topological energy, the solutions of the perturbed ASD equation will still have non-negative topological energy. Having fixed the above, another technical issue would be to know whether the solutions of the perturbed ASD equation with vanishing topological energy are cut out regularly. If we happen to know that our chosen perturbation has this additional property, then we can proceed as above to show that the map $\mathrm{I}(D(W))$ is an isomorphism. However, the authors have not checked whether there is a time-independent perturbation with this property.

6. Thurston geometries and ribbon homology cobordisms

In this section, we provide further obstructions to ribbon $\mathbb{Q}$-homology cobordisms between compact 3-manifolds that arise from the Thurston geometries that these manifolds admit.

We first prove a homology version of Theorem 1.1.

Lemma 6.1. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W\colon Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism. Then*

(1) *The inclusion of Y_- into W induces an inclusion on H_1; and*
(2) *The inclusion of Y_+ into W induces a surjection on H_1.*

Proof. For (1), view W as constructed by attaching 1- and 2-handles to Y_-; the fact that W is a ribbon $\mathbb{Q}$-homology cobordism implies that the attaching circles of the 2-handles are linearly independent in $H_1(Y_- \,\sharp\, m(S^1 \times S^2))/H_1(Y_-)$, implying that $H_2(W, Y_-) = 0$. The statement now follows from the long exact sequence associated to the pair (W, Y_-).

The statement (2) follows from Abelianizing the statement of Theorem 1.1 (2). □

Lemma 6.2. *Let $\mathcal{P}$ be one of the following properties of groups:*

(1) *Finite;*
(2) *Cyclic;*
(3) *Abelian;*
(4) *Nilpotent;*
(5) *Solvable; or*
(6) *Virtually $\mathcal{P}'$, where $\mathcal{P}'$ is one of the properties above.*

Let Y_- and Y_+ be compact 3-manifolds. Suppose that $\pi_1(Y_+)$ has property $\mathcal{P}$, while $\pi_1(Y_-)$ does not. Then there does not exist a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+.

Proof. Suppose that there exists a ribbon $\mathbb{Q}$-homology cobordism $W\colon Y_- \to Y_+$. By Theorem 1.1, $\pi_1(Y_-)$ is a subgroup of $\pi_1(W)$, which is a quotient of $\pi_1(Y_+)$. For (1) to (5), the lemma is now evident. For (6), a simple algebraic argument shows that if $\mathcal{P}'$ is a property inherited by subgroups (resp. quotients), then so is the property "virtually $\mathcal{P}'$". □

Let Y be a compact 3-manifold with empty or toroidal boundary. Then according to [AFW15, Theorem 1.11.1], Y belongs to one of the classes in Figure 5 (if Y is closed) or Figure 6 (if Y has toroidal boundary). Indeed, if Y is spherical or has a finite solvable cover $\widetilde{Y}$ that is a torus bundle, then Y is obviously closed. In the latter case, by [AFW15, Theorem 1.10.1], $\widetilde{Y}$ admits either a Euclidean, Nil-, or Sol-geometry; by [AFW15, Theorem 1.9.3], Y is itself geometric, and, according to [AFW15, Table 1.1], also admits one of these geometries. That the last rows of Figure 5 and Figure 6 encompass all remaining cases is a consequence of the Geometric Decomposition Theorem; see [AFW15, Theorem 1.9.1]. Note that five out of seven $(S^2 \times \mathbb{R})$-manifolds [Sco83, p. 457] either have S^2 as a boundary component or are not orientable; the other two are $S^1 \times S^2$ and $\mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$. Also, if Y is geometric and has toroidal boundary, and is not homeomorphic to $K \mathbin{\widetilde{\times}} I$, $S^1 \times D^2$, or $T^2 \times I$, then it must have $(\mathbb{H}^2 \times \mathbb{R})$-, $\widetilde{\mathrm{SL}(2, \mathbb{R})}$-, or hyperbolic geometry.

Theorem 6.3. *Suppose that Y_- and Y_+ are compact 3-manifolds with empty or toroidal boundary that belong to distinct classes in Figure 5 or Figure 6, such that there does not exist a sequence of arrows from the class of Y_- to the class of Y_+. Then there does not exist a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+.*

Proof. We begin by inspecting Figure 5, which consists of two columns corresponding to whether $\pi_1(Y)$ is finite; we call them the *finite column* and the *infinite column* respectively. Focusing on each of these columns separately, successive application of Lemma 6.2 shows that there are no arrows that point up. Of course, one must check that the manifolds in each class indeed have fundamental groups that are characterized by the property on the left. For the finite column, Y is a lens space if and only if $\pi_1(Y)$ is cyclic, and $\pi_1(Y)$ is solvable unless it is the direct sum of a cyclic group and the binary icosahedral group P_{120} (in which case Y is known as a *type-***I** *manifold*); the only spherical

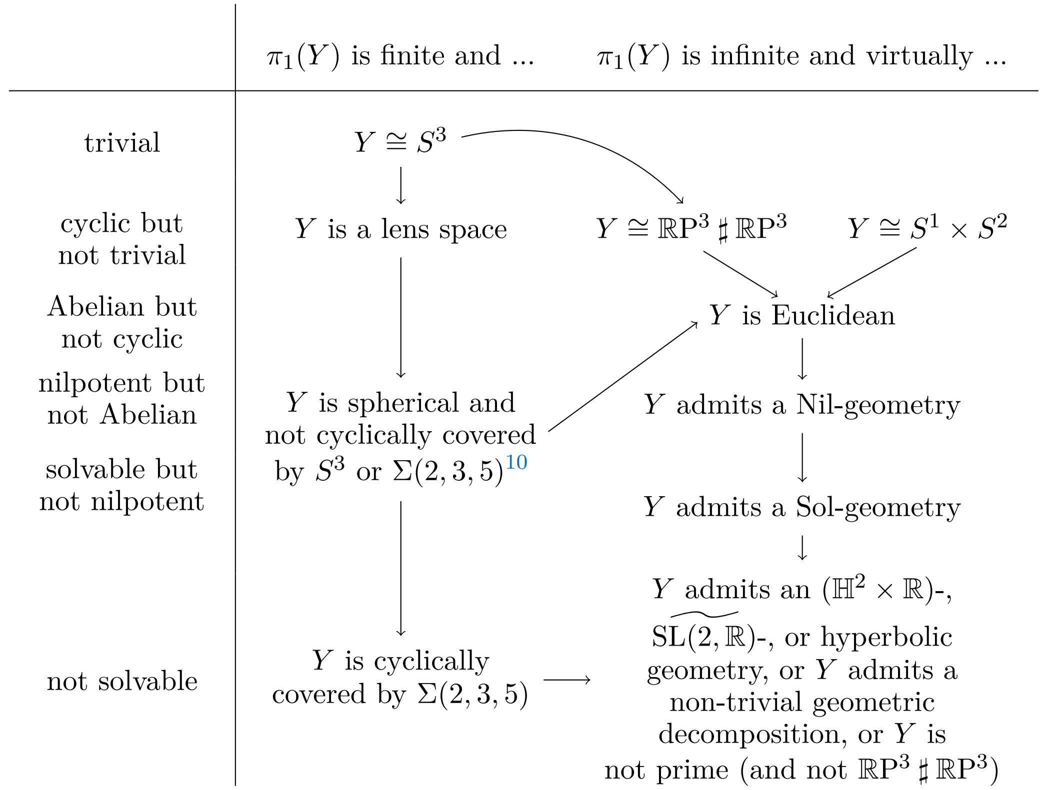

FIGURE 5. Hierarchy of ribbon $\mathbb{Q}$-homology cobordisms of closed 3-manifolds. For 3-manifolds with infinite π_1, the adverb "virtually" applies to all adjectives in the leftmost column. (For example, the fundamental group of a Euclidean manifold is virtually Abelian but not virtually cyclic.)

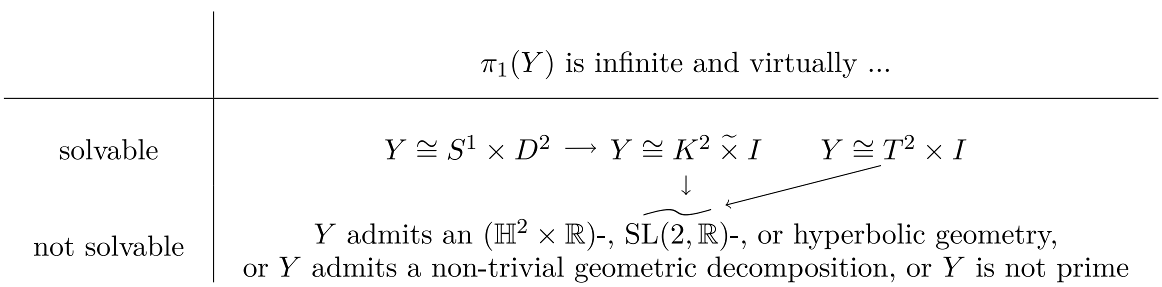

FIGURE 6. Hierarchy of ribbon $\mathbb{Q}$-homology cobordisms of compact 3-manifolds with toroidal boundary.

3-manifold with fundamental group P_{120} is the Poincaré homology sphere $\Sigma(2,3,5)$. See [AFW15, Section 1.7] for a discussion. For the infinite column, the classification by π_1 follows from [AFW15, Table 1.1 and Table 1.2]; the fact there are no arrows between $\mathbb{RP}^3 \sharp \mathbb{RP}^3$ and $S^1 \times S^2$ reflects the fact that their $\mathbb{Q}$-homologies have different ranks.

[10]For this class, $\pi_1(Y)$ is solvable but not Abelian; it is nilpotent if and only if it is a direct sum of a cyclic group and the generalized quaternion group Q_{2^n}; all such manifolds Y are prism (i.e. type-**D**) manifolds. One could accordingly stratify the class into two classes with an arrow between them.

We now move on to arrows between the two columns. First, there are clearly no arrows from the infinite to the finite column. Also, the ranks of the $\mathbb{Q}$-homologies obstruct any arrow from the finite column to $S^1 \times S^2$. The only remaining obstructions are as follows. There are no arrows

(1) From lens spaces to $\mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$. Indeed, Lemma 6.1 implies that $H_1(Y_-)$ is a subgroup of a quotient of $H_1(\mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3)$, and thus can only be the trivial group, $\mathbb{Z}/2$, or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since Y_- is a lens space, it has non-trivial cyclic H_1; thus, $H_1(Y_-) \cong \mathbb{Z}/2$. Suppose there exists a ribbon $\mathbb{Q}$-homology cobordism from Y_- to $\mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$; then $(-Y_-) \,\sharp\, \mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$ bounds a $\mathbb{Q}$-homology ball. This implies that $|H_1(-Y_- \,\sharp\, \mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3)| = 8$ is a perfect square, which is a contradiction.
(2) From spherical manifolds that are not cyclically covered by S^3 or $\Sigma(2,3,5)$ to $\mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$. As before, Lemma 6.1 implies that $H_1(Y_-)$ is either $\mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, and, supposing that there exists a ribbon $\mathbb{Q}$-homology cobordism $W \colon Y_- \to \mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$, an argument involving perfect squares shows that $H_1(Y_-) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Note that this implies that W is in fact a $\mathbb{Z}$-homology cobordism, and so the set of d-invariants [OSz03b] of Y_- must coincide with that of $\mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$. According to [Sei33], the only spherical 3-manifolds Y_- with $H_1(Y_-) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ are type-**D** manifolds. By [Doi15, Example 15], the set of d-invariants of such manifolds is never $\{-1/2, 1/2, 0, 0\}$, which is the set of d-invariants of $\mathbb{RP}^3 \,\sharp\, \mathbb{RP}^3$; thus, we get a contradiction.
(3) From type-**I** manifolds to any manifold with solvable π_1; for manifolds in the infinite column, these are exactly the ones with virtually solvable π_1 (see [AFW15, Theorem 1.11.1]), i.e. all classes except the one in the last row.

For Figure 6, it suffices to observe that, in the first row, the $\mathbb{Q}$-homology of $T^2 \times I$ differs from those of $S^1 \times D^2$ and $K^2 \,\widetilde{\times}\, I$, and Lemma 6.1 shows that there is no ribbon $\mathbb{Q}$-homology cobordism from $K^2 \,\widetilde{\times}\, I$ to $S^1 \times D^2$. □

Remark 6.4. It is easy to construct a ribbon $\mathbb{Q}$-homology cobordism from $S^1 \times D^2$ to $K^2 \,\widetilde{\times}\, I$.

Remark 6.5. Boyer, Gordon, and Watson [BGW13, Theorem 2] show that all $\mathbb{Q}$-homology spheres with Sol-geometry are L-spaces. By Corollary 1.28, there do not exist ribbon $\mathbb{Z}/2$-homology cobordisms from any $\mathbb{Q}$-homology sphere that is not an L-space to a manifold that admits a Sol-geometry. Observe that this is consistent with Figure 5, since $\mathbb{Q}$-homology spheres with spherical, $(S^2 \times \mathbb{R})$-, Euclidean, and Nil-geometry are also L-spaces [BGW13, Proposition 5].

7. Surgery obstructions

In this section, we give some applications of the work above on ribbon $\mathbb{Q}$-homology cobordisms to reducible Dehn surgery problems. Recall that a closed 3-manifold Y is aspherical if it is irreducible and has infinite, torsion-free fundamental group. For an aspherical 3-manifold Y, we have $H^3(\pi_1(Y);\mathbb{Z}) \cong H^3(Y)$, where on the left we are computing group cohomology with coefficients in $\mathbb{Z}$ equipped with the trivial module structure. If in addition Y is closed, then $H^3(\pi_1(Y);\mathbb{Z}) \cong \mathbb{Z}$. (Recall our convention that all manifolds are oriented.)

Proposition 7.1. *Suppose that Y is an aspherical $\mathbb{Q}$-homology sphere, K is a null-homotopic knot in Y, and $Y_0(K) \cong N \,\sharp\, (S^1 \times S^2)$. Then N covers Y; in particular, $N \ncong S^3$.*

Proof. First, we observe that since K is a null-homotopic knot, there exists a degree-1 map from $Y_0(K)$ to Y. Thus, $\pi_1(N \,\sharp\, (S^1 \times S^2))$ surjects onto $\pi_1(Y)$. In particular, we see that $N \ncong S^3$, since $\pi_1(Y)$ is not a quotient of $\mathbb{Z}$. By the Geometrization Theorem, we have that $\pi_1(N) \neq 0$.

Consider the prime decomposition $N \cong N_1 \,\sharp \cdots \sharp\, N_k$, for some $k \geq 1$. Since $b_1(Y_0(K)) = 1$, we see that N cannot contain any $S^1 \times S^2$ summands. Thus, N_i is irreducible for each $i \in \{1, \dots, k\}$.

Consider now the cobordism $W\colon Y \to N$ obtained by attaching a 0-framed 2-handle along K, and then a 3-handle along some $\{p\} \times S^2$ in the $S^1 \times S^2$ summand. Since the 2-handle is attached along a null-homotopic knot, and the 3-handle does not affect π_1, we see that the inclusion $\iota\colon Y \to W$ induces an isomorphism on π_1. By flipping W upside down and reversing orientation, we obtain a ribbon $\mathbb{Q}$-homology cobordism $-W\colon N \to Y$, which implies that $\pi_1(N)$ embeds as a subgroup of $\pi_1(W) \cong \pi_1(Y)$ by Theorem 1.1. Since Y is aspherical, $\pi_1(Y)$ is an infinite torsion-free group; this implies that $\pi_1(N)$, and hence $\pi_1(N_i)$ for each $i \in \{1, \dots, k\}$, are also infinite torsion-free groups. In other words, N_i is an aspherical 3-manifold.

We now claim that, in fact, $k = 1$. Indeed, let $\widetilde{Y}$ be the connected cover of Y corresponding to $\pi_1(N)$. On the one hand, $H^3(\widetilde{Y})$ is isomorphic to either $\mathbb{Z}$ or 0, depending on whether $\widetilde{Y}$ is a finite- or infinite-sheeted cover of Y. On the other hand, since $\widetilde{Y}$ covers Y, it is also aspherical, and so

$$H^3(\widetilde{Y}) \cong H^3(\pi_1(\widetilde{Y}); \mathbb{Z}) \cong H^3(\pi_1(N); \mathbb{Z}) \cong H^3(\pi_1(N_1); \mathbb{Z}) \oplus \cdots \oplus H^3(\pi_1(N_k); \mathbb{Z}) \cong \mathbb{Z}^k,$$

where we have used the fact that $K(G_1 * G_2, 1) \simeq K(G_1, 1) \vee K(G_2, 1)$, which implies that $H^i(G_1 * G_2; \mathbb{Z}) \cong H^i(G_1; \mathbb{Z}) \oplus H^i(G_2; \mathbb{Z})$ for $i \neq 0$. This shows that N is in fact irreducible, and thus aspherical.

Since $\widetilde{Y}$ and N are both closed, aspherical 3-manifolds with isomorphic fundamental groups, they are homeomorphic. □

Remark 7.2. Gordon [Gor81] asks whether the Gromov norm of Y_+ is necessarily at least that of Y_- if there exists a ribbon $\mathbb{Q}$-homology cobordism from Y_- to Y_+. (Technically, he asks this only in the context of ribbon concordances of knots, but the more general question seems equally reasonable.) Assuming an affirmative answer, in the set-up in Proposition 7.1, we would have that if Y is hyperbolic, then $\mathrm{vol}(Y) \geq \mathrm{vol}(N) = \deg(p) \cdot \mathrm{vol}(Y)$, where $p\colon N \to Y$ is the covering map; this implies that p must have degree 1, and so p is a homeomorphism from N to Y.

While it is not clear whether one could prove a statement analogous to Proposition 7.1 for non-aspherical Y in general, in the case that Y is an L-space homology sphere, we obtain an even stronger conclusion without any restriction on the homotopy class of K.

Proposition 7.3. *Suppose that Y is an L-space homology sphere and K is a non-trivial knot in Y. Then $Y_0(K)$ does not contain an $S^1 \times S^2$ summand.*

Proof. Suppose that $Y_0(K) \cong N \,\sharp\, (S^1 \times S^2)$. Note that N is necessarily a $\mathbb{Z}$-homology sphere, and possibly S^3. Consider the cobordism $W\colon Y \to N$ obtained by attaching a 0-framed 2-handle along K, and then a 3-handle along some $\{p\} \times S^2$ in the $S^1 \times S^2$ summand. Since W is a $\mathbb{Z}$-homology cobordism, we have that $d(Y) = d(N)$. Flipping W upside down and reversing its orientation, we obtain a ribbon $\mathbb{Z}$-homology cobordism $-W\colon N \to Y$. Since Y is an L-space, Corollary 1.28 implies that N is also an L-space. It follows from the surgery exact triangle in Heegaard Floer homology [OSz03b, Lemma 3.1] that $Y_{1/n}(K)$ is an L-space homology sphere for any integer n. Recall that for knots in S^3, if r-surgery yields an L-space, then $|r| \geq 2g(K) - 1$ [OSz11, Corollary 1.4]. The same argument in fact applies in any L-space homology sphere; therefore, we conclude that K is trivial. □

It can be shown that the proposition above is true for an arbitrary L-space Y with the additional assumption that K is null-homologous. (Without this assumption, the conclusion is obviously false.)

Remark 7.4. Gabai's proof of Property R shows that no surgery on a non-trivial knot in S^3 can produce an $S^1 \times S^2$ summand. We leave it as an elementary exercise to prove (without using Property R or Floer homology, but instead using the Geometrization Theorem) that no surgery on a knot in S^3 can produce $N \sharp (S^1 \times S^2)$ where $N \not\cong S^3$, using Theorem 1.1. (Of course, this does not give a proof of Property R.)

Finally, we conclude the section by proving a similar statement for Seifert fibered homology spheres.

Proof of Theorem 1.34. If $Y \cong S^3$, then $N \cong S^3$ by Gabai's proof of the Poénaru Conjecture [Gab87, Corollary 8.3]. If $Y \cong \Sigma(2,3,5)$, then the result follows from Proposition 7.3. Now assume that Y is aspherical. Proposition 7.1 implies that N covers Y. For homology reasons, N is also a $\mathbb{Z}$-homology sphere. Of course, N is Seifert fibered as well. As in the proof of Proposition 7.1, there exists a ribbon $\mathbb{Q}$-homology cobordism from N to Y, which is in fact a $\mathbb{Z}$-homology cobordism because N and Y are both $\mathbb{Z}$-homology spheres. (See Remark 1.10.) By Theorem 1.4, we have

$$\dim_{\mathbb{F}} \mathrm{HF}^{\mathrm{red}}(N) \leq \dim_{\mathbb{F}} \mathrm{HF}^{\mathrm{red}}(Y).$$

Let $p\colon N \to Y$ denote the covering map. By [KL15, Theorem 1.9],

$$\dim_{\mathbb{F}} \mathrm{HF}^{\mathrm{red}}(N) \geq \deg(p) \cdot \dim_{\mathbb{F}} \mathrm{HF}^{\mathrm{red}}(Y).$$

Since aspherical Seifert fibered homology spheres have non-trivial reduced Heegaard Floer homology [Eft09], we deduce that $\deg(p) = 1$; in other words, p is a homeomorphism. ☐

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