Take-Home Exam I

1. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is the map

$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2y - x + yz \\ xyz + x + 2y + z^2 \end{bmatrix}.$$

(i) Show that this function is differentiable at the point $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ using the definition of differentiable functions. (Do not cite the theorem on continuity of partial derivatives or any other result that does not require the definition of differentiability. You may use our discussion in the class on the examples of little-o functions.)

(ii) Show that this function is differentiable at any point $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ using the definition of differentiable functions. (As in the first part, do not cite the theorem on continuity of partial derivatives or any other result that does not require the definition of differentiability. You may use our discussion in the class on the examples of little-o functions.)

2. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers which is bounded. Define:

$$b_k = \inf\{a_i \mid i \geq k\}.$$

Show that the sequence $\{b_k\}_{k=1}^{\infty}$ is convergent. (Hint: You may want to use the result of Problem 2 in Problem Set 1.)

3. Suppose A is an $m \times n$ matrix. Suppose also $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear map $T(x) = Ax$.

   (i) Show that the image of T can be identified with the column space of A.

   (ii) Show that the kernel of T can be identified with the orthogonal complement of the row space of A.

   (iii) Use parts (i) and (ii) to show that the dimension of the row space and the column space of A are equal to each other. (Hint: There are two identities which are useful in proving this claim. First recall that

   $$\dim(\ker(T)) + \dim(\text{image}(T)) = n.$$

   The other identity is what you proved in Problem 6 of Problem Set 3: if V is a subspace of $\mathbb{R}^n$, then:

   $$\dim(V) + \dim(V^{\perp}) = n$$

   where $V^{\perp}$ is the orthogonal complement of V.)