Practice Exam for Midterm II

1. In each part determine whether W is a linear subspace of V. Justify your answer carefully; if you think W is a subspace of V check all the necessary conditions and if you think W is not a subspace of V explain which condition fails by giving an example.

   (a) $V = P_2$, $W := \{Q \in P_2 \mid Q(1) = 0\}$

   (b) $V = \mathbb{R}^3$, $W := \{(x, y, z) \in \mathbb{R}^3 \mid x + y \leq z\}$

2. In each part, is $\{v_1, v_2, v_3\}$ is a basis for $\text{span}(v_1, v_2, v_3)$? If not, find a subset of the vectors $\{v_1, v_2, v_3\}$ which is a basis for $\text{span}(v_1, v_2, v_3)$.

   (a)

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   (b)

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \quad v_4 = \begin{bmatrix} 9 \\ 11 \\ 13 \\ 15 \end{bmatrix}$$

3. Find bases for the image and the kernel of the linear transformation defined by the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

4. Let A be the following 2×2 matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

and the basis $\mathfrak{B} = \{v_1, v_2\}$ consists of the following vectors:

$$v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B}$:

(a) by the formula $B = S^{-1}AS$.

(b) by constructing B column by column.

5. Show that the following maps are linear transformation. Are they isomorphism? Justify your answer.

(a) $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined as:

$$T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$$

(b) $T : P_2 \to \mathbb{R}^3$:

$$T(Q) = \begin{bmatrix} Q(1) \\ Q(2) \\ Q(4) \end{bmatrix}$$

6. (a) Show that:

$$\mathfrak{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

Is a basis for $\mathbb{R}^{2\times 2}$.

(b) Find the $\mathfrak{B}$-coordinate of the following matrix:

$$\mathfrak{B} := \begin{bmatrix} 3 & 9 \\ -1 & 1 \end{bmatrix}$$