Problem Set 8

1. Let $M = \{(x,y,z) \in \mathbb{R}^3 \, : \, x^2 + y^2 = z^2, \; z > 0\}$.

   (i) Show that M is a smooth manifold.

   (ii) What is the dimension of M?

   (iii) Find the tangent space of M at the point (3,4,5).

2. Let

$$M = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1 \text{ and } x^3 + y^3 + z^3 + xyz = 0\}.$$

(i) Show that M is a smooth manifold.

(ii) What is the dimension of M?

(iii) Find the tangent space of M at the point $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$.

3. Let $U = \{(s,t) \in \mathbb{R}^2 \,:\, 0 < s < \pi, 0 < t < 2\pi\}$. Define $\phi : U \to \mathbb{R}^3$ by

$$\phi(s,t) = (\sin(s)\cos(t), \sin(s)\sin(t), \cos(s)),$$

and let $M = \phi(U)$.

(i) Give a simple geometric description of M.

(ii) Use the parameterization ϕ to show that M is a smooth manifold. (Be sure to use the parameterization and the corresponding theorem (p.174 of the textbook, for reference). Do not just redefine M using your answer to part (a).)

4. Let

$$S = \{(r,\theta) \,:\, -\frac{1}{2} < r < \frac{1}{2},\ 0 \leq \theta < 2\pi\}$$

and

$$S' = \{(r,\theta) \,:\, -\frac{1}{2} < r < \frac{1}{2},\ 0 < \theta < 2\pi\}$$

(so S' is the interior of S). Define $\gamma : S \to \mathbb{R}^3$ by

$$\gamma(r,\theta) = \begin{bmatrix} 2\cos\theta + r\cos(\theta/2) \\ 2\sin\theta + r\sin(\theta/2) \\ r\sin(\theta/2) \end{bmatrix}.$$

Then the image $M = \gamma(S)$ is a Mobius strip (graph this using software if you want to see for yourself). You may assume without proof that γ is injective on S, and $D(\gamma)(\vec{c})$ is injective for each $\vec{c} \in S$. However, this doesn't make γ a valid parameterization, because S is not open. Instead, we restrict the domain of γ to S' (which is open).

(i) Which points of M are missing from the image $\gamma(S')$? Explain, and describe the missing set geometrically.

(ii) Find another open set $T \subseteq \mathbb{R}^2$ and another parameterization $\phi : T \to M$ such that $\phi(T)$ is "most of" M (in the same sense that $\gamma(S')$ is "most of" M), but the set of points missing from $\phi(T)$ is different from the set of points missing from $\gamma(S')$. Explain why your T and ϕ work, and describe the missing points of M. (You don't need to prove that ϕ and $D(\phi)$ are injective.)

That is, the goal of this problem is to "patch over" the part of M that's missing from $\gamma(S')$, thus parameterizing all of M.