MAT 132

Midterm 1 Spring 2017

Name: ____________________ **ID:** ____________________

Problem	1 (10 pts)	2 (10 pts)	3 (10 pts)	4 (10 pts)	5 (10 pts)	6 (10 pts)	7 (15 pts)	8 (25 pts)	Total (100 pts)
Score									

Instructions:

(1) Fill in your name and Stony Brook ID number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no calculators, no phones.

(3) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4) and simplify them as much as possible (e.g. 1/2, not 2/4) to receive full credit.

(4) Answer all questions in the space provided. If you need more room use the blank backs of the pages.

(5) Show your work; correct answers alone will receive only partial credit.

Evaluate the following integrals. Each part worths 10 points:

1. $\displaystyle\int_0^1 \frac{\arctan^2(x)+1}{x^2+1}\,dx$

Solution. We use u-substitution where $u = \arctan(x)$. Then $du = \frac{1}{x^2+1}dx$ and we have:

$$\begin{aligned}
\int_0^1 \frac{\arctan^2(x)+1}{x^2+1}\,dx &= \int_{\arctan(0)}^{\arctan(1)} u^2+1\,du \\
&= \int_0^{\frac{\pi}{4}} u^2+1\,du \\
&= \frac{u^3}{3}+u\Big|_0^{\frac{\pi}{4}} \\
&= \frac{(\frac{\pi}{4})^3}{3}+\frac{\pi}{4} \\
&= \frac{\pi^3}{192}+\frac{\pi}{4}
\end{aligned}$$

Grading: 3 points for finding u and du, 7 points for writing the integral in terms of u and computing the integral. You would lose 3 points if you didn't change your bounds after u-substitution, and 1 or 2 points if you didn't simplify the final answer.

2. $\int e^{3t}\cos(2t)\,dt$

Solution. We use integration by parts where $u = e^{3t}$ and $dv = \cos(2t)dt$. That implies that $du = 3e^{3t}dt$ and $v = \frac{\sin(2t)}{2}$. Therefore, we have:

$$\begin{aligned}\int e^{3t}\cos(2t)\,dt &= e^{3t}\frac{\sin(2t)}{2} - \int 3e^{3t}\frac{\sin(2t)}{2}\,dt \\ &= e^{3t}\frac{\sin(2t)}{2} - \frac{3}{2}\int e^{3t}\sin(2t)\,dt \qquad (1)\end{aligned}$$

Then we pply integration by parts again to the integral in (1) where $u = e^{3t}$ and $dv = \sin(2t)dt$. We have $du = 3e^{3t}dt$ and $v = -\frac{\cos(2t)}{2}$ which implies that:

$$\begin{aligned}e^{3t}\frac{\sin(2t)}{2} - \frac{3}{2}\int e^{3t}\sin(2t)\,dt &= e^{3t}\frac{\sin(2t)}{2} - \frac{3}{2}(-e^{3t}\frac{\cos(2t)}{2} - \int -3e^{3t}\frac{\cos(2t)}{2}\,dt) \\ &= e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) - \frac{9}{4}\int e^{3t}\cos(2t)\,dt)\end{aligned}$$

In summary, we have:

$$\int e^{3t}\cos(2t)\,dt = e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) - \frac{9}{4}\int e^{3t}\cos(2t)\,dt)$$

which implies that:

$$\begin{gathered}(1+\frac{9}{4})\int e^{3t}\cos(2t)\,dt = e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) \implies \\ \frac{13}{4}\int e^{3t}\cos(2t)\,dt = e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) \implies \\ \int e^{3t}\cos(2t)\,dt = \frac{2}{13}e^{3t}\sin(2t) + \frac{3}{13}e^{3t}\cos(2t) + C\end{gathered}$$

At the end, we included the arbitrary constant of integration, because our integral is indefinite.

Grading: 3 points for each of integration by parts, 1 point for carrying out each of integration by parts correctly, 1 point for using the result to compute the integral and 1 point for arbitrary constant of integration

3. $\int_1^3 \frac{3x+1}{x^2-2x-15}\,dx$

Solution. The denominator of this fraction can be factorized as $(x-5)(x+3)$. Therefore, we can use partial fraction decomposition to compute this integral:

$$\frac{3x+1}{x^2-2x-15} = \frac{A}{x-5} + \frac{B}{x+3} \implies$$

$$\frac{3x+1}{x^2-2x-15} = \frac{A(x+3)+B(x-5)}{(x-5)(x+3)} \implies$$

$$3x+1 = A(x+3)+B(x-5) \tag{2}$$

Identity (2) has to hold for all values of x. In particular, we can evaluate it at $x=5$ and $x=-3$:

$$x=5: \qquad 3\times 5+1 = A(5+3)+B(5-5) \implies$$

$$16 = 8\times A \implies A = 2$$

$$x=-3: \qquad 3\times(-3)+1 = A(-3+3)+B(-3-5) \implies$$

$$-8 = -8\times B \implies B = 1$$

In order to find A and B, we can follow the following alternative approach. The equation (2) can be rewritten as:

$$3x+1 = (A+B)x + (3A-5B) \implies$$

$$\begin{cases} A+B=3 \\ 3A-5B=1 \end{cases}$$

By multiplying the first equation by 5 and then adding it up to the second equation, we obtain:

$$8A = 16 \implies A = 2.$$

Similarly, we can multiply the first equation by 3 and then subtract it from the second equation. This implies that:

$$-8B = -8 \implies B = 1.$$

In any case, we have:

$$\begin{aligned}
\int_1^3 \frac{3x+1}{x^2-2x-15}\,dx &= \int_1^3 \frac{2}{x-5} + \frac{1}{x+3}\,dx \\
&= 2\ln(|x-5|) + \ln(|x+3|)|_1^3 \\
&= (2\ln(|3-5|) + \ln(|3+3|)) - (2\ln(|1-5|) + \ln(|1+3|)) \\
&= 2\ln(2) + \ln(6) - 2\ln(4) - \ln(4) \\
&= \ln(\frac{2^2\times 6}{4^2\times 4}) \\
&= \ln(\frac{3}{8})
\end{aligned}$$

Grading: 3 points for factorization of the denominator and setting up partial fraction decomposition, 3 points for finding A and B, 2 points finding the antiderivatives, 2 pointing for computing the integral and simplifying the answer

4. $\int x \ln(x)^2 \, dx$

Solution. Integration by parts can be used to simplified this integral. Let $u = \ln(x)^2$ and $dv = xdx$. Then $du = 2\frac{\ln(x)}{x}$ and $v = \frac{x^2}{2}$:

$$\int x \ln(x)^2 \, dx = \ln(x)^2 \frac{x^2}{2} - \int 2\frac{\ln(x)}{x}\frac{x^2}{2} \, dx$$
$$= \frac{x^2 \ln(x)^2}{2} - \int x \ln(x) \, dx$$

The last integral can be computed by another application of integration by parts. If we pick $u = \ln(x)$ and $dv = xdx$, then $du = \frac{1}{x}$ and $v = \frac{x^2}{2}$. Therefore, we have:

$$\frac{x^2 \ln(x)^2}{2} - \int x \ln(x) \, dx = \frac{x^2 \ln(x)^2}{2} - (\ln(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2} \, dx)$$
$$= \frac{x^2 \ln(x)^2}{2} - \frac{x^2 \ln(x)}{2} + \int \frac{x}{2} \, dx)$$
$$= \frac{x^2 \ln(x)^2}{2} - \frac{x^2 \ln(x)}{2} + \frac{x^2}{4} + C$$

Grading: 8 points for setting up the integration by parts correctly and 2 points for evaluating integration by parts. You would lose 0.5 points for not writing the arbitrary constant of integration. Some of the students interpreted $\ln(x)^2$ as $\ln(x^2)$. They got complete score if they took the integral correctly. Note that $x\ln(x^2) = 2x\ln(x)$ and the above method (integration by parts with $u = \ln(x)$ and $dv = 2xdx$) can be used to take this alternative version of the integral.

5. $\displaystyle\int \frac{\cos(x)\sin(x)}{2-\cos(x)}\,dx$

Solution. We can use u-substitution with $u = \cos(x)$. Then $du = -\sin(x)dx$ and we have:

$$\int \frac{\cos(x)\sin(x)}{2-\cos(x)}\,dx = \int \frac{u}{u-2}\,du$$

The expression in inside the integral on the left hand side can be simplified as:

$$\frac{u}{u-2} = \frac{u-2+2}{u-2} = \frac{u-2}{u-2} + \frac{2}{u-2} = 1 + \frac{2}{u-2} \tag{3}$$

Therefore, we can write::

$$\begin{aligned}\int \frac{u}{u-2}\,du &= \int 1 + \frac{2}{u-2}\,du \\ &= \int 1\,du + 2\int \frac{1}{u-2}\,du \\ &= u + 2\ln(|u-2|) + C \\ &= \cos(x) + 2\ln(2-\cos(x)) + C\end{aligned}$$

In the last step, we plug in $\cos(x)$ for u.

Grading: 2 points for finding u, 1 point for writing correct integral after u-substitution, 2 points for decomposing as in (3), 4 points for computing the integral and 1 point for writing the final answer in terms of x

6. $\int_0^2 t^3 e^{t^2}\, dt$

Solution. Firstly, use u-substitution with $u = t^2$. Then $du = 2tdt$ and we have:

$$\begin{aligned}\int_0^2 t^3 e^{t^2}\, dt &= \int_{0^2}^{2^2} ue^u \,\frac{du}{2} \\ &= \frac{1}{2}\int_0^4 ue^u\, du\end{aligned}$$

The latter integral can be computed using integration by parts. Let $r = u$ and $ds = e^u du$. Then $r = du$ and $s = e^u$, and we can rewrite the last expression as:

$$\begin{aligned}\frac{1}{2}\int_0^4 ue^u\, du &= \frac{1}{2}(ue^u|_0^4 - \int_0^4 e^u\, du) \\ &= \frac{1}{2}(4 \times e^4 - 0 \times e^0 - e^u|_0^4) \\ &= \frac{1}{2}(4e^4 - e^4 + e^0) \\ &= \frac{1}{2}(3e^4 + 1)\end{aligned}$$

Grading: You would lose 1, 2 or 3 points, if you used the above strategy to compute the integral but you got the wrong answer: 1 point for a small oversight, 2 points if you forgot to change the bounds of integration after a u-substitution, and 3 points for a mix of these. You would get 2 points if you attempted an integration by parts that lead nowhere, or 3 points if you set up the problem with the correct integration by parts but couldn't do anything else.

7. (15 points) Albert's boomerang has the shape of the region enclosed by the parabolas $y = x^2 - 3x + 3$ and $y = 2x^2 - 6x + 5$. Find the area of his boomerang.

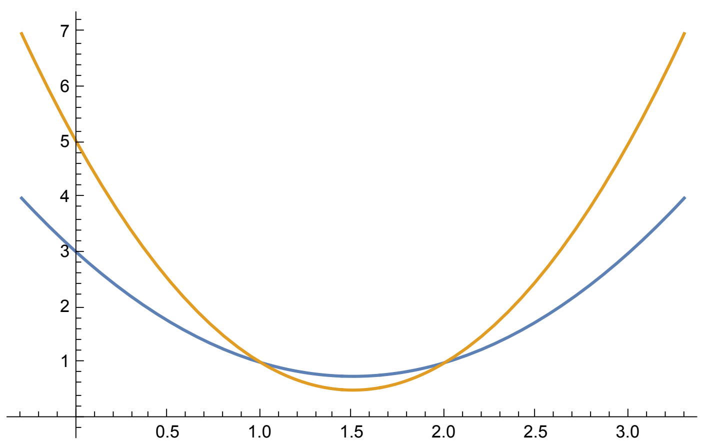
Solution. Firstly, we need to find the intersection points of the two parabolas. If (x,y) lies on the graph of these two curves, then:

$$x^2 - 3x + 3 = 2x^2 - 6x + 5 \implies$$

$$0 = x^2 - 3x + 2 \implies$$

$$x = 1, 2$$

Therefore, the two intersection points are (1,1) and (2,1). We slicing the region enclosed by the



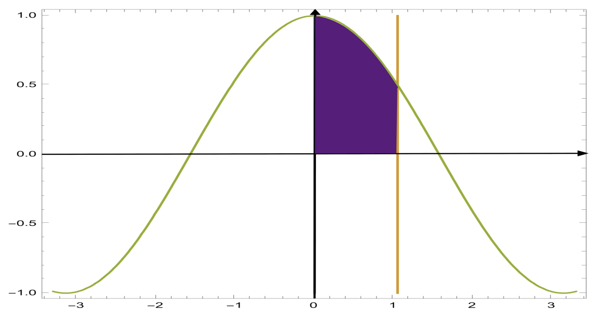
two parabolas vertically. Therefore, we have to use the x-coordinate to parametrize our slices and the possible values of x lie in the interval [1,2]. For $x \in$ [1,2], the length of the slice is equal to $(x^2 - 3x + 3) - (2x^2 - 6x + 5) = 3x - 2 - x^2$. (In order to see which graph is on top in the interval [1,2], we can evaluate our functions at an arbitrary point in (1,2) like $\frac{3}{2}$.) Therefore, the area is equal to:

$$\begin{aligned}
\int_1^2 3x - 2 - x^2 \, dx &= 3\frac{x^2}{2} - 2x - \frac{x^3}{3}|_1^2 \\
&= (3 \times \frac{2^2}{2} - 2 \times 2 - \frac{2^3}{3}) - (3 \times \frac{1^2}{2} - 2 \times 1 - \frac{1^3}{3}) \\
&= (6 - 4 - \frac{8}{3}) - (\frac{3}{2} - 2 - \frac{1}{3}) \\
&= \frac{1}{6}
\end{aligned}$$

Grading: 5 points for finding the intersection points, 5 points for deciding which parabola is on top and 5 points for writing the integral and its evaluation.

8. (25 points) Let $\mathcal{R}$ be the region obtained by rotating the region enclosed by the x-axis, y-axis, $x = \frac{\pi}{3}$, and the curve $y = \cos(x)$.

(a) Sketch the shape of this region in the coordinate plane.



Grading: 5 points for the figure

(b) Let $\mathcal{S}$ be the solid given by rotating the region $\mathcal{R}$ about the y-axis. Find the volume of $\mathcal{S}$.

Solution. We slice the region $\mathcal{R}$ vertically. Thus we have to use the x-axis to parametrize our slices, and for each value of $x \in [0, \frac{\pi}{3}]$ we have a slice. Each such slice determines a cylindrical shell in the solid $\mathcal{S}$. The height of this shell is $\cos(x)$ and the radius is equal to x. Therefore, the volume of $\mathcal{S}$ is equal to:

$$\int_0^{\frac{\pi}{3}} 2\pi x \cos(x)\, dx$$

We can use integration by parts to compute this integral. Define the parts by $u = 2\pi x$ and $dv = \cos(x)dx$. Therefore, we have $du = 2\pi dx$ and $v = \sin(x)$:

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} 2\pi x \cos(x)\, dx &= 2\pi x \sin(x)|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 2\pi \sin(x)\, dx \\
&= 2\pi\frac{\pi}{3}\sin(\frac{\pi}{3}) - 2\pi \times 0 \sin(0) - (-2\pi\cos(x)|_0^{\frac{\pi}{3}}) \\
&= 2\frac{\pi^2}{3}\frac{\sqrt{3}}{2} + 2\pi\cos(\frac{\pi}{3}) - 2\pi\cos(0) \\
&= \frac{\sqrt{3}\pi^2}{3} + 2\pi\frac{1}{2} - 2\pi \\
&= \frac{\sqrt{3}\pi^2}{3} - \pi
\end{aligned}$$

Grading: 4 points for finding the correct integral, 5 points for computing the integral, 1 point for the final answer

(c) Let $\mathcal{T}$ be the solid given by rotating the region $\mathcal{R}$ about the horizontal line $y = 2$. Find the volume of $\mathcal{T}$.

Solution. We slice the region $\mathcal{R}$ vertically again and for each $x \in [0, \frac{\pi}{3}]$ we obtain one slice. However, such slice in this case gives rise to a washer because we are rotating a vertical slice about a horizontal line. The inner radius of each slice $2 - \cos(x)$ and the outer radius is equal to

2. Therefore, volume of a slice with thickness Δx at the point $x \in [0, \frac{\pi}{3}]$:

$$(\pi 2^2 - \pi(2 - \cos(x))^2)\Delta x$$

Therefore, the volume of the solid is equal to:

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} (\pi 2^2 - \pi(2-\cos(x))^2)\, dx &= \int_0^{\frac{\pi}{3}} (4\pi - \pi(4 - 4\cos(x) - \cos(x)^2))\, dx \\
&= \int_0^{\frac{\pi}{3}} 4\pi \cos(x) - \pi \cos(x)^2 \, dx \\
&= 4\pi \int_0^{\frac{\pi}{3}} \cos(x)\, dx - \pi \int_0^{\frac{\pi}{3}} \cos(x)^2\, dx \\
&= 4\pi \sin(x)|_0^{\frac{\pi}{3}} - \pi \int_0^{\frac{\pi}{3}} \frac{1+\cos(2x)}{2}\, dx \\
&= 4\pi \sin(\frac{\pi}{3}) - 4\pi\sin(0) - \frac{\pi}{2}\int_0^{\frac{\pi}{3}} 1\, dx - \frac{\pi}{2}\int_0^{\frac{\pi}{3}} \cos(2x)\, dx \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi}{2}\frac{\pi}{3} - \frac{\pi}{2}\int_{2\times 0}^{2\times\frac{\pi}{3}} \cos(u)\, \frac{du}{2} \qquad (4) \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi^2}{6} - \frac{\pi}{4}(\sin(u)|_0^{\frac{2\pi}{3}}) \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi^2}{6} - \frac{\pi}{4}(\sin(\frac{2\pi}{3}) - \sin(0)) \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi^2}{6} - \frac{\pi}{4}\frac{\sqrt{3}}{2} \\
&= 15\pi\frac{\sqrt{3}}{8} - \frac{\pi^2}{6}
\end{aligned}$$

In step (4), we use integration by substitution with $u = 2x$

Grading: 6 points for finding the correct integral, 4 points for computing the integral