Problem Set 2

1. *(i) Find a subset of the following vectors which is a basis for* $\mathbb{R}^3$

$$S = \left\{ v_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 37 \\ -17 \end{bmatrix}, v_5 = \begin{bmatrix} -3 \\ -5 \\ 8 \end{bmatrix} \right\}.$$

Solution. *Since the dimension of* $\mathbb{R}^3$ *is three, we need to find three linearly independent vectors. We start from the vector* v_1. *This vector is non-zero. So it is linearly independent. The vector* v_2 *is not a multiple of* v_1. *Thus* v_1 *and* v_2 *are not linearly independent. Next we need to see if* v_3 *can be written as a linear combination of* v_1 *and* v_2. *It turns out that* v_3 *is equal to* $-4 \times v_1$. *Thus the three vectors* v_1, v_2 *and* v_3 *are linearly dependent because:*

$$4v_1 + 0 \cdot v_2 + v_3 = 0.$$

So we skip v_3 *and consider the possibility of adding* v_4 *to* $\{v_1, v_2\}$ *to form a linearly independent set. We have* $v_4 = (-3)v_1 + 7v_2$, *which means that* v_1, v_2 *and* v_4 *are also linearly dependent*

$$(-3)v_1 + 7v_2 - v_4 = 0.$$

So we skip v_4 *and our last hope is that* $\{v_1, v_2, v_5\}$ *are linearly independent. Because we already know that* $\{v_1, v_2\}$ *is linearly independent, we just need to check that if* v_5 *is a linear combination of* v_1 *and* v_2. *This is equivalent to the existence of a solution for the following system.*

$$\begin{cases} 2x + y = -3 \\ (-3)x + 4y = -5 \\ x - 2y = 8 \end{cases} \implies \begin{cases} 2x + y = -3 \\ (-3)x + 4y = -5 \\ 5x = 2 \end{cases} \implies \begin{cases} y = -3 - \frac{4}{5} \\ (-3)x + 4y = -5 \\ x = \frac{2}{5} \end{cases}$$

To get to the second system from the first system we add twice of the first equation to the last equation, and to get to the third system from the second system we subtracted $\frac{2}{5}$ *times the last equation from the first equation. It is easy to see that* $x = \frac{2}{5}$ *and* $y = -\frac{19}{5}$ *do not satisfy the second equation of the last equation, which shows that our system does not have any solution and hence* $\{v_1, v_2, v_5\}$ *are linearly independent. Since these are three vectors, they form a basis for* $\mathbb{R}^3$.

(ii) The following vectors are linearly independent. By adding vectors to it, extend this set into a basis

$$S = \left\{ w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Solution. *We add the vectors*

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

to S. *Then we have:*

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \left(\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) + d \left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \implies$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = (a - 3c - 2d)e_1 + (b - c - d)e_2 + dw_1 + cw_2$$

Therefore, any vector in $\mathbb{R}^4$ *can be written as a linear combination of* $\{e_1, e_2, w_1, w_2\}$. *Since this set consists of* $4 = \dim(\mathbb{R}^4)$ *vectors, it is also linearly independent and hence is a basis for* $\mathbb{R}^4$.

2. Find the boundary of each of the following sets. You don't need to justify your answer.

(i) The unit circle $C = \{(x,y) \,:\, x^2 + y^2 = 1\}$

Solution. The circle C.

(ii) The upper half-plane $H = \{(x,y) \,:\, y > 0\}$

Solution. The x-axis $\{(x,y) \,:\, y = 0\}$.

(iii) The one-point set $S = \{(1,1)\}$

Solution. The set $S = \{(1,1)\}$.

(iv) The set $\mathbb{Q}^2 = \{(x,y) \,:\, x,y \in \mathbb{Q}\}$ (where $\mathbb{Q}$ denotes the set of rational numbers)

Solution. $\mathbb{R}^2$.

3. *Use the Gram-Schmidt process to turn the following basis of $\mathbb{R}^4$ into an orthonormal basis. (We saw the Gram-Schmidt in the context of an example on Friday without mentioning its name. We'll review this process in more generality on Monday.)*

$$S = \left\{ x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Solution. **Step 1.** *Define* $u_1 = \dfrac{x_1}{\|x_1\|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}.$

Step 2. *Firstly form* $w_2 = x_2 - (x_2 \cdot u_1)u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - 0 \times u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. *Next let* $u_2 = \dfrac{w_2}{\|w_2\|} =$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Step 3. *Firstly form:*

$$w_3 = x_3 - (x_3 \cdot u_1)u_1 - (x_3 \cdot u_2)u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Next let $u_3 = \dfrac{w_3}{\|w_3\|} = w_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$

Step 4. *Firstly form:*

$$w_4 = x_4 - (x_4 \cdot u_1)u_1 - (x_4 \cdot u_2)u_2 - (x_4 \cdot u_4)u_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - 0 \times \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Next let $u_4 = \dfrac{w_4}{\|w_4\|} = w_4 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$

In Summary, the vectors $\left\{ u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, u_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, u_4 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$ *is the desired orthonormal basis.*

4. *(i) Find a basis for the following subspace of $\mathbb{R}^4$:*

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \mid x - y + 2z + w = 0,\ 2x - 3y - z + 7w = 0 \right\}$$

Solution. *We wish to solve the following system of equations*

$$\begin{cases} x - y + 2z + w = 0 \\ 2x - 3y - z + 7w = 0 \end{cases} \implies \begin{cases} x - y + 2z + w = 0 \\ -y - 5z + 5w = 0 \end{cases} \implies \begin{cases} x + 7z - 4w = 0 \\ y + 5z - 5w = 0 \end{cases}$$

where the second system is obtained from the first system by subtracting twice of the first equation from the second equation. The third system is obtained from the second one by subtracting the second equation from the first one and then multiplying the second equation by -1. Now it is easy to solve the system; we determine z and w arbitrarily and then use the first equation to solve for x and the second equation to solve for y. So a general solution of this system has the following form:

$$\begin{bmatrix} 4t - 7s \\ 5t - 5s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -7 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

It is clear that the two vectors

$$x_1 = \begin{bmatrix} -7 \\ -5 \\ 1 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent because neither of them is a multiple of the other one. Therefore, they form a basis for W.

(ii) Use the basis in the last part to find an orthonormal basis for W.

Solution. *We apply the Gram-Schmidt process to the basis $\{x_1, x_2\}$ obtained in the last part.*

Step 1. *Define* $u_1 = \dfrac{x_1}{\|x_1\|} = \dfrac{1}{5\sqrt{3}} \begin{bmatrix} -7 \\ -5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{5\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{5\sqrt{3}} \\ 0 \end{bmatrix}.$

Step 2. *Firstly form*

$$w_2 = x_2 - (x_2 \cdot u_1)u_1 = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 1 \end{bmatrix} + \frac{53}{5\sqrt{3}} \times \begin{bmatrix} -\frac{7}{5\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{5\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{71}{75} \\ -\frac{22}{15} \\ \frac{53}{75} \\ 1 \end{bmatrix}.$$

Next let $u_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{15}{\sqrt{1023}} \begin{bmatrix} -\frac{71}{75} \\ -\frac{22}{15} \\ \frac{53}{75} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{71}{5\sqrt{1023}} \\ -\frac{22}{\sqrt{1023}} \\ \frac{53}{5\sqrt{1023}} \\ \frac{15}{\sqrt{1023}} \end{bmatrix}.$

Therefore, the vectors $\{u_1, u_2\}$ provide an orthonormal basis for W.

5. *Suppose S is a subset of a vector space V which has* $\dim(V)$ *elements. Using the properties of bases that we discussed in the class, show that S is a generating set if and only if it is linearly independent.*

Solution. *In class, we saw that a set S with* $\dim(V)$ *elements is a basis if we show one of the two conditions:*

- *S is a generating set;*
- *S is a linearly independent set.*

Now suppose that S is a set with $\dim(V)$ *elements which is generating. Then the above fact shows that S is a basis and hence it is linearly independent. Similarly if S is a set with* $\dim(V)$ *elements which is linearly independent, then the above fact shows that S is a basis and hence it is generating.*

6. *Suppose* $S_0 = \{x_1, x_2, \ldots, x_n\}$ *is a basis for a vector space* V*. Show that each of the following sets is also a basis for* V*.*

   (i) $S_1 = \{x_1 + x_2, x_2, x_3, \ldots, x_n\}$.

   Solution. *Since* S_0 *is a basis for* V*, the dimension of* V *is* n*. So to show that* S_1 *is a basis for* V*, it suffices to show that it is a generating set. Suppose* v *is an arbitrary vector in* V*. Since* S_0 *is a basis for* V*, we can find scalars* $a_1, a_2, \ldots, a_n$ *such that*

$$\begin{aligned} v &= a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \\ &= a_1(x_1 + x_2) + (a_2 - a_1)x_2 + a_3x_3 + \cdots + a_nx_n \end{aligned}$$

   This shows that any vector can be written as a linear combination of S_1*.*

   (i) $S_2 = \{\lambda \cdot x_1, x_2, x_3, \ldots, x_n\}$*, where* λ *is a non-zero number.*

   Solution. *To show that* S_2 *is a basis for* V*, it suffices to show that it is a generating set because it has* $n = \dim(V)$ *elements. Suppose* v *is an arbitrary vector in* V*. Since* S_0 *is a basis for* V*, we can find scalars* $a_1, a_2, \ldots, a_n$ *such that*

$$\begin{aligned} v &= a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \\ &= \frac{a_1}{\lambda}(\lambda x_1) + a_2x_2 + a_3x_3 + \cdots + a_nx_n \end{aligned}$$

   This shows that any vector can be written as a linear combination of S_2*.*