Problem Set 3

1. Determine the kernel of the following linear transformations:

(i) $$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y + 5z - w \\ 5x + 3y + z + w \end{bmatrix}.$$

(ii) $$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 7y + z \\ 2x + z + 3y \\ z + 3x - y \end{bmatrix}.$$

2. Verify whether the vector v is in the column space of the matrix A. Show your work.

(i) $v = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 1 \\ 3 & 1 & -7 \end{bmatrix}.$

(ii) $v = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 5 \\ 0 & 3 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}.$

3. Determine whether the given functions $f : \mathbb{R}^2 \to \mathbb{R}$ are continuous at the point (0,0)

(i) $$f(x,y) = \begin{cases} \frac{\sin(x)\cos(y)}{x} & x \neq 0 \\ \cos(y) & x = 0 \end{cases}.$$

(ii) $$f(x,y) = \begin{cases} \frac{x^3 y}{x^4+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

4. For any $m \times n$ matrix A show that $T : \mathbb{R}^n \to \mathbb{R}^m$, defined as follows is continuous.

$$T(x) = Ax.$$

(Hint: Firstly show that it suffices to show that $m = 1$, and then show that for any ϵ we can take $\delta = \frac{\epsilon}{a \cdot n}$ where a is the maximum of the magnitudes of the entries of A.)

5. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^m$ be a function.

(i) Show that if f is continuous then the inverse image $f^{-1}(V)$ for any open subset V of $\mathbb{R}^m$ is an open set.

(ii) Show that if for any open subset V of $\mathbb{R}^m$, the inverse image set $f^{-1}(V)$ is open, then f is continuous.

6. Let V be a subspace of $\mathbb{R}^n$ and $V^{\perp}$ be its orthogonal complement. Let $S_1 = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for V.

(i) Show that any vector x in $\mathbb{R}^n$ can be written as

$$x - (x \cdot v_1)v_1 - (x \cdot v_2)v_2 - \cdots - (x \cdot v_k)v_k$$

belongs to $V^{\perp}$.

(ii) Let S_2 be an orthonormal basis for $V^{\perp}$. Use the last part to show that $S_1 \cup S_2$ is a generating set for $\mathbb{R}^n$.

(iii) Show that $S_1 \cup S_2$ is linearly independent. (Hint: Use the fact that any two vectors in $S_1 \cup S_2$ are orthogonal to each other.)

(iv) Conclude that:

$$\dim(V) + \dim(V^{\perp}) = n.$$