1. (10 points) In each part, the augmented matrix of a linear system is given. Determine whether the solution is unique, there are no solutions or whether there are infinitely many. If the solution is unique give it. If there are infinitely many give the solution parametrically.

(i) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right]$

Solution.(5 points) This system has 4 variables, x_1, x_2, x_3 and x_4 where x_1 and x_3 are leading variables and x_2 and x_4 are free variables. Therefore, there are infinitely many solutions which are given as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5-2s-3t \\ s \\ 6-4t \\ t \end{bmatrix}$$

(You would lose 1 point if you don't mention explicitly that there are infinitely many solutions.)

(ii) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution.(5 points) This system has 3 leading variables x_1, x_2, x_3. Because the last equation is $0 = 0$, the system is consistent. Therefore, the system has a unique solution which is equal to:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix}$$

2. (10 points) Solve the following linear system using an augmented matrix. State whether the solution is unique, there are no solutions or whether there are infinitely many. If the solution is unique give it. If there are infinitely many give the solution parametrically.

$$\begin{cases} \phantom{2x_1 -{}} x_2 + 2x_3 = 3 \\ 2x_1 - 3x_2 + 2x_3 = -1 \\ 2x_1 + 3x_2 + 14x_3 = 18 \end{cases}$$

Solution. The augmented matrix of this system has the following form:

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 2 & -3 & 2 & -1 \\ 2 & 3 & 14 & 18 \end{array}\right]$$

We firstly switch the first two rows to obtain a non-zero term in the first entry. Then we proceed as follows:

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 14 & 18 \end{array}\right]\div 2 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 14 & 18 \end{array}\right]_{-2(\mathrm{I})} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 12 & 19 \end{array}\right]\begin{matrix} \frac{3}{2}(\mathrm{II}) \\ \\ 6(\mathrm{II}) \end{matrix} \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last equation asserts that $0 = 1$. Therefore, the system does not have any solution. (You would lose 2 points if you make an arithmetic mistake.)

3. (a) Find the rank of the following matrix (10 points):

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

Solution. (8 points) We need to turn this matrix into the reduced row echelon form. Firstly switch the first two rows to obtain a non-zero term in the first entry:

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 0 \end{bmatrix} \div -1 \quad \rightsquigarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ -2 & -3 & 0 \end{bmatrix} 2(\mathrm{I}) \quad \rightsquigarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix} 3(\mathrm{II}) \quad \rightsquigarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Consequently, the rank of the matrix is equal to 2. (You would lose 2 points if you make an arithmetic mistake.)

(b) Is this matrix invertible? If so, what is the inverse?

Solution. (2 points) The rank of this matrix is not equal to 3. Therefore, it is not invertible.

4. (15 points) We know the following values of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^4$:

$$T(\begin{bmatrix} 7 \\ 6 \end{bmatrix}) = \begin{bmatrix} 5 \\ 1 \\ -2 \\ 10 \end{bmatrix} \qquad T(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = \begin{bmatrix} 4 \\ 8 \\ 2 \\ 8 \end{bmatrix}$$

Find the matrix of this linear transformation. (Hint: Firstly compute $T(e_1)$ and $T(e_2)$.)

Solution. We firstly write down e_1 and e_2 as linear combinations of:

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

That is to say, we need to find x_1, x_2, y_1 and y_2 such that:

$$x_1 \begin{bmatrix} 7 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad y_1 \begin{bmatrix} 7 \\ 6 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, x_1, x_2, y_1 and y_2 satisfy the following systems:

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where A is equal to the following matrix:

$$\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

We can solve these systems using the associated augmented matrix. Alternatively, we can compute the inverse of A as:

$$A^{-1} = \frac{1}{9} \begin{bmatrix} 3 & -2 \\ -6 & 7 \end{bmatrix}$$

Then x_1, x_2, y_1 and y_2 can be computed as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 3 & -2 \\ -6 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 3 & -2 \\ -6 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} \\ \frac{7}{9} \end{bmatrix}$$

Consequently:

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = T(\frac{1}{3}\begin{bmatrix} 7 \\ 6 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = \frac{1}{3}T(\begin{bmatrix} 7 \\ 6 \end{bmatrix}) - \frac{2}{3}T(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = \frac{1}{3}\begin{bmatrix} 5 \\ 1 \\ -2 \\ 10 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 4 \\ 8 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -2 \\ -2 \end{bmatrix}$$

Similarly:

$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = T(-\frac{2}{9}\begin{bmatrix} 7 \\ 6 \end{bmatrix} + \frac{7}{9}\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = -\frac{2}{9}T(\begin{bmatrix} 7 \\ 6 \end{bmatrix}) + \frac{7}{9}T(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = -\frac{2}{9}\begin{bmatrix} 5 \\ 1 \\ -2 \\ 10 \end{bmatrix} + \frac{7}{9}\begin{bmatrix} 4 \\ 8 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \end{bmatrix}$$

Therefore, the matrix of T is equal to:

$$\left[\ T(e_1) \mid T(e_2)\ \right] = \begin{bmatrix} -1 & 2 \\ -5 & 6 \\ -2 & 2 \\ -2 & 4 \end{bmatrix}$$

5. (15 points)

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that rotates a vector in $\mathbb{R}^2$ through the angle $\frac{\pi}{4}$ clockwise. Let A be the matrix of this transformation. Compute AB when B is the following matrix:

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 6 & 0 \end{bmatrix}$$

Solution. (5 points) In general, the matrix of the counterclockwise rotation in $\mathbb{R}^2$ through the angle θ is equal to:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{1}$$

Therefore, A is given by the above matrix for $\theta = -\frac{\pi}{4}$:

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

As the final step, we need to compute AB:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 4 & 6 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}+2\sqrt{2} & \frac{\sqrt{2}}{2}+3\sqrt{2} & \sqrt{2} \\ -\sqrt{2}+2\sqrt{2} & -\frac{\sqrt{2}}{2}+3\sqrt{2} & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & \frac{7\sqrt{2}}{2} & \sqrt{2} \\ \sqrt{2} & \frac{5\sqrt{2}}{2} & -\sqrt{2} \end{bmatrix}$$

(2 points for finding A and 2 points for computing AB. You would lose 1 point for not simplifying the answer.)

(b) Suppose l is the line whose angle with the x-axis is equal to $\frac{\pi}{3}$. The linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ maps a vector to its reflection about the line l. Compute the matrix of this linear transformation.

Solution. (5 points) In general, the matrix of the reflection about a line that passes through the vector $(w_1,w_2) \in \mathbb{R}^2$ is given by the following matrix:

$$\frac{1}{w_1^2+w_2^2} \begin{bmatrix} w_1^2 - w_2^2 & 2w_1w_2 \\ 2w_1w_2 & w_2^2 - w_1^2 \end{bmatrix}$$

Therefore, the matrix of $S$is given by the above matrix for $(w_1,w_2) = (\cos(\pi/3), \sin(\pi/3))$:

$$\begin{bmatrix} \cos^2(\frac{\pi}{3}) - \sin^2(\frac{\pi}{3}) & 2\cos(\frac{\pi}{3})\sin(\frac{\pi}{3}) \\ 2\cos(\frac{\pi}{3})\sin(\frac{\pi}{3}) & \sin^2(\frac{\pi}{3}) - \cos^2(\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & -\cos(\frac{2\pi}{3}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(You would lose 1 point for not simplifying the answer.)

(c) $U : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that rotates a vector though the angle $\frac{\pi}{2}$ counter-clockwise and then projects the resulting vector to the x-axis. Compute the matrix of this linear transformation.

Solution. (5 points) The matrix of the counterclockwise rotation in $\mathbb{R}^2$ through the angle $\frac{\pi}{2}$ is given by (1) when $\theta = \frac{\pi}{2}$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The reflection to the x-axis maps the vector (x,y) to $(x,0)$. Therefore, the matrix of this projection is equal to:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The matrix of U is the product of these two matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

(2 points for finding each matrix and 1 point for computing the product.)

6. (10 points) For what values of λ, the following matrix is invertible? When it is invertible, find the inverse.

$$\begin{bmatrix} 2 & 9-\lambda \\ 8 & 4 \end{bmatrix}$$

Solution. A 2×2 matrix is invertible, if its determinant is non-zero. The determinant of the above matrix is equal to:

$$2 \times 4 - 8(9-\lambda) = 8\lambda - 64$$

Therefore, the determinant is zero if and only if $\lambda = 8$. In the case that the determinant is non-zero (*i.e.* when the above matrix is invertible), the inverse is given by the following matrix:

$$\frac{1}{8\lambda - 64} \begin{bmatrix} 4 & \lambda - 9 \\ -8 & 2 \end{bmatrix}$$

(5 points for finding λ and 5 points for computing the inverse. Computing the inverse for a special value of λ would get at most 3 points out of 5.)

7. (10 points) Determine whether the following matrix is invertible. If it is, compute the inverse:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

Solution. The combined augmented matrix associated to this matrix is equal to:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -1 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

This matrix can be simplified to the reduced row echelon form in the following way:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -1 & 4 & 1 & 0 & 0 & 1 \end{array}\right]\begin{matrix} \\ -3(\mathrm{I}) \\ +(\mathrm{I}) \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 4 & 1 & 1 & 0 & 1 \end{array}\right]\begin{matrix} \\ \\ -4(\mathrm{II}) \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 13 & -4 & 1 \end{array}\right]$$

Therefore, (2) is invertible and its inverse is equal to:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 13 & -4 & 1 \end{bmatrix}$$

(5 points for showing it is invertible and 5 points for computing the inverse. You would lose 2 points if you make an arithmetic mistake.)

8. (10 points) A, B, and C are 3×3 matrices. Their inverses are given as follows:

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

Find the inverse of ABC.

Solution. In general, if A, B, and C are invertible, then ABC is invertible and its inverse is equal to the product $C^{-1}B^{-1}A^{-1}$, because:

$$(ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = AB\,\mathrm{I}\,B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A\,\mathrm{I}\,A^{-1} = AA^{-1} = \mathrm{I}$$

Similarly, you can show that $(C^{-1}B^{-1}A^{-1})(ABC)$ is equal to the identity matrix. This verifies that the inverse of ABC is equal to $C^{-1}B^{-1}A^{-1}$. For this problem, we need to compute the following product:

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 3 & 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 3 & 7 & 4 \end{bmatrix}$$

Alternatively, one can find the inverses of A^{-1}, B^{-1} and C^{-1}. These matrices are equal to A, B and C, respectively. Then you can compute ABC and take the inverse of the resulting 3×3 matrix. Clearly, this approach takes much more work.

(5 points for noting that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ and 5 points for computing the product. You wouldn't get any point if you compute $A^{-1}B^{-1}C^{-1}$ as the final answer.)