Problem Set 2

1. (i) Find a subset of the following vectors which is a basis for $\mathbb{R}^3$

$$S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 37 \\ -17 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 8 \end{bmatrix} \right\}$$

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(ii) The following vectors are linearly independent. By adding vectors to it, extend this set into a basis

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

2. Find the boundary of each of the following sets. You don't need to justify your answer.

(i) The unit circle $C = \{(x,y) \,:\, x^2 + y^2 = 1\}$

(ii) The upper half-plane $H = \{(x,y) \,:\, y > 0\}$

(i) The one-point set $S = \{(1,1)\}$

(i) The set $\mathbb{Q}^2 = \{(x,y) \,:\, x,y \in \mathbb{Q}\}$ (where $\mathbb{Q}$ denotes the set of rational numbers)

3. Use the Gram-Schmidt process to turn the following basis of $\mathbb{R}^4$ into an orthonormal basis. (We saw the Gram-Schmidt in the context of an example on Friday without mentioning its name. We'll review this process in more generality on Monday.)

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. (i) Find a basis for the following subspace of $\mathbb{R}^4$:

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \mid x - y + 2z + w = 0,\ 2x - 3y - z + 7w = 0 \right\}$$

(ii) Use the basis in the last part to find an orthonormal basis for W.

5. Suppose S is a subset of a vector space V which has $\dim(V)$ elements. Using the properties of bases that we discussed in the class, show that S is a generating set if and only if it is linearly independent.

6. Suppose $\{x_1,x_2,\ldots,x_n\}$ is a basis for a vector space V. Show that each of the following sets is also a basis for V.

   (i) $\{x_1 + x_2, x_2, x_3, \ldots, x_n\}$.

   (i) $\{\lambda \cdot x_1, x_2, x_3, \ldots, x_n\}$, where λ is a non-zero number.