Practice Exam for the Final

1. (I) Which one of the following integrals is not improper?

   (a) $\displaystyle\int_0^2 \frac{1}{x-3}\,dx$

   (b) $\displaystyle\int_{-\infty}^1 \frac{1}{x^2+1}\,dx$

   (c) $\displaystyle\int_0^2 \frac{1}{x^2-1}\,dx$

   (d) $\displaystyle\int_0^\infty \cos(x)\,dx$

   (II) The Maclaurin series of a function $f(x)$ is equal to $\displaystyle\sum_{n=0}^{\infty}(n-1)x^n$. What is the 15^{th} derivative of f evaluated at 0:

   (a) $f^{(15)}(0) = 0$ (b) $f^{(15)}(0) = 14{\cdot}15!$ (c) $f^{(15)}(0) = \dfrac{15}{15!}$ (d) $f^{(15)}(0) = \dfrac{14}{15!}$

   (III) A power series is convergent at 3 and divergent at 6. Which one of the following points could be the center of this power series:

   (a) 0 (b) 5 (c) 7 (d) 10

   (IV) Write down a convergent series with infinitely many non-zero terms such that the sum is equal to $\frac{1}{3}$.

2. (a) Find the general solution of the the following second-order differential equation:

$$y'' - 13y' + 36y = 0$$

(b) Solve the following initial value problem:

$$xy' + 5y = 7x^2 \qquad\qquad y(2) = 5$$

3. A large vat contains 100 tablespoons of sugar dissolved in 100 liters of water. Sugar-water containing 5 tablespoons of sugar per liter enters the vat through a pipe at a rate of 2 liters per minute. At the same time, sugar-water is pumped out of the vat at a rate of 2 liters per minute. The vat is kept well mixed and the concentration of sugar in the vat is uniform. How long does it take until the vat contains 400 tablespoons of sugar?

4. (a) Find the interval of convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$$

(b) Find a power series representation for $f(x) = x^2 \ln(1 + x^3)$. What is the radius of convergence?

5. We want to use Taylor polynomials of $f(x) = \sqrt[5]{x}$ to approximate $\sqrt[5]{33}$.

   (a) What would be a good choice for the center of the Taylor polynomial? Why?

   (b) Use the Taylor polynomial of degree 2 centered at the point that you found in the previous part to write an estimate for $\sqrt[5]{33}$.

   (c) Find an upper bound for the error of your estimate.

6. (a) Determine whether the following series is convergent or divergent. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n!e^n}$$

(b) Determine whether the following series is convergent or divergent. State the test you use, and show your work:

$$\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^4 + 2}$$

7. (a) Find the volume obtained by rotating the region which is enclosed by $y = x$ and $y = x^2$ about the line $x = 4$.

(b) Find the length of the curve $x(t) = e^{\frac{t}{2}} \cdot \cos(t)$, $y = e^{\frac{t}{2}} \cdot \sin(t)$ for $\frac{\pi}{2} \leq t \leq \pi$.

8. Compute the following integrals

(a) $\displaystyle\int_1^4 \frac{x^3}{x^2+1}\,dx$

(b) $\displaystyle\int \frac{2x-1}{x^2-x-6}\,dx$

9. Find a number t in $[0, \frac{\pi}{2}]$ such that the average of the following function $f(x) = \sin(x) + x\cos(x)$ in the interval $[0, t]$ is equal to $\frac{1}{2}$.