Problem Set 6

1. (i) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be the function $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = e^x \sin(y) + \cos(x + y - 2z)$. Find the degree 3 Taylor polynomial of F at the point $\mathbf{c} = \begin{bmatrix} 2 \\ \frac{\pi}{6} \\ 1 \end{bmatrix}$.

(ii) Find the degree 3 Taylor polynomial of $F : \mathbb{R}^3 \to \mathbb{R}$ given by $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = e^{x+2y^2+3z^3}$ centered at the point $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

2. Let $f(x,y) = x^2 - y(y-1)^2$, and let $V = \{(x,y) : f(x,y) = 0\}$. In the last problem set, you studied the set V using the implicit function theorem.

   (i) Show that in a neighborhood of $(0,0) \in V$, the set V can be described locally as the graph of a function $y = \phi(x)$. (Recall that last week you showed at any point in V, except (0,0) and (0,1), the set V can be described locally as the graph of a function $x = \psi(y)$.)

   (ii) Find $\phi'(0)$ for the function that you found in the first part.

   (iii) Find the degree 3 Taylor polynomial of ϕ centered at 0.

3. Suppose $V \subset \mathbb{R}^3$ is the set of points (x,s,t) which satisfy the following equation

$$F(x,s,t) = x^3 + xs + t^2 - 2.$$

Notice that the point $(-1,1,2)$ is an element of V.

(i) Use the implicit function theorem to show that there are neighborhoods B of $(1,2)$ and W of -1 and a function $\varphi : B \to W$ such that $V \cap (W \times B)$ is given by the graph of the function ϕ:

$$G_\phi = \{(\phi(s,t),s,t) \mid (s,t) \in B\}$$

(ii) Find the degree 2 Taylor polynomial of φ centered at $(1,2)$.

4. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is a C^1 function and $V = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 0\}$, and for $\mathbf{c} \in V$

$$J_{\vec{c}}(F) = \begin{bmatrix} 4 & 5 & -6 \\ -6 & 8 & 9 \end{bmatrix}.$$

(i) Which variables (out of x, y, and z) can be chosen as the free variable so that the implicit function theorem implies that at $\vec{c}$, V is locally the graph of a function of the free variable? Explain briefly.

(ii) Give an example of a 2×3 matrix A such that if $J_{\vec{c}}(F) = A$ in the setup above, then *exactly one* of the variables can be free, but not either of the other two. Explain briefly.

5. Suppose $P : \mathbb{R}^3 \to \mathbb{R}$ is the polynomial $p(\vec{h}) = \vec{h}^{\alpha}$ for $\alpha \in \mathbb{N}^3$. To be more specific, if $\vec{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then $p(\vec{h}) = h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3}$. Find partial derivatives of arbitrary oder of the function p at the point $\vec{h} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

6. Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are C^∞ functions. Suppose $F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0$. Suppose the degree 2 Taylor polynomial of F at the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is

$$T_2(\begin{bmatrix} h \\ k \end{bmatrix}) = a_1 h + a_2 k + b_1 h^2 + b_2 hk + b_3 k^2$$

and the degree 2 Taylor polynomial of g at 0 is

$$S_2(l) = c + dl + el^2.$$

(i) Determine the constants a_i, b_j, c, d and e in terms of partial derivatives of F at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the derivatives of g at 0.

(ii) Suppose R is the polynomial of degree 2 obtained by forming the composition $S_2 \circ T_2$ and then erasing the terms of degree greater than 2. Use your answer to the first part to determine R in terms of the partial derivatives of F at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the derivatives of g at 0.

(iii) Show that R is equal to the degree 2 Taylor polynomial of the composed function $g \circ F : \mathbb{R}^2 \to \mathbb{R}$. (Hint: To show this you have to relate the coefficients of R to the partial derivatives of $g \circ F$ at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Use chain rule (for one variable functions) to establish this relation.)

Notice that this is a special case of a general result about Taylor polynomials of composed functions that we stated in the class. The proof of the general case is similar.