Problem Set 1

1. (a) Compute the inner product $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

   (b) Two vectors are orthogonal to each other if their inner product is equal to 0. Let $x = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find the scalar $c \in \mathbb{R}$ such that the inner product of $x - cy$ and y is zero.

   (c) Let $x = \begin{bmatrix} 4 \\ -8 \\ 2 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Find scalars $c_1, c_2 \in \mathbb{R}$ such that the inner product of $x - c_1 e_1 - c_2 e_2$ with any of the two vectors e_1 and e_2 is zero.

2. Suppose $\{a_i\}_{i\in\mathbb{N}}$ is a non-decreasing sequence of real numbers. That is to say:

$$a_0 \leq a_1 \leq a_2 \leq \dots.$$

We want to show any such sequence is convergent. Let z denote the least upper bound of the set $\{a_0, a_1, a_2, \dots\}$.

(i) Show that for any positive real number δ, there is N such that $z - \delta \leq a_N \leq z$.

(ii) Use the assumption on $\{a_i\}_{i\in\mathbb{N}}$ and the definition of N to show that for any $n \geq N$, we have $z - \delta \leq a_n \leq z$.

(iii) Use the last part to conclude that the sequence is convergent to z.

3. We want to show that the following set does not have a least upper bound in the set of rational numbers

$$A = \{\frac{m}{n} \in \mathbb{Q} \mid (\frac{m}{n})^2 < 2\}.$$

We assume that there is a least upper bound z for this set and then we get a contradiction.

(i) Firstly let $z^2 < 2$. Show that $\frac{3z+2}{z+3}$ is a rational number strictly larger than z and $(\frac{3z+2}{z+3})^2 < 2$. Why is it impossible?

(ii) If $z^2 > 2$, then show that $\frac{3z+2}{z+3}$ is a positive rational number which is less than z and is an upper bound for the set A. Why is it impossible?

(iii) Using the last two parts and the fact that there is no rational number which squares to 2 (no need to prove this fact), conclude that A does not have a rational least upper bound z.

4. (i) Show that intersection of finitely many open sets is again an open set.

(ii) Find infinitely many open subsets $X_1, X_2, \ldots$ of $\mathbb{R}^2$ such that:

$$\bigcap_{i=1}^{\infty} X_i$$

is not an open set.