Problem Set 1

1. (a) *Compute the inner product* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

   Solution.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 1 \times (-1) + 2 \times 1 + 3 \times 2 = 7$$

   (b) *Two vectors are orthogonal to each other if their inner product is equal to* 0. *Let* $x = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ *and* $y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. *Find the scalar* $c \in \mathbb{R}$ *such that the inner product of* $x - cy$ *and* y *is zero.*

   Solution.

$$0 = (x - cy) \cdot y = \left(\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} - c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = (3-c) \times 1 + (-2) \times 0 + (-1+c) \times (-1) \implies$$

$$0 = 4 - 2c \implies \boxed{c = 2}$$

   (c) *Let* $x = \begin{bmatrix} 4 \\ -8 \\ 2 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. *Find scalars* $c_1, c_2 \in \mathbb{R}$ *such that the inner product of* $x - c_1 e_1 - c_2 e_2$ *with any of the two vectors* e_1 *and* e_2 *is zero.*

   Solution.

$$0 = (x - c_1 e_1 - c_2 e_2) \cdot e_1 = \begin{bmatrix} 4 - c_1 \\ -8 - c_2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (4 - c_1) \implies \boxed{c_1 = 4}$$

$$0 = (x - c_1 e_1 - c_2 e_2) \cdot e_2 = \begin{bmatrix} 4 - c_1 \\ -8 - c_2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (-8 - c_2) \implies \boxed{c_2 = -8}$$

2. *Suppose* $\{a_i\}_{i\in\mathbb{N}}$ *is a non-decreasing sequence of real numbers. That is to say:*

$$a_0 \le a_1 \le a_2 \le \dots .$$

We want to show any such sequence is convergent. Let z *denote the least upper bound of the set* $\{a_0, a_1, a_2, \dots\}$.

   (i) *Show that for any positive real number* δ*, there is* N *such that* $z - \delta \le a_N \le z$.

   Solution. *Since* z *is the least upper bound of the set* $\{a_0, a_1, a_2, \dots\}$*, there is an element of this set, say* a_N*, which is larger than* $z - \delta$*. Otherwise,* $z - \delta$ *would be another upper bound for this set which contradicts the assumption that* z *is the least upper bound. Clearly,* $a_N \le z$ *because* z *is an upper bound for the set* $\{a_0, a_1, a_2, \dots\}$*. Thus the chosen* a_N *satisfies the desired inequalities* $z - \delta \le a_N \le z$.

   (ii) *Use the assumption on* $\{a_i\}_{i\in\mathbb{N}}$ *and the definition of* N *to show that for any* $n \ge N$*, we have* $z - \delta \le a_n \le z$.

   Solution. *Let* n *be greater than or equal to the integer* N *obtained in the last part. Since the sequence* $\{a_0, a_1, a_2, \dots\}$ *is increasing* $a_n \ge a_N$*. In particular,* $a_n \ge z - \delta$*. We again have* $a_n \le z$ *because* z *is an upper bound for the set* $\{a_0, a_1, a_2, \dots\}$*. In summary,* $z - \delta \le a_n \le z$.

   (iii) *Use the last part to conclude that the sequence is convergent to* z.

   Solution. *Let* $\delta > 0$ *be an arbitrary real number. By the definition of convergence, we need to find an integer* N_0 *such that for* $n \ge N_0$ *we have* $|z - a_n| \le \delta$*. We let* N_0 *be the integer* N *obtained in part (i). Then part (ii) shows our desired claim for any* $n \ge N$.

3. *We want to show that the following set does not have a least upper bound in the set of rational numbers*

$$A = \{\frac{m}{n} \in \mathbb{Q} \mid (\frac{m}{n})^2 < 2\}.$$

We assume that there is a least upper bound z for this set and then we get a contradiction.

   (i) *Firstly let $z^2 < 2$. Show that $\frac{3z+2}{z+3}$ is a rational number strictly larger than z and $(\frac{3z+2}{z+3})^2 < 2$. Why is it impossible?*

   Solution. *Since z is a rational number, clearly $\frac{3z+2}{z+3}$ is also a rational number.*

$$\frac{3z+2}{z+3} - z = \frac{2z+2-z^2-3z}{z+3} = \frac{2-z^2}{z+3} > 0$$

   where in the last inequality we use the fact that $z^2 < 2$. We also have:

$$2 - \left(\frac{3z+2}{z+3}\right)^2 = \frac{2z^2+12z+18-9z^2-12z-4}{z^2+6z+9} = \frac{7(2-z^2)}{(z+3)^2} > 0$$

   where the last inequality is again a consequence of $z^2 < 2$. This shows that $\frac{3z+2}{z+3}$ belongs to the set A. On the other hand, it is greater than z, an upper bound for A, which is a contradiction.

   (ii) *If $z^2 > 2$, then show that $\frac{3z+2}{z+3}$ is a positive rational number which is less than z and is an upper bound for the set A. Why is it impossible?*

   Solution. *Note that $0 \in A$ and hence z is a non-negative number. Since z is a non-negative rational number, $\frac{3z+2}{z+3}$ is clearly a positive rational number.*

$$\frac{3z+2}{z+3} - z = \frac{2z+2-z^2-3z}{z+3} = \frac{2-z^2}{z+3} < 0 \tag{1}$$

   where the last follows from $z^2 > 2$. We also have:

$$\left(\frac{3z+2}{z+3}\right)^2 - 2 = \frac{9z^2+12z+4-2z^2-12z-18}{z^2+6z+9} = \frac{7(z^2-2)}{(z+3)^2} > 0 \tag{2}$$

   where the last inequality is again a consequence of $z^2 > 2$. Inequality (2) and positivity of $\frac{3z+2}{z+3}$ shows that $\frac{3z+2}{z+3}$ is an upper bound for the set A. On the other hand, (1) asserts that this upper bound is strictly less than the least upper bound z, which is a contradiction.

   (iii) *Using the last two parts and the fact that there is no rational number which squares to 2 (no need to prove this fact), conclude that A does not have a rational least upper bound z.*

   Solution. *By the last two parts, z^2 cannot be greater than or smaller than 2. Therefore, $z^2 = 2$ which is again impossible. Therefore, the initial assumption on the existence of the least upper bound in the set of rational numbers is wrong.*

4. *(i) Show that intersection of finitely many open sets is again an open set.*

Solution. *Suppose U_1, U_2, $\ldots$, U_k are open sets in $\mathbb{R}^n$, and x is an arbitrary point in the intersection $\bigcap_{i=1}^{k} U_i$. Since $x \in U_i$ and U_i is open, there is $r_i > 0$ such that $B_{r_i}(x) \subset U_i$. Define:*

$$r = \min(r_1, r_2, \ldots, r_k).$$

Then we have:

$$B_r(x) \subset B_{r_i}(x) \subset U_i.$$

Therefore, $B_r(x)$ is a subset of the intersection $\bigcap_{i=1}^{k} U_i$ and hence x is an interior point of the intersection. This shows that $\bigcap_{i=1}^{k} U_i$ is an open set.

(ii) Find infinitely many open subsets X_1, X_2, $\ldots$ of $\mathbb{R}^2$ such that:

$$\bigcap_{i=1}^{\infty} X_i$$

is not an open set.

Solution. *Take $X_i = B_{\frac{1}{i}(0)}$. Then $\bigcap_{i=1}^{\infty} X_i = \{0\}$. The single element set $\{0\}$ is not open because there is no ball centered at the origin 0 which is contained in the set $\{0\}$.*