Problem Set 3

1. *Determine the kernel of the following linear transformations:*

(i) $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 3x+2y+5z-w \\ 5x+3y+z+w \end{bmatrix}.$

Solution. *Suppose* $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ *is in the kernel of* T. *Then we have:*

$$\begin{cases} 3x+2y+5z-w=0 \\ 5x+3y+z+w=0 \end{cases} \implies \begin{cases} 3x+2y+5z-w=0 \\ 8x+5y+6z=0 \end{cases} \implies \begin{cases} \frac{1}{8}y+\frac{11}{4}z-w=0 \\ 8x+5y+6z=0 \end{cases}$$

The second system is obtained from the first by adding the first equation to the second equation. Then subtracting $\frac{3}{8}$ *times the second equation from the first one gives the third system. Now if we pick* y *and* z *arbitrarily, then the second equation can be used to determine* x *and the first equation can be used to determine* w. *Therefore, a general solution to the above system is given as:*

$$\left\{\begin{bmatrix} \frac{1}{8}s+\frac{11}{4}t \\ s \\ t \\ -\frac{5}{8}s-\frac{3}{4}t \end{bmatrix} \mid s,t \in \mathbb{R}\right\}.$$

(ii) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+7y+z \\ 2x+z+3y \\ z+3x-y \end{bmatrix}.$

Solution. *Suppose* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *is in the kernel of* T. *Then we have:*

$$\begin{cases} x+7y+z=0 \\ 2x+3y+z=0 \\ 3x-y+z=0 \end{cases} \implies \begin{cases} x+7y+z=0 \\ x-4y=0 \\ 2x-8y=0 \end{cases} \implies \begin{cases} 11y+z=0 \\ x-4y=0 \\ 0=0 \end{cases}$$

The second system is obtained from the first one by subtracting the first equation from the second and the third equations. Then subtracting multiples of the second equation from the first and the third equations gives rise to the third system. Now if we pick y *arbitrarily, then the first equation can be used to determine* z, *the second equation can be used to determine* x *and the third equation is always satisfied. Therefore, a general solution to the above system is given as:*

$$\left\{\begin{bmatrix} 4t \\ t \\ -11t \end{bmatrix} \mid t \in \mathbb{R}\right\}$$

2. *Verify whether the vector v is in the column space of the matrix A. Show your work.*

(i) $v = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 1 \\ 3 & 1 & -7 \end{bmatrix}.$

Solution. *If the vector v is equal to $A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then x, y and z need to satisfy the following*

$$x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix} \implies \begin{cases} x - y + 5z = 1 \\ 2x + z = 7 \\ 3x + y - 7z = 9 \end{cases} \implies \begin{cases} 4x - 2z = 10 \\ 2z = 2 \\ 3x + y - 7z = 9 \end{cases}$$

To obtain the second system, we subtract halves of the first and the third equations from the second equation and add then add the third equation to the first equation. The second equation of the second system implies that $z = 1$. Using the first equation next we can conclude that $x = 3$. Finally $y = 7$ is the consequence of the third equation. In particular, these choices of x, y and z satisfy the equations and v is in the column space of A.

(ii) $v = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 5 \\ 0 & 3 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}.$

Solution. *If the vector v is equal to $A\begin{bmatrix} x \\ y \end{bmatrix}$, then x and y need to satisfy the following*

$$x\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 5 \\ 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix} \implies \begin{cases} x + 5y = 8 \\ 3y = 3 \\ x + 3y = -1 \\ 2y = 1 \end{cases}$$

The above system doesn't have any solution because the second equation implies that $y = 1$ and the last equation implies that $y = \frac{1}{2}$. So v is in the column space of A.

3. *Determine whether the given functions* $f : \mathbb{R}^2 \to \mathbb{R}$ *are continuous at the point* $(0{,}0)$

(i) $f(x{,}y) = \begin{cases} \frac{\sin(x)\cos(y)}{x} & x \neq 0 \\ \cos(y) & x = 0 \end{cases}$.

Solution. *The function* f *can be written as the product*

$$f(x{,}y) = g(x)h(y)$$

where

$$g(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad h(y) = \cos(y).$$

The function h *is clearly continuous at* 0 *and we can see that* g *is continuous at* 0 *using l'Hospital's rule:*

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1.$$

Notice that we can use l'Hospital's rule because both the numerator and the denominator of $\frac{\sin(x)}{x}$ *are converging to* 0 *as* x *goes to* 0.

(ii) $f(x{,}y) = \begin{cases} \frac{x^3 y}{x^4+y^4} & (x{,}y) \neq (0{,}0) \\ 0 & (x{,}y) = (0{,}0) \end{cases}$.

Solution. *The restriction of this function to the line* $x = y$ *has the form:*

$$f(x{,}x) = \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

for $x \neq 0$ *and* $f(0{,}0) = 0$. *Therefore, if we pick a sequence of points* $\{(x_i{,}x_i)\}_i$ *on the line* $x = y$ *which converges to* $(0{,}0)$ *but* $(x_i{,}x_i) \neq (0{,}0)$, *then* $f(x_i{,}x_i) = \frac{1}{2}$ *does not converge to* $f(0{,}0) = 0$. *This shows that* f *is not continuous at* $(0{,}0)$.

4. *For any $m \times n$ matrix A show that $T : \mathbb{R}^n \to \mathbb{R}^m$, defined as follows is continuous.*

$$T(x) = Ax.$$

Solution. *We just have to show each coordinate function of T is continuous. Any such coordinate function is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$. This reduces the problem to the case that $n = 1$. In this case, the function has the form*

$$T(x_1,x_2,\dots,x_n) = a_1x_1 + a_2x_2 + \dots a_nx_n$$

Fix $\mathbf{x} = (x_1,x_2,\dots,x_n)$ and $\epsilon > 0$ and take $\delta = \frac{\epsilon}{a \cdot n}$ where a is the maximum of the magnitudes of the constants a_i. For $\mathbf{y} = (y_1,y_2,\dots,y_n) \in \mathbb{R}^n$ with

$$\|(x_1 - y_1,x_2 - y_2,\dots,x_n - y_n)\| \leq \delta = \frac{\epsilon}{a \cdot n}$$

we have

$$\begin{aligned}|T(x_1,x_2,\dots,x_n) - T(y_1,y_2,\dots,y_n)| &\leq |a_1||x_1 - y_1| + |a_2||x_2 - y_2| + \dots |a_n||x_n - y_n| \\ &\leq (|a_1| + |a_2| + \dots |a_n|)\|(x_1 - y_1,x_2 - y_2,\dots,x_n - y_n)\| \\ &< (|a_1| + |a_2| + \dots |a_n|)\frac{\epsilon}{a \cdot n} \\ &\leq na\frac{\epsilon}{a \cdot n} \\ &= \epsilon.\end{aligned}$$

In the second inequality, we use the inequality

$$|x_i - y_i| \leq \|(x_1 - y_1,x_2 - y_2,\dots,x_n - y_n)\|$$

5. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^m$ be a function.*

   (i) Show that if f is continuous then the inverse image $f^{-1}(V)$ for any open subset V of $\mathbb{R}^m$ is an open set.

   Solution. *Let $x \in f^{-1}(V)$. This means that $f(x)$ is a point of the open set V. Therefore, there is $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset V$. Continuity of f implies that there is $\delta > 0$ such that if $y \in B_\delta(x)$, then $f(y) \in B_\epsilon(f(x))$. That is to say, $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Since $f^{-1}(B_\epsilon(f(x))) \subset f^{-1}(V)$, we conclude that $B_\delta(x) \subset f^{-1}(V)$, which means that x is an interior point of $f^{-1}(V)$. This shows that $f^{-1}(V)$ is open.*

   (ii) Show that if for any open subset V of $\mathbb{R}^m$, the inverse image set $f^{-1}(V)$ is open, then f is continuous.

   Solution. *Let $x \in \mathbb{R}^n$ and $\epsilon > 0$ be given. Then $B_\epsilon(f(x))$ is an open subspace of $\mathbb{R}^m$, and hence $f^{-1}(B_\epsilon(f(x)))$ has to be open. Since $x \in f^{-1}(B_\epsilon(f(x)))$, openness of $f^{-1}(B_\epsilon(f(x)))$ implies that there should exist $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. This in turn means that if $y \in B_\delta(x)$, then $f(y) \in B_\epsilon(f(x))$. Thus f is continuous.*

6. *Let V be a subspace of $\mathbb{R}^n$ and $V^\perp$ be its orthogonal complement. Let $S_1 = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for V.*

   (i) *Show that any vector x in $\mathbb{R}^n$, the vector*

   $$x - (x \cdot v_1)v_1 - (x \cdot v_2)v_2 - \cdots - (x \cdot v_k)v_k$$

   belongs to $V^\perp$.

   Solution. *We want to show that the vector*

   $$y = x - (x \cdot v_1)v_1 - (x \cdot v_2)v_2 - \cdots - (x \cdot v_k)v_k$$

   is in the orthogonal complement of V. Since $\{v_1, \ldots, v_k\}$ is a basis for V, it suffices to show that y is orthogonal to v_i for any i:

   $$\begin{aligned} y \cdot v_i =& x \cdot v_i - (x \cdot v_1)(v_1 \cdot v_i) - \cdots - (x \cdot v_{i-1})(v_{i-1} \cdot v_i) - (x \cdot v_i)(v_i \cdot v_i) - \ldots \cdots - (x \cdot v_k)(v_k \cdot v_i) \\ =& x \cdot v_i - 0 - \cdots - 0 - (x \cdot v_i) - 0 - \cdots - 0 \\ =& 0. \end{aligned}$$

   (ii) *Let S_2 be an orthonormal basis for $V^\perp$. Use the last part to show that $S_1 \cup S_2$ is a generating set for $\mathbb{R}^n$.*

   Solution. *Suppose $S_2 = \{v_1, v_2, \ldots, v_l\}$. For an arbitrary vector $x \in \mathbb{R}^n$, the last part implies that*

   $$y = x - (x \cdot v_1)v_1 - (x \cdot v_2)v_2 - \cdots - (x \cdot v_k)v_k$$

   is an element of $V^\perp$ and hence it can be written as a linear combination of the elements of S_2

   $$y = b_1 u_1 + \cdots + b_l u_l.$$

   Thus we have:

   $$\begin{aligned} x =& (x - y) + y \\ =& ((x \cdot v_1)v_1 + (x \cdot v_2)v_2 + \cdots + (x \cdot v_k)v_k) + (b_1 u_1 + \cdots + b_l u_l) \end{aligned}$$

   So x can be written as a linear combination of the elements of $S_1 \cup S_2$.

   (iii) *Show that $S_1 \cup S_2$ is linearly independent. (Hint: Use the fact that any two vectors in $S_1 \cup S_2$ are orthogonal to each other.)*

   Solution. *Note that any element of $S_1 \cup S_2$ has norm one and any two different elements are orthogonal to each other. To show that $S_1 \cup S_2$ is linearly independent, let*

   $$a_1 v_1 + \cdots + a_k u_k + b_1 u_1 + \cdots + b_l u_l = 0.$$

   The inner product of the above identity with v_i implies that

   $$a_i(v_i \cdot v_i) = 0 \implies a_i = 0$$

   and its inner product with u_j implies that

   $$b_j(u_j \cdot u_j) = 0 \implies b_j = 0.$$

   This proves the claim.

(iv) Conclude that:

$$\dim(V) + \dim(V^{\perp}) = n.$$

Solution. *Since $S_1 = \{v_1, v_2, \ldots, v_k\}$ is a basis for V and $S_2 = \{u_1, u_2, \ldots, u_l\}$, the dimensions of V and $V^{\perp}$ are equal to k and l. The sum $k + l$ is equal to n because $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_l\}$ is a basis for $\mathbb{R}^n$. Therefore, we have:*

$$\dim(V) + \dim(V^{\perp}) = n.$$