MAT 132

Midterm 2 Spring 2017

Name: ______________________ **ID:** ______________________

Recitation: ______________________

List of Recitations:

R01	MW 12:00 pm-12:53 pm	Frederik Benirschke
R02	TuTh 10:00 am-10:53 am	Juan Ysimura
R03	TuTh 8:30am- 9:23am	Erik Gallegos Baños
R04	MW 11:00am-11:53am	Paul Frigge
R20	TuTh 5:30pm-6:23pm	Jean-François Arbour
R21	10:00am- 10:53am	Lisandra Hernandez Vazquez
R22	TuTh 11:30am- 12:23pm	Robert Abramovic

Instructions:

(1) Fill in your name and Stony Brook ID number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no calculators, no phones.

(3) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4) and simplify them as much as possible (e.g. $1/2$, not $2/4$) to receive full credit.

(4) Answer all questions in the space provided. If you need more room use the blank backs of the pages.

(5) Show your work; correct answers alone will receive only partial credit.

(6) This exam has 5 extra credit points.

Problem	1 (10 pts)	2 (10 pts)	3 (10 pts)	4 (10 pts)	5 (15 pts)	6 (15 pts)	7 (15 pts)	8 (20 pts)	Total (105 pts)
Score									

1. (I) (2 points) Which one of the following options is correct?

(a) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ are both convergent.

(b) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ are both divergent.

(c) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

(d) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is divergent and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is convergent.

Solution. The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent to 0. On the other hand, the series $\sum_{n=1}^{\infty}\frac{1}{n}$, which is a p-series with $p = 1$, is divergent. Therefore, choice (c) is correct.

Grading: Full point for choice (c) and zero otherwise.

(II) (2 points) Of the following series listed below, select ALL which are geometric series. (There are exactly two correct answers.)

(a) $\sum_{n=1}^{\infty}\frac{e^n}{n}$ (b) $\sum_{n=1}^{\infty}2^{\frac{1}{n}}$ (c) $\sum_{n=0}^{\infty}\frac{3^{n+1}}{4^{n-1}}$ (d) $\sum_{n=1}^{\infty}n^{\frac{1}{2}}$ (e) $\sum_{n=1}^{\infty}e^{2n+5}$

Solution. Choice (c) and (e) are geometric series with common ratios $\frac{3}{4}$ and e^2.

Grading: Each correct choice worths 1 point. If you choose more than two choices you would receive:

$$\max(0, \#\text{correct choices} - \#\text{wrong choices})$$

(III) (3 points) Give an example of a divergent series $\sum_{n=1}^{\infty}a_n$ such that $\sum_{n=1}^{\infty}a_n^2$ is convergent. Explain briefly why your series satisfies these two conditions.

Solution. If we pick $a_n = \frac{1}{n}$, then the series $\sum_{n=1}^{\infty}a_n$ is a p-series with $p = 1$ and hence it is divergent. In this case, the series $\sum_{n=1}^{\infty}a_n^2$ is a p-series with $p = 2$. Therefore, it is convergent.

Grading: If you give a correct answer for a_n without justifying, you would receive 2 points. If you give the answer $a_n = \frac{1}{x}$ you would receive only one point.

(IV) (3 points) Write the number $3.\overline{48} = 3.484848\ldots$ as the ratio of two integer numbers in a reduced form. (Just give a fraction as the final answer. You do not need to justify your answer.)

Solution. We have:

$$\begin{aligned}
3.484848\ldots &= 3 + \frac{48}{100} + \frac{48}{100^2} + \frac{48}{100^3} + \ldots \\
&= 3 + \frac{48}{100}(1 + \frac{1}{100} + \frac{1}{100^2} + \ldots) \\
&= 3 + \frac{48}{100}(\frac{1}{1 - \frac{1}{100}}) \\
&= 3 + \frac{48}{99} \\
&= 3 + \frac{16}{33} \\
&= \frac{115}{33}
\end{aligned}$$

Grading: If you give your answer in an unreduced form (e.g. $\frac{345}{99}$), you would lose one point. If you don't give the correct final answer, but your approach is correct and you make only arithmetic mistakes, you would get one point.

Determine whether the following three series converge or not. State the tests that you are using and show your work. (10 points for each series)

2. $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$

Solution. We apply the integral test to $f(x) = \frac{1}{x(\ln(x))^2}$. Observe that when $x \geq 3$, f is non-negative, decreasing and continuous. Moreover, we have:

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x(\ln(x))^2}\,dx &= \lim_{t\to\infty} \int_3^{t} \frac{1}{x(\ln(x))^2}\,dx \\
&= \lim_{t\to\infty} \int_{\ln(3)}^{\ln(t)} \frac{1}{u^2}\,du && \text{substitution using } u = \ln(x) \\
&= \lim_{t\to\infty} (-\frac{1}{u})|_{\ln(3)}^{\ln(t)} \\
&= \lim_{t\to\infty} \frac{1}{\ln(3)} - \frac{1}{\ln(t)} \\
&= \frac{1}{\ln(3)}
\end{aligned}$$

Therefore, by the Integral test, the series $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ is also convergent.

Grading scheme: If conditions for the Integral Test were not stated, one point is lost. You would lose another point, if the test is not mentioned. If the improper integral is misevaluated (for instance as divergent), you would lose two points. Failure to take limits to evaluate the indefinite integral or not paying attention to the change of upper limit and lower limit of the integral after substitution would cost you one point. Note that you could solve $\int_{\ln(3)}^{\infty} \frac{1}{u^2}\,du$ in terms of p-integrals (or p-series), but one point would be taken off unless you specify $\ln(3) > 1$. Answers with more serious errors (such as misevaluating $\int \frac{1}{u^2}\,du$) would receive partial credit, and as long as you guessed correctly, then a minimum of one point would be attained.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3}{2n^3+3n-1}$

Solution.

Using the divergence test (the n^{th}-term test): We can use the divergence testto show that the series is divergent (2 points for mentioning the name of the test used). The limit of the terms of the series can be computed as

$$\lim_{n\to\infty} \frac{3n^3}{2n^3+3n-1} = \lim_{n\to\infty} \frac{3\frac{n^3}{n^3}}{2\frac{n^3}{n^3}+3\frac{n}{n^3}-\frac{1}{n^3}} = \lim_{n\to\infty} \frac{3}{2+3\frac{1}{n^2}-\frac{1}{n^3}} = \frac{3}{2} \qquad \text{3 points}$$

Since this limit is not equal to 0, the series is divergent by the divergence test (5 points).

Using limit comparison Test: We can use limit comparison test to show the divergence of this series (2 points mentioning the name of the test used). Let $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{2}$. Then:

$$\lim_{n\to\infty} \frac{\frac{3n^3}{2n^3+3n-1}}{\frac{3}{2}} = \frac{2}{3}\lim_{n\to\infty} \frac{3\frac{n^3}{n^3}}{2\frac{n^3}{n^3}+3\frac{n}{n^3}-\frac{1}{n^3}} = \frac{2}{3}\lim_{n\to\infty} \frac{3}{2+3\frac{1}{n^2}-\frac{1}{n^3}} = 1 \qquad \text{2 points} \qquad (1)$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2}$ is divergent, because the terms of the series are constant and non-zero (2 points). Since the limit in (1) is not equal to 0 or ∞, the limit comparison test implies that the series is divergent (4 points).

4. $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n+2}{n^4}}$

Solution.

Using comparison Test: We have $n+2 \leq 3n$ for all $n \geq 1$. Therefore, for $n \geq 1$, we can write:

$$n+2 \leq n+2n = 3n \implies \qquad \text{2 points}$$

$$0 \leq \sqrt{\frac{n+2}{n^4}} \leq \sqrt{\frac{3n}{n^4}} = \sqrt{3}\frac{1}{n^{\frac{3}{2}}} \qquad \text{2 points}$$

Since $\displaystyle\sum_{n=1}^{\infty}\sqrt{3}\frac{1}{n^{\frac{3}{2}}} = \sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is a multiple of a p-series with $p = \frac{3}{2}$, it is convergent (3 points). Therefore, we can apply the comparison test to conclude that $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n+2}{n^4}}$ is also convergent (3 points).

Using Limit comparison Test: We can use limit comparison test with the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ (3 points) to show that our series is convergent:

$$\lim_{n\to\infty}\frac{\sqrt{\frac{n+2}{n^4}}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty}\sqrt{\frac{n+2}{n}} = \lim_{n\to\infty}\sqrt{1+\frac{2}{n}} = 1 \qquad \text{3 points}$$

Since this limit is between 0 and ∞, the series $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n+2}{n^4}}$ converges if and only if $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges (2 points). The latter series is a p-series with $p = \frac{3}{2}$. Thereofre, it is convergent (2 points).

5. (15 points) For each of the following improper integrals, determine whether it is convergent or not. If it is convergent, evaluate the integral:

(a) (7 points) $\int_1^\infty xe^{-x^2}\,dx$

Solution. We use the substitution $u = x^2$ to compute this improper integral. Note that we have $du = 2xdx$.

$$\begin{aligned}
\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx && \text{2 points}\\
&= \lim_{t\to\infty}\int_1^{t^2}\frac{1}{2}e^{-u}\,du && \text{2 points}\\
&= \lim_{t\to\infty}\frac{1}{2}(-e^{-u})|_1^{t^2}\\
&= \lim_{t\to\infty}\frac{1}{2}(e^{-1}-e^{-t^2})\\
&= \frac{1}{2}e^{-1} && \text{3 points}
\end{aligned}$$

(b) (8 points) $\int_0^1 \frac{2+\sin(x)}{x^3}\,dx$

Solution. We use comparison to show that this improper integral is divergent. Since $-1 \leq \sin(x) \leq 1$ for all x, we have:

$$\frac{2+\sin(x)}{x^3} \geq \frac{2-1}{x^3} \geq \frac{1}{x^3} \qquad \text{2 points}$$

$$\begin{aligned}
\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx && \text{2 points}\\
&= \lim_{t\to\infty}\int_1^{t^2}\frac{1}{2}e^{-u}\,du && \text{2 points}\\
&= \lim_{t\to\infty}\frac{1}{2}(-e^{-u})|_1^{t^2}\\
&= \lim_{t\to\infty}\frac{1}{2}(e^{-1}-e^{-t^2})\\
&= \frac{1}{2}e^{-1} && \text{3 points}
\end{aligned}$$

Since the p-integral $\int_0^1 \frac{1}{x^3}\,dx$ is divergent (3 points), the integral $\int_0^1 \frac{2+\sin(x)}{x^3}\,dx$ is also divergent by comparison test (2 points).

6. (15 points) The temperature of Stony Brook on March 29 can be modeled by the following function:

$$T(t) = 2 + 14\sin^2(\frac{\pi t}{24})$$

where $T(t)$ denotes the temperature t hours after 12:00 am. (The temperature is measured in °C.) Find the average temperature of Stony Brook on that day.

Solution.

$$\text{Average temperature} = \frac{1}{24}\int_0^{24} 2 + 14\sin^2\left(\frac{\pi t}{24}\right) dt \qquad \text{4 points. See 1 below.}$$

$$= 2 + \frac{14}{24}\int_0^{24} \sin^2\left(\frac{\pi t}{24}\right) dt \qquad \text{See 2.}$$

$$= 2 + \frac{14}{\pi}\int_0^{\pi} \sin^2 u du \qquad \text{substitution using } u = \frac{\pi t}{24}$$

$$= 2 + \frac{14}{\pi}\int_0^{\pi} \frac{1-\cos 2u}{2} du$$

$$= 2 + 7 \qquad \text{1 point See 3.}$$

Some comments about how I assigned points for each part.

1. 4 points is for stating the formula for average value and the interval on which they have to integrate, i.e. the endpoints of the integral.

2. 10 points for the total computation of the integral if they separated the integral, 11 otherwise. For the particular way showed in this solution the points are given using the criterion 2 points for the computation of the average of the constant 2, 3 points for the change of variable in the integral of the trigonometric function $\sin^2(\frac{\pi t}{24})$, 2 for the trigonometric identity $\sin^2 u = \frac{1-\cos 2u}{2}$, and 3 for the computation of the last integral.

3. 1 point for the final answer. This applied as follows. The points of 2, are basically for separating the integrals $\int 2$ and $\int \sin^2(\frac{\pi t}{24})$. They just have to rewrite together the values of such integrals.

7. Find the length of the arc which is parametrized as $x = t^3$ and $y = \frac{2}{3}t^{\frac{9}{2}}$ for $t \in [0\,,2]$.

Solution. The length of this arc is given by the following formula:

$$\begin{aligned}
\int_0^2 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt &= \int_0^2 \sqrt{(3t^2)^2 + (3t^{\frac{7}{2}})^2}\,dt \\
&= \int_0^2 \sqrt{9t^4 + 9t^7}\,dt \\
&= \int_0^2 3t^2\sqrt{1+t^3}\,dt \\
&= \int_{1+0^2}^{1+2^2} \sqrt{u}\,du && \text{substitution using } u = 1+t^3 \\
&= \frac{2}{3}u^{\frac{3}{2}}|_1^5 \\
&= \frac{2}{3}(5^{\frac{3}{2}} - 1)
\end{aligned}$$

8. A flat plate with uniform density has the shape of the region enclosed by $y = \dfrac{\ln(x)}{\sqrt{x}}$, $x = e$, $x = e^2$ and $y = 0$. Find the coordinates of the center of mass of the plate. **Solution.** The plate has the form of the shape under the graph of $y = \dfrac{\ln(x)}{\sqrt{x}}$ from e to e^2. Therefore, the coordinates of the center of mass are given by the following formulas:

$$(\overline{x},\overline{y}) = (\frac{\int_e^{e^2} x\frac{\ln(x)}{\sqrt{x}}\,dx}{\int_e^{e^2} \frac{\ln(x)}{\sqrt{x}}\,dx},\frac{\int_e^{e^2} \frac{1}{2}(\frac{\ln(x)}{\sqrt{x}})^2\,dx}{\int_e^{e^2} \frac{\ln(x)}{\sqrt{x}}\,dx}) \tag{2}$$

We firstly compute the integral $\int_e^{e^2} \frac{\ln(x)}{\sqrt{x}}\,dx$. We use integration by parts with $u = \ln(x)$ and $dv = \frac{1}{\sqrt{x}}dx$. That implies that $du = \frac{1}{x}dx$ and $v = 2\sqrt{x}$. Therefore, we have:

$$\begin{aligned}\int_e^{e^2} \frac{\ln(x)}{\sqrt{x}}\,dx &= 2\sqrt{x}\ln(x)|_e^{e^2} - \int_e^{e^2} 2\frac{1}{\sqrt{x}}\,dx \\ &= 2\sqrt{e^2}\ln(e^2) - 2\sqrt{e}\ln(e) - (4\sqrt{x}|_e^{e^2}) \\ &= 4e - 2\sqrt{e} - (4e - 4\sqrt{e}) \\ &= 2\sqrt{e}\end{aligned}$$

Similarly, we use integration by parts with $u = \ln(x)$ and $dv = \sqrt{x}dx$ to compute the integral $\int_e^{e^2} \sqrt{x}\ln(x)\,dx$. We have $du = \frac{1}{x}dx$ and $v = \frac{2}{3}x^{\frac{3}{2}}$:

$$\begin{aligned}\int_e^{e^2} \sqrt{x}\ln(x)\,dx &= \frac{2}{3}x^{\frac{3}{2}}\ln(x) - \int_e^{e^2} \frac{2}{3}\sqrt{x}\,dx \\ &= \frac{2}{3}(e^2)^{\frac{3}{2}}\ln(e^2) - \frac{2}{3}e^{\frac{3}{2}}\ln(e) - ((\frac{2}{3})^2 x^{\frac{3}{2}}|_e^{e^2}) \\ &= \frac{4}{3}e^3 - \frac{2}{3}e^{\frac{3}{2}} - (\frac{4}{9}e^3 - \frac{4}{9}e^{\frac{3}{2}}) \\ &= \frac{8}{9}e^3 - \frac{2}{9}e^{\frac{3}{2}}\end{aligned}$$

Finally, we need to compute $\int_e^{e^2} \frac{1}{2}\frac{\ln(x)^2}{x}\,dx$. For this integral, we use the substitution $u = \ln(x)$. Then we have $du = \frac{1}{x}dx$:

$$\begin{aligned}\int_e^{e^2} \frac{1}{2}\frac{\ln(x)^2}{x}\,dx &= \frac{1}{2}\int_{\ln(e)}^{\ln(e^2)} u^2\,du \\ &= \frac{1}{2}(\frac{u^3}{3}|_1^2) \\ &= \frac{7}{6}\end{aligned}$$

Consequently, we can use (2), to find the coordinates of the center of mass:

$$(\overline{x},\overline{y}) = (\frac{\frac{8}{9}e^3 - \frac{2}{9}e^{\frac{3}{2}}}{2\sqrt{e}},\frac{\frac{7}{6}}{2\sqrt{e}}) = (\frac{4}{9}e^{\frac{5}{2}} - \frac{1}{9}e,\frac{7}{12}e^{-\frac{1}{2}}) \tag{3}$$

Grading: 5 points for the center of mass formula (2). 5 points for computing each integral,