Problem Set 6

1. *(i) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be the function $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = e^x \sin(y) + \cos(x + y - 2z)$. Find the degree 3 Taylor polynomial of F at the point $\mathbf{c} = \begin{bmatrix} 2 \\ \frac{\pi}{6} \\ 1 \end{bmatrix}$.*

Solution. *The Taylor polynomial of the sum of two functions is equal to the sum of Taylor polynomials. So we can compute the Taylor polynomials R_3 and S_3 of the functions $G(x,y,z) = e^x \sin(y)$ and $H(x,y,z) = \cos(x + y - 2z)$ centered at $\mathbf{c}$ separately and then add them up to find the Taylor polynomial T_3 of F centered at $\mathbf{c}$. To start, we firstly compute partial derivatives of G with order at most three. Notice that G depends only on x and y and the partial derivative with respect to z and higher order partial derivatives which contain z are equal to zero. So we focus on partial derivatives with respect to x and y.*

	at (x,y,z)	*at* c
G	$e^x \sin(y)$	$\frac{1}{2}e^2$
$\frac{\partial G}{\partial x}$	$e^x \sin(y)$	$\frac{1}{2}e^2$
$\frac{\partial G}{\partial y}$	$e^x \cos(y)$	$\frac{\sqrt{3}}{2}e^2$
$\frac{\partial^2 G}{\partial x^2}$	$e^x \sin(y)$	$\frac{1}{2}e^2$
$\frac{\partial^2 G}{\partial y^2}$	$-e^x \sin(y)$	$-\frac{1}{2}e^2$
$\frac{\partial^2 G}{\partial x \partial y}$	$e^x \cos(y)$	$\frac{\sqrt{3}}{2}e^2$

	at (x,y,z)	*at* c
$\frac{\partial^3 G}{\partial x^3}$	$e^x \sin(y)$	$\frac{1}{2}e^2$
$\frac{\partial^3 G}{\partial x^2 \partial y}$	$e^x \cos(y)$	$\frac{\sqrt{3}}{2}e^2$
$\frac{\partial^3 G}{\partial x \partial y^2}$	$-e^x \sin(y)$	$-\frac{1}{2}e^2$
$\frac{\partial^3 G}{\partial y^3}$	$e^x \cos(y)$	$-\frac{\sqrt{3}}{2}e^2$

Using general formula for the Taylor polynomials we can write

$$R_3(\mathbf{c}+\begin{bmatrix} h \\ k \\ l \end{bmatrix}) = \frac{1}{2}e^2+\frac{1}{2}e^2h+\frac{\sqrt{3}}{2}e^2k+\frac{1}{4}e^2h^2+\frac{\sqrt{3}}{2}e^2hk-\frac{1}{4}e^2k^2+\frac{1}{12}e^2h^3+\frac{\sqrt{3}}{4}e^2h^2k-\frac{1}{4}e^2hk^2-\frac{\sqrt{3}}{12}e^2k^3.$$

To compute S_3, we note that the function G is composition of the linear function $x+y-2z$. The Taylor polynomial S_3' of the linear function $x + y - 2z$ at $\mathbf{c}$ is given by itself and hence we have

$$S_3'(\mathbf{c} + \begin{bmatrix} h \\ k \\ l \end{bmatrix}) = (2 + h) + (\frac{\pi}{6} + k) - 2(1 + l) = \frac{\pi}{6} + h + k - 2l.$$

The degree three Taylor polynomial S_3'' of $\cos(x)$ centered at the point $\frac{\pi}{6} = 2 + \frac{\pi}{6} - 2 \times 1$ is equal to

$$S_3''(\frac{\pi}{6} + t) = \cos(\frac{\pi}{6}) - \sin(\frac{\pi}{6})t - \frac{\cos(\frac{\pi}{6})}{2}t^2 + \frac{\sin(\frac{\pi}{6})}{6}t^3 = \frac{\sqrt{3}}{2} - \frac{1}{2}t - \frac{\sqrt{3}}{4}t^2 + \frac{1}{12}t^3$$

Therefore, the Taylor polynomial of the composed function G is given as

$$S_3(\mathbf{c}+\begin{bmatrix} h \\ k \\ l \end{bmatrix}) = \frac{\sqrt{3}}{2} - \frac{1}{2}(h+k-2l) - \frac{\sqrt{3}}{4}(h+k-2l)^2 + \frac{1}{12}(h+k-2l)^3$$

By adding up $R_3(\mathbf{c}+\begin{bmatrix} h \\ k \\ l \end{bmatrix})$ *and* $S_3(\mathbf{c}+\begin{bmatrix} h \\ k \\ l \end{bmatrix})$ *we obtain the desired Taylor polynomial* $T_3(\mathbf{c}+\begin{bmatrix} h \\ k \\ l \end{bmatrix})$.

(ii) Find the degree 3 Taylor polynomial of $F:\mathbb{R}^3 \to \mathbb{R}$ *given by* $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = e^{x+2y^2+3z^3}$ *centered at the point* $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution. *Since* $G(x,y,z) = x+2y^2+3z^3$ *is a polynomial of degree 3, the degree 3 polynomial* S_3 *of* $x+2y^2+3z^3$ *centered at the point* $\mathbf{c}$ *is*

$$S_3(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = x+2y^2+3z^3.$$

The degree Taylor polynomial R_3 *of* $h(t) = e^t$ *at the point* $G(\mathbf{c}) = 0$ *is equal to*

$$R_3(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}.$$

Thus the degree Taylor polynomial T_3 *of* F *at* $\mathbf{c}$ *is given by composing* R_3 *and* S_3 *and then dropping terms of degree higher than 3.*

$$R_3 \circ S_3(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = R_3(x+2y^2+3z^3) = 1+(x+2y^2+3z^3)+\frac{(x+2y^2+3z^3)^2}{2}+\frac{(x+2y^2+3z^3)^3}{6} =$$

$$1+(x+2y^2+3z^3)+\frac{(x+2y^2)^2}{2}+\frac{x^3}{6}+o(\| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \|^3) = 1+(x+2y^2+3z^3)+\frac{x^2+4xy^2}{2}+\frac{x^3}{6}+o(\| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \|^3)$$

Consequently, we have

$$T_3(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = 1 + x + \frac{x^2}{2} + 2y^2 + \frac{x^3}{6} + \frac{4xy^2}{2} + 3z^3.$$

2. *Let $f(x,y) = x^2 - y(y-1)^2$, and let $V = \{(x,y) : f(x,y) = 0\}$. In the last problem set, you studied the set V using the implicit function theorem.*

   (i) *Show that in a neighborhood of $(0,0) \in V$, the set V can be described locally as the graph of a function $y = \phi(x)$. (Recall that last week you showed at any point in V, except $(0,0)$ and $(0,1)$, the set V can be described locally as the graph of a function $x = \psi(y)$.)*

   Solution.
$$J_{(x,y)}f = \begin{bmatrix} 2x & -3y^2 + 4y - 1 \end{bmatrix} \implies J_{(0,0)}f = \begin{bmatrix} 0 & -1 \end{bmatrix}$$
   Since the second entry of $J_{(0,0)}f$, which is $\frac{\partial f}{\partial y}(0,0)$ is non-zero, the implicit function theorem implies that the set V in a neighborhood of $(0,0)$ can be described as the graph of a function $y = \phi(x)$ with $\phi(0) = 0$.

   (ii) *Find $\phi'(0)$ for the function that you found in the first part.*

   Solution. *The implicit function theorem implies that $\phi'(0) = -LR^{-1}$, where R and L are given by*
$$J_{(0,0)}f = \begin{bmatrix} 0 & -1 \end{bmatrix} = \begin{bmatrix} L & R \end{bmatrix}.$$
   Therefore, $\phi'(0) = 0$.

   (iii) *Find the degree 3 Taylor polynomial of ϕ centered at 0.*

   Solution. *Since $\phi(0) = 0$ and $\phi'(0) = 0$, the the degree 3 Taylor polynomial T_3 of ϕ centered at 0 has the following form*
$$T_3(x) = ax^2 + cx^3.$$
   Using the properties of Taylor polynomials for functions defined implicitly, which we discussed in the class, we know that $f(x,T_3(x))$ does not have any term of degree lower than 4. That is to say:
$$x^2 - (ax^2 + cx^3)(ax^2 + cx^3 - 1)^2 = o(x^3) \implies (1-a)x^2 - cx^3 = o(x^3)$$
   This shows that $a = 1$ and $c = 0$.

3. *Suppose* $V \subset \mathbb{R}^3$ *is the set of points* (x,s,t) *which satisfy the following equation*

$$F(x,s,t) = x^3 + xs + t^2 - 2.$$

Notice that the point $(-1,1,2)$ *is an element of* V*.*

(i) *Use the implicit function theorem to show that there are neighborhoods* B *of* $(1,2)$ *and* W *of* -1 *and a function* $\varphi : B \to W$ *such that* $V \cap (W \times B)$ *is given by the graph of the function* ϕ*:*

$$G_\phi = \{(\phi(s,t),s,t) \mid (s,t) \in B\}$$

Solution.

$$J_{(x,s,t)}F = \begin{bmatrix} 3x^2 + s & x & 2t \end{bmatrix} \implies J_{(-1,1,2)}F = \begin{bmatrix} 4 & -1 & 4 \end{bmatrix}$$

Since the first entry of $J_{(-1,1,2)}F$*, which is* $\frac{\partial f}{\partial x}(-1,1,2)$ *is non-zero, the implicit function theorem implies that the set* V *in a neighborhood of* $(-1,1,2)$ *can be described as the graph of a function* $x = \phi(s,t)$ *with* $\phi(1,2) = -1$*.*

(ii) *Find the degree* 2 *Taylor polynomial of* φ *centered at* $(1,2)$*.*

Solution. *The implicit function theorem implies that* $J_{(1,2)}\phi = -L^{-1}R$*, where* R *and* L *are given by*

$$J_{(-1,1,2)}F = \begin{bmatrix} 4 & -1 & 4 \end{bmatrix} = \begin{bmatrix} L & R \end{bmatrix} \implies L = \begin{bmatrix} 4 \end{bmatrix},\ R = \begin{bmatrix} -1 & 4 \end{bmatrix}.$$

Therefore, $J_{(1,2)}\phi = \begin{bmatrix} \frac{1}{4} & -1 \end{bmatrix}$.

The computation of $\phi(1,2)$ *and* $J_{(1,2)}\phi$ *shows that the degree* 2 *Taylor polynomial* T_2 *of* φ *centered at* $(1,2)$ *has the following form.*

$$T_2(1+h,2+k) = -1 + \frac{1}{4}h - k + ah^2 + bhk + ck^2.$$

In order to determine a*,* b *and* c*, we use the fact that if we plug in* $T_2(1+h,2+k)$ *for* x*,* $1+h$ *for* s *and* $2+k$ *for* t *in* $F(x,s,t) = x^3 + xs + t^2 - 2$*, there does not exists in any term in terms of* h *and* k *with degree less than* 2

$$F(T_2(1+h,2+k),1+h,2+k) =$$

$$= (-1 + \frac{1}{4}h - k + ah^2 + bhk + ck^2)^3 + (-1 + \frac{1}{4}h - k + ah^2 + bhk + ck^2)(1+h) + (2+k)^2 - 2$$

After dropping terms with degree higher than 2*, we obtain*

$$-3(\frac{1}{4}h-k)^2+4(ah^2+bhk+ck^2)+h(\frac{1}{4}h-k)+k^2 = 0 \implies (\frac{1}{16}+4a)h^2+(\frac{1}{2}+4b)hk+(-2+4c)k^2 = 0$$

The last identity shows that $a = -\frac{1}{64}$*,* $b = -\frac{1}{8}$ *and* $c = \frac{1}{2}$*.*

4. *Suppose* $F : \mathbb{R}^3 \to \mathbb{R}^2$ *is a* C^1 *function and* $V = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 0\}$, *and for* $\mathbf{c} \in V$

$$J_{\vec{c}}(F) = \begin{bmatrix} 4 & 5 & -6 \\ -6 & 8 & 9 \end{bmatrix}.$$

(i) *Which variables (out of* x, y, *and* z*) can be chosen as the free variable so that the implicit function theorem implies that at* $\vec{c}$, V *is locally the graph of a function of the free variable? Explain briefly.*

Solution. *The* 2×2 *matrices*

$$\begin{bmatrix} 4 & 5 \\ -6 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix}$$

obtained by removing the last column (corresponding to the variable z*) and the first column (corresponding to the variable* x*) from* $J_{\vec{c}}(F)$ *are invertible because*

$$\det \begin{bmatrix} 4 & 5 \\ -6 & 8 \end{bmatrix} = 62 \neq 0, \qquad \begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} = 93 \neq 0.$$

Therefore, by the implicit function theorem, we may use x *and* z *as the free variables so that* V *in a neighborhood of* c *can be describsed as the graph of a function of the variable* x *or* z. *However, we cannot use the implicit function theorem for the variable* y *because the matrix obtained from* $J_{\vec{c}}(F)$ *by removing the second column (corresponding to the variable* y*) is*

$$\begin{bmatrix} 4 & -6 \\ -6 & 9 \end{bmatrix}$$

which has vanishing determinant and hence it is not invertible.

(ii) *Give an example of a* 2×3 *matrix* A *such that if* $J_{\vec{c}}(F) = A$ *in the setup above, then exactly one of the variables can be free, but not either of the other two. Explain briefly.*

Solution. *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

The 2×2 *matrix*

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

obtained from removing the last column of A *is invertible because its determinant is equal to* 1. *Thus we may use the variable* z *to represent* V *in a neighborhood of* c *as the graph of a function of* z. *However, we may not use the other two variables because the matrices obtained from removing the first or the second column are not invertible:*

$$\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

5. Suppose $P : \mathbb{R}^3 \to \mathbb{R}$ is the polynomial $p(\vec{h}) = \vec{h}^{\alpha}$ for $\alpha \in \mathbb{N}^3$. To be more specific, if $\vec{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$ and $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, then $p(\vec{h}) = h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3}$. Find partial derivatives of arbitrary oder of the function p at the point $\vec{h} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution. We have

$$\frac{\partial^{i+j+k} p}{\partial^i h_1 \partial^j h_2 \partial^k h_3}(\vec{h}) = \alpha_1(\alpha_1-1)\ldots(\alpha_1-i+1)\alpha_2(\alpha_2-1)\ldots(\alpha_2-j+1)\alpha_3(\alpha_3-1)\ldots(\alpha_3-k+1)h_1^{\alpha_1-i}h_2^{\alpha_2-j}h_3^{\alpha_3-k}.$$

From this it can be easily seen that

$$\frac{\partial^{i+j+k} p}{\partial^i h_1 \partial^j h_2 \partial^k h_3}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{cases} \alpha_1!\alpha_2!\alpha_3! & i = \alpha_1, j = \alpha_2, k = \alpha_k \\ 0 & \text{otherwise} \end{cases}.$$

6. *Suppose* $F : \mathbb{R}^2 \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are* C^∞ *functions. Suppose* $F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0$. *Suppose the degree* 2 *Taylor polynomial of* F *at the point* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *is*

$$T_2(\begin{bmatrix} h \\ k \end{bmatrix}) = a_1 h + a_2 k + b_1 h^2 + b_2 hk + b_3 k^2$$

and the degree 2 *Taylor polynomial of* g *at 0 is*

$$S_2(l) = c + dl + el^2.$$

(i) *Determine the constants* a_i, b_j, c, d *and* e *in terms of partial derivatives of* F *at* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and the derivatives of* g *at* 0.

Solution. *The formulas for Taylor polynomials imply that*

$$a_1 = \frac{\partial F}{\partial x}(0,0), \quad a_2 = \frac{\partial F}{\partial y}(0,0), \quad b_1 = \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(0,0), \quad b_2 = \frac{\partial^2 F}{\partial x \partial x}(0,0), \quad b_3 = \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(0,0),$$

and

$$c = g(0), \quad d = g'(0), \quad e = \frac{1}{2}g''(0).$$

(ii) *Suppose* R *is the polynomial of degree* 2 *obtained by forming the composition* $S_2 \circ T_2$ *and then erasing the terms of degree greater than* 2. *Use your answer to the first part to determine* R *in terms of the partial derivatives of* F *at* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and the derivatives of* g *at* 0.

Solution.

$$S_2 \circ T_2(\begin{bmatrix} h \\ k \end{bmatrix}) = c + d(a_1 h + a_2 k + b_1 h^2 + b_2 hk + b_3 k^2) + e(a_1 h + a_2 k + b_1 h^2 + b_2 hk + b_3 k^2)^2$$

$$\implies R_2(\begin{bmatrix} h \\ k \end{bmatrix}) = c + (da_1)h + (da_2)k + (db_1 + ea_1^2)h^2 + (db_2 + 2ea_1a_2)hk + (db_3 + ea_2^2)k^2 =$$

$$= g(0) + (g'(0)\frac{\partial F}{\partial x}(0,0))h + (g'(0)\frac{\partial F}{\partial y}(0,0))k + (g'(0)\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(0,0) + e\frac{\partial F}{\partial x}(0,0)^2)h^2 +$$

$$+(g'(0)\frac{\partial^2 F}{\partial x \partial y}(0,0) + 2e\frac{\partial F}{\partial x}(0,0)\frac{\partial F}{\partial y}(0,0))hk + (\frac{1}{2}g'(0)\frac{\partial^2 F}{\partial y^2}(0,0) + e\frac{\partial F}{\partial y}(0,0)^2)k^2$$

(iii) *Show that* R *is equal to the degree* 2 *Taylor polynomial of the composed function* $g \circ F : \mathbb{R}^2 \to \mathbb{R}$. *(Hint: To show this you have to relate the coefficients of* R *to the partial derivatives of* $g \circ F$ *at* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *Use chain rule (for one variable functions) to establish this relation.)*

Notice that this is a special case of a general result about Taylor polynomials of composed functions that we stated in the class. The proof of the general case is similar.

Solution. *Clearly we have*

$$g \circ F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = g(0). \tag{1}$$

This shows that the constant term in R_2 *matches with the constant term in the degree* 2 *Taylor polynomial of* $g \circ F$ *centered at* $[\begin{bmatrix} 0 \\ 0 \end{bmatrix}]$.

Using chain rule we may write:

$$\frac{\partial(g \circ F)}{\partial x}(x,y) = g'(F(x,y))\frac{\partial F}{\partial x}(x,y), \qquad \frac{\partial(g \circ F)}{\partial y}(x,y) = g'(F(x,y))\frac{\partial F}{\partial y}(x,y).$$

By plugging in (0,0) *for* (x,y) *in the above identities we can see that the coefficient of* h *and* k *in* R_2 *and the degree* 2 *Taylor polynomial of* $g \circ F$ *centered at* $[\begin{bmatrix} 0 \\ 0 \end{bmatrix}]$ *match.*

By differentiating the two identities in (1) *and applying the product rule and the chain rule we obtain*

$$\frac{\partial^2(g \circ F)}{\partial x^2}(x,y) = g'(F(x,y))\frac{\partial^2 F}{\partial x^2}(x,y) + g''(F(x,y))(\frac{\partial F}{\partial x}(x,y))^2,$$

$$\frac{\partial^2(g \circ F)}{\partial x \partial y}(x,y) = g'(F(x,y))\frac{\partial^2 F}{\partial x \partial y}(x,y) + g''(F(x,y))\frac{\partial F}{\partial x}(x,y)\frac{\partial F}{\partial y}(x,y)$$

and

$$\frac{\partial^2(g \circ F)}{\partial y^2}(x,y) = g'(F(x,y))\frac{\partial^2 F}{\partial y^2}(x,y) + g''(F(x,y))(\frac{\partial F}{\partial y}(x,y))^2.$$

By plugging in (0,0) *for* (x,y) *in the above identities we can see that the coefficient of* h^2, hk *and* k^2 *in* R_2 *and the degree* 2 *Taylor polynomial of* $g \circ F$ *centered at* $[\begin{bmatrix} 0 \\ 0 \end{bmatrix}]$ *match.*