Problem Set 5

1. In each part, use the chain rule to find the Jaobian of the function $F \circ G$ at the point $\mathbf{c}$.

(i) $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2y + y^2z + z^2x$, $G(x) = \begin{bmatrix} e^x \sin(x) \\ \frac{x}{x^2+1} \\ \tan(x) \end{bmatrix}$, $\mathbf{c} = \frac{\pi}{4}$.

(ii) $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4x^2y^2 + \sin(xz) \\ \frac{e^{xy}}{z} \end{bmatrix}$, $G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x^2 + y^2 \\ xy^2 + x^2y \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

2. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x^3 - 3xy^2 \\ 3x^2y - y^3 \end{bmatrix}$$

(i) Show that F is a local diffeomorphism at any point other than the origin.

(ii) Suppose $\mathbf{c} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{b} \in \mathbb{R}^2$ is the point $F(\mathbf{c}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, F is a diffeomorphism on the ball $B_r(\mathbf{c})$ and G is the inverse of $F|_{B_r(\mathbf{c})}$. Use the inverse function theorem to find $J_{\mathbf{b}}(G)$

3. Let $Q = \{(x,y) \,:\, x > 0,\, y > 0\} \subseteq \mathbb{R}^2$. The **hyperbolic coordinate system** on Q is defined by assigning to each point (x,y) new coordinates (u,v), where $u = \ln\sqrt{\frac{x}{y}}$ and $v = \sqrt{xy}$.

(i) Prove that the function $f : Q \to \mathbb{R}^2$ defined by $f(x,y) = (u,v)$ is a local diffeomorphism at every point of Q.

(ii) Find the (global) inverse function $f^{-1}(u,v)$.

4. Let $f(x,y) = x^2 - y(y-1)^2$, and let $V = \{(x,y) \,:\, f(x,y) = 0\}$.

(i) Sketch the set V in $\mathbb{R}^2$. (You can use computer assistance; no need to show work.)

(ii) Use the Implicit Function Theorem to find all points on V for which V is, locally, the graph of a function $x = \varphi(y)$. Mark these points on your graph in part (a).

5. Suppose $F : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and $\varphi : \mathbb{R}^d \to \mathbb{R}^n$ are continuously differentiable functions such that

$$F\left(\begin{bmatrix} \phi(y) \\ y \end{bmatrix}\right) = \vec{0}_n$$

for any $y \in \mathbb{R}^d$. Suppose also $\mathbf{c} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^d$ is a point such that $\mathbf{a} = \varphi(\mathbf{b})$ and if we present the Jacobian of F at $\mathbf{c}$ as

$$J_{\mathbf{c}}(F) = \begin{bmatrix} L \vdots R \end{bmatrix}$$

where $L \in M_{n\times n}(\mathbb{R})$ and $R \in M_{n\times d}(\mathbb{R})$, then L is an invertible matrix. Use the chain rule to show that:

$$J_{\mathbf{b}}(\varphi) = -L^{-1}R.$$

(So this problem asks you to prove the second part of the implicit function theorem)

6. Suppose $U(r_1,r_2)$ is the ring given by the points in $\mathbb{R}^2$ which belong to the region between the circles of radii r_1 and r_2 centered at the origin.

   (i) Sketch the region $U(r_1,r_2)$.

   (ii) Find a diffeomorphism from $U(2,3)$ to $U(1,4)$.