1. *In each part, a basis $\mathfrak{B}$ of $\mathbb{R}^2$ is given (you don't need to show $\mathfrak{B}$ is a basis). Find he $\mathfrak{B}$-coordinate of the vector v.*

   (a) $\mathfrak{B} = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\}, \ v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

   Solution.(5 points) We have:

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 0 \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \times \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

   Therefore, the $\mathfrak{B}$-coordinate of v is equal to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (You would lose 1 point if you don't give $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the final answer or give $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the final answer.)

   (b) $\mathfrak{B} = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \end{bmatrix}\}, \ v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

   Solution.(10 points) We need to find c_1 and c_2 such that:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} \tag{1}$$

   That is we want to solve the following system:

$$\underbrace{\begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \tag{2}$$

   We can find apply our general formula to find the inverse of the 2×2 matrix A:

$$A^{-1} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$$

   Consequently:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 6 \end{bmatrix}$$

   We can find apply our general formula to find the inverse of the 2×2 matrix A:

$$A^{-1} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$$

   Therefore, the $\mathfrak{B}$-coordinate of v is equal to $\begin{bmatrix} 26 \\ 6 \end{bmatrix}$.

   (5 points for forming the system (1) or (2). 5 points for solving the system.)

2. *Find bases for the image and the kernel of the linear transformation defined by the following matrix:*

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 3 & 4 & -6 \\ 4 & 5 & -7 \end{bmatrix}$$

Solution.(15 points) We firstly find the RREF of A.

$$\begin{bmatrix} 2 & 2 & -2 \\ 3 & 4 & -6 \\ 4 & 5 & -7 \end{bmatrix} \begin{matrix} \div -2 \\ \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -6 \\ 4 & 5 & -7 \end{bmatrix} \begin{matrix} \\ -3(\mathrm{I}) \\ -4(\mathrm{I}) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \begin{matrix} -(\mathrm{II}) \\ \\ -(\mathrm{II}) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there are leading ones in the first two columns of the RREF of A, the first two columns of A form a basis for the image of A. That is to say, a basis for the image of A is given by the following vectors:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

We can also use the RREF of A to find any vector in the kernel. Any such vector has the following form:

$$\begin{bmatrix} -2t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

Therefore, the following vector gives a basis for the kernel of A;

$$\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

(5 points for finding the RREF, 5 points for finding a basis for the image and 5 points for finding a basis for the kernel. You would lose 2 points if you make an arithmetic mistake in computing RREF. You would lose 3 points if you give the first two columns of the RREF of A (instead of the first two columns of A) as a basis for the image of A.)

3. *Determine whether each statement below is correct or not. You don't need to justify your answer; just give a "yes" or "no" answer.*

(a) *Let $C(\mathbb{R},\mathbb{R})$ be the vector space that consists of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define W as:*

$$W = \{f \in C(\mathbb{R},\mathbb{R}) \mid f(20) = 16\}$$

Is W a vector subspace of $C(\mathbb{R},\mathbb{R})$?

Solution.(2 points) No, because the function $f(x) = 0$ is not an element of W.

(b) *Let $W \subset \mathbb{R}^3$ be defined as:*

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid x \le y \le 2z\}$$

Is W a vector subspace of $\mathbb{R}^3$?

Solution.(2 points) No, because, for example, (1,1,1) is in W while $(-1,-1,-1) = -1 \times (1,1,1)$ is not an element of W.

(c) *Let P_2 be the vector space of degree at most 2 polynomials and $\mathfrak{B}$ be the basis for P_2 that consists of 1, x, and x^2. Is:*

$$T : P_2 \to \mathbb{R}^3$$

$$T(Q) = [Q]_{\mathfrak{B}}$$

an isomorphism?

Solution.(2 points) Yes. In general given a basis $\mathfrak{B}$ for a vector basis, the $\mathfrak{B}$-coordinate map is an isomorphism.

(d) *Let l^∞ be the linear space of all infinite sequences of numbers where addition and scalar multiplication is defined as:*

$$(x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots) := (x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots)$$

$$k(x_0, x_1, x_2, \dots) := (kx_0, kx_1, kx_2, \dots)$$

Consider the map $T : l^\infty \to l^\infty$:

$$T(x_0, x_1, x_2, \dots) := (0, x_1, x_2, x_3, \dots)$$

Is the vector $(0, 1, 2, 3, \dots)$ in the image of T?

Solution.(2 points) Yes. For example $T(10, 1, 2, 3, \dots) = (0, 1, 2, 3, \dots)$.

(e) *Let T be defined as in the previous part. Is the vector $(0, 1, 2, 3, \dots)$ in the kernel of T?*

Solution.(2 points) No. $T(0, 1, 2, 3, \dots)$ is equal to $(0, 1, 2, 3, \dots)$ which is not zero.

4. *In each part, is the given set of vectors a basis for $\mathbb{R}^3$? If not, is it possible to find a subset of the given vectors to form a basis? Justify your answer; if you think it is not possible explain why. If you think it is possible, find a subset which is a basis.*

(a)

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}$$

Solution.(5 points) No, this set doesn't form a basis. Because a basis for $\mathbb{R}^3$ has three elements. Obviously removing one or two elements from this set of vector also would leave us less than three vectors. Therefore, we cannot form a basis by removing one or two of these vectors.

(3 points for explaining that v_1 and v_2 is not a basis. 2 points for showing that there is not a subset which is a basis. You would lose 1 point if you say v_1 and v_2 do not span $\mathbb{R}^3$ but you don't explain why.)

(b)

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Solution.(5 points) This set has 3 elements which is equal to the dimension of $\mathbb{R}^3$. Thus to show that it gives us a basis, it suffices to show that these vectors are linearly independent:

$$c_1v_1 + c_2v_2 + c_3v_3 = 0 \implies \begin{bmatrix} c_1 + 3c_2 + 2c_3 \\ 5c_2 + 4c_3 \\ 6c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The last equation implies that $c_2 = 0$. Then the second equation asserts that c_3 also has to be zero. Finally, we can use the first equation to show that c_1 is also zero. Therefore these three vectors are linearly independent and they form a basis.

(3 points for showing that the three vectors are linearly independent. 2 points for showing they span $\mathbb{R}^3$ or mentioning that there are the same number of vectors as the dimension of $\mathbb{R}^3$.)

(c)

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 6 \\ 11 \\ 5 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

Solution.(10 points) No, this set doesn't form a basis. Because a basis for $\mathbb{R}^3$ has three elements. However there is a chance to come up with a basis after removing one element of this set of vectors. To do that, it's helpful to form a matrix which has v_1, v_2, v_3 and v_4 as its column and then find the RREF of the resulting matrix:

$$\begin{bmatrix} 1 & -2 & 6 & 1 \\ 2 & -3 & 11 & 3 \\ 1 & -1 & 5 & 3 \end{bmatrix}\begin{matrix} \\ -2(\mathrm{I}) \\ -(\mathrm{I}) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 6 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix}\begin{matrix} 2(\mathrm{II}) \\ \\ -(\mathrm{II}) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{matrix} -3(\mathrm{III}) \\ -(\mathrm{III}) \\ \\ \end{matrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first, the second and the last row of the RREF have a leading one. Therefore, v_1, v_2, v_4 form a basis for $\mathbb{R}^3$.

(2 points for arguing that v_1, v_2, v_3 and v_4 is not a basis for $\mathbb{R}^3$. 5 points for finding RREF. 3 points for the subset which is a basis. You would lose 2 points if you make an arithmetic mistake in computing RREF.)

5. *Let A be the following 2×2 matrix:*

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix}$$

and the basis $\mathfrak{B} = \{v_1, v_2\}$ consists of the following vectors:

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B}$.

Solution.(15 points) We can find the matrix B either column-by-column or by using the formula $B = S^{-1}AS$ where S is the matrix whose columns are v_1 and v_2

$$S = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$$

We use the second method. Therefore, we have to find the inverse of S:

$$S^{-1} = \frac{1}{2 \times 4 - 7 \times 1} \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$$

Consequently:

$$B = S^{-1}AS = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -42 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -40 & -161 \\ 12 & 48 \end{bmatrix}$$

(If you use $B = S^{-1}AS$, then 2 points for the formula $B = S^{-1}AS$, 3 points for forming S, 4 points for finding S^{-1}, and 6 points for computing the product. If you find B column by column, then 3 points for the general formula for the columns of B and 6 points for finding each column.)

6. *Different parts of this question are not necessarily related to each other. So if you can't do one part, don't give up on the other parts.*

   (a) *Show that the following set determines a basis for $\mathbb{R}^{2\times 2}$: (Hint: You can use the fact that $\dim(\mathbb{R}^{2\times 2}) = 4$.)*

$$\mathfrak{B} := \left\{ f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, f_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, f_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

Solution.(5 points) The dimension of $\mathbb{R}^{2\times 2}$ is equal to 4 and there are 4 vectors in $\mathcal{B}$. Therefore, it suffices to show that the elements of $\mathfrak{B}$ are linearly independent:

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0 \implies \begin{bmatrix} c_1 & c_3 + c_4 \\ c_3 - c_4 & c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This implies that $c_1 = 0$, $c_2 = 0$, $c_3 + c_4 = 0$ and $c_3 - c_4 = 0$. By adding up the last two equations we can conclude that $c_3 = 0$. The difference of those two equations assert that $c_4 = 0$. Therefore, $c_1 = c_2 = c_3 = c_4 = 0$. That is to say, the elements of $\mathcal{B}$ are linearly independent.

(If you just say these matrices are linearly independent without justification, you would lose 3 points)

(b) *Find the $\mathfrak{B}$-coordinate of the following matrix:*

$$\begin{bmatrix} 1 & 3 \\ 9 & 5 \end{bmatrix}$$

Solution.(5points)

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 9 & 5 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} c_1 & c_3 + c_4 \\ c_3 - c_4 & c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 9 & 5 \end{bmatrix} \Longrightarrow \begin{cases} c_1 = 1 \\ c_2 = 5 \\ c_3 + c_4 = 3 \\ c_3 - c_4 = 9 \end{cases} \Longrightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 5 \\ 2c_3 = 12 \Longrightarrow c_3 = 6 \\ 2c_4 = -6 \Longrightarrow c_4 = -3 \end{array}$$

Therefore, the $\mathfrak{B}$-coordinate of the given matrix is equal to:

$$\begin{bmatrix} 1 \\ 5 \\ 6 \\ -3 \end{bmatrix}$$

(2 points for forming the equation, 2 points for solving the equation and 1 point for giving the final answer as a vector in $\mathbb{R}^4$.)

(c) *Show $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, defined as follows, is a linear transformation.*

$$T(M) = M \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} M$$

Solution.(5points) We have two check the two properties of a linear transformation. First:

$$T(M_1 + M_2) = (M_1 + M_2) \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} (M_1 + M_2) =$$

$$M_1 \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} + M_2 \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} M_1 - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} M_2 =$$

$$\left(M_1 \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} M_1 \right) + \left(M_2 \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} M_2 \right) = T(M_1) + T(M_2)$$

and then:

$$T(cM) = (cM) \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} (cM) = c(M \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} M) = cT(M)$$

(3 points for checking the first property and 2 points for checking the second property.)

(d) *Is T an isomorphism? Justify your answer.*

Solution.(5points) Consider the identity matrix:

$$id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then:

$$T(id) = id \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} id = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = 0$$

Therefore, the non-zero matrix id is in the kernel of T. That implies that T is not an isomorphism.

(e) *Find the $\mathfrak{B}$-matrix of the linear transformation T.*

Solution.(5points) The $\mathfrak{B}$-matrix is given by the following formula:

$$\left[\ [T(f_1)]_{\mathfrak{B}} \ \vdots\ [T(f_2)]_{\mathfrak{B}} \ \vdots\ [T(f_3)]_{\mathfrak{B}} \ \vdots\ [T(f_4)]_{\mathfrak{B}}\ \right]$$

where f_1, f_2, f_3 and f_4 are given in the first part.

$$T(f_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$$

$$T(f_2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix}$$

$$T(f_3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & -4 \end{bmatrix}$$

$$T(f_4) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & -6 \end{bmatrix}$$

As in part (b) we can find the $\mathfrak{B}$-coordinate of each of these vectors:

$$[T(f_1)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 3 \end{bmatrix} \qquad [T(f_2)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \end{bmatrix} \qquad [T(f_3)]_{\mathfrak{B}} = \begin{bmatrix} 4 \\ -4 \\ 0 \\ 1 \end{bmatrix} \qquad [T(f_4)]_{\mathfrak{B}} = \begin{bmatrix} 6 \\ -6 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, the $\mathfrak{B}$-matrix is equal to:

$$\begin{bmatrix} 0 & 0 & 4 & 6 \\ 0 & 0 & -4 & -6 \\ -2 & 2 & 0 & 1 \\ 3 & -3 & 1 & 0 \end{bmatrix}$$

(f) *(Bonus question) Find a basis for the kernel and the image of T.*

Solution.(10points) We firstly find the RREF of the matrix from the previous part. As the first step we replace the first and the third row:

$$\begin{bmatrix} -2 & 2 & 0 & 1 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & 4 & 6 \\ 3 & -3 & 1 & 0 \end{bmatrix}\begin{matrix} +(\mathrm{IV}) \\ \\ \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & 4 & 6 \\ 3 & -3 & 1 & 0 \end{bmatrix}\begin{matrix} \\ \\ \\ -3(\mathrm{I}) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & -2 & -3 \end{bmatrix}\begin{matrix} \\ \div -4 \\ \\ \\ \end{matrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1.5 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & -2 & -3 \end{bmatrix}\begin{matrix} -(\mathrm{II}) \\ \\ -4(\mathrm{II}) \\ 2(\mathrm{II}) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & -0.5 \\ 0 & 0 & 1 & 1.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, a basis for the image of the $\mathfrak{B}$-matrix is given by the first and the third column. These two columns are $[T(f_1)]_{\mathfrak{B}}$ and $[T(f_3)]_{\mathfrak{B}}$. Consequently, a basis for the image of T is given by the 2×2 matrices $T(f_1)$ and $T(f_3)$. An element in the kernel of the $\mathfrak{B}$-matrix has the following form:

$$\begin{bmatrix} s + 0.5t \\ s \\ -1.5t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0.5 \\ 0 \\ -1.5 \\ 1 \end{bmatrix}$$

Therefore, a basis for the kernel of the $\mathfrak{B}$-matrix is:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0.5 \\ 0 \\ -1.5 \\ 1 \end{bmatrix}$$

These are the $\mathfrak{B}$-coordinate of the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0.5 & -0.5 \\ -2.5 & 0 \end{bmatrix}$$

These two matrices give a basis for the kernel of T.

(3 points for finding the RREF, 2 points for finding the $\mathfrak{B}$-coordinate of a basis for the image of T, 3 points for finding the $\mathfrak{B}$-coordinate of a basis for the kernel of T, and 2 points for fining 2×2 matrices with given $\mathfrak{B}$-coordinates.)